

The equilibrium consequences of indexing*

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Abstract

We develop a benchmark model to study the equilibrium consequences of indexing in a standard rational expectations setting (Grossman and Stiglitz (1980); Hellwig (1980); Diamond and Verrecchia (1981)). Individuals must incur costs to participate in financial markets, and these costs are lower for individuals who restrict themselves to indexing strategies. Individuals' participation decisions exhibit strategic complementarity. As indexing becomes cheaper (1) indexing increases, while individual stock trading decreases; (2) aggregate price efficiency falls, while relative price efficiency increases; (3) the welfare of relatively uninformed traders increases; (4) for well-informed traders, the share of trading gains stemming from market timing increases, and the share of gains from stock selection decreases; (5) market-wide reversals become more pronounced. We discuss empirical evidence for these predictions.

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1 Introduction

The standard investment recommendation that academic financial economists offer to individual retail investors is to purchase a low-fee index mutual fund or exchange-traded fund (ETF). An increasing number of such products are available, and are increasingly inexpensive and accessible, and an increasing number of investors follow this advice. In this paper, we develop a benchmark model to analyze the equilibrium consequences of an increase in “indexing,” paying particular attention to consequences for welfare.¹

To preview our results: First, and unsurprisingly, a reduction in the cost of indexing leads to more indexing, and less trading of individual stocks. Second, this reduces price efficiency of the aggregate stock market (i.e., market price movements become more divorced from cash flow news), but increases the efficiency of the relative prices of individual securities. Third, the welfare of relatively uninformed “retail” investors increases, precisely because price efficiency has fallen. Fourth, and in contrast, the welfare of relatively well-informed “institutional” investors may either increase or decrease: they gain more from aggregate market timing trades, but less from individual security trades. Fifth, as indexing increases, the extent to which high market-wide prices today forecast low market returns in the future (reversal) increases. As we review below, these predictions are largely consistent with empirical evidence.

The indexing recommendation is frequently justified by the observation that retail investors are unlikely to be the most informed traders in the market.² Because we want to

¹Our focus on welfare differentiates us from related papers on index futures and exchange traded funds by Subrahmanyam (1991), Cong and Xu (2016), Bhattacharya and O’Hara (2017), all of which employ models with exogenous “noise” trade.

²For example, Cochrane (2013) writes: “The average investor theorem is an important benchmark: The average investor must hold the value-weighted market portfolio. Alpha, relative to the market portfolio, is by definition a zero-sum game. For every investor who over-weights a security or invests in a fund that earns positive alpha, some other investor must underweight the same security and earn the same negative alpha. Collectively, we cannot even rebalance. And each of us can protect ourselves from being the negative-alpha mark with a simple strategy: hold the market portfolio, buy or sell only the portfolio in its entirety, and refuse to trade away from its weights, no matter what price is offered. If every uninformed trader followed this strategy, informed traders could never profit at our expense.” French (2008) makes a similar argument. At the same time, note also that Pedersen (forthcoming) documents that even indexing requires a considerable amount of trading, consistent with the prediction of our model that even indexing agents care about price

characterize welfare, a necessary first step in our analysis is to characterize welfare in an economy of the type introduced by Diamond and Verrecchia (1981). Like Grossman and Stiglitz (1980) and Hellwig (1980), these authors analyze trade between differentially informed agents, but different from these papers, there are no exogenous “noise” or “liquidity” trades. Instead, agents have heterogeneous and privately observed exposures to risk. Consequently, financial markets hold the potential to increase welfare by allowing agents to redistribute risk. Perhaps surprisingly, and although a model of this type has been analyzed by a significant number of authors, results on welfare are scarce, as we review below.

Specifically, in the first part of the paper we analyze an economy of this type with participation costs and heterogeneous precision levels of private signals, so that agents with relatively imprecise signals correspond to retail investors. As one would expect, only agents with sufficiently precise private signals participate, and a fall in participation costs increases participation by reducing the cutoff level of precision associated with participation. Increased participation by relatively uninformed traders in turn reduces price efficiency. Our main result in this first part of the paper is to show that increased participation leads to a Pareto improvement in welfare, precisely because of the fall in price efficiency. Consequently, individuals’ participation decisions exhibit strategic complementarity: the more other relatively uninformed agents participate, the greater the gains of participation. This result is related to the so-called “Hirshleifer effect” (Hirshleifer (1971)), but does not follow directly from it (see subsection 2.2 below, and also related discussions in Marín and Rahi (1999) and Dow and Rahi (2003)).

For transparency of exposition, we derive the above results in an economy with a single risky asset. In the second part of the paper, we extend our model to one with multiple (specifically, two) risky assets, and apply our results to understand the equilibrium consequences of indexing.

A number of our predictions are consistent with empirical evidence. Israeli, Lee, and Sridharan
efficiency.

(2017) document a negative correlation between ETF ownership and price efficiency.³ Israeli, Lee, and Sridharan (2017) also document that analyst coverage of individual stocks declines in ETF activity, consistent with reduced participation by relatively well-informed investors in individual securities. Consistent with our prediction that indexing increases the benefits that relatively well-informed investors obtain from market-timing strategies, but reduce the benefits they obtain from asset-selection strategies, AQR document that the correlation between hedge fund returns and market returns has risen from 0.6 to 0.9 over the last two decades.⁴ Related, Stambaugh (2014) documents a decline in asset-selection strategies by active mutual funds over the same period. Also related, and using data since 2000, Gerakos, Linnainmaa, and Morse (2017) show that a significant fraction of returns generated by active mutual funds stem from market timing strategies.

Asides from its implications for the equilibrium effects of indexing, our paper also contributes to the wider debate of the extent to which the financial sector contributes to social welfare (see, e.g, Baumol (1965)). In particular, we work with a canonical model in which a financial market exists because it facilitates risk-sharing, and show that informed trading generally worsens this risk-sharing function, while uninformed trading improves it. While we believe there is considerable value in isolating the effect of informed trading on a specific function of the financial sector, we also fully acknowledge that our analysis is silent on how informed trading affects other possible functions of the financial sector. For example, we do not speak to the question of whether information produced by financial markets is valuable in incentive contracts or in guiding resource allocation decisions (see Bond, Goldstein, and Edmans (2012) for a survey).

Related literature: In addition to the papers noted in footnote 1, our paper is also related to Stambaugh (2014), who considers the implications of a decline in noise trade in individual

³For full disclosure, we note that Glosten, Nallareddy, and Zou (2016) document just the opposite. As discussed by Israeli, Lee, and Sridharan (2017), the difference in results stems from the time period over which changes in ETF ownership is measured.

⁴See “Hedge fund correlation risk alarms investors,” Financial Times, June 29th, 2014.

assets. In our paper, a somewhat analogous decline in trading by relatively uninformed agents occurs. In contrast to Stambaugh's paper, this decline is an endogenous reaction to a decline in the cost of indexing strategies. Moreover, many of our results relate to welfare, which is absent in Stambaugh's analysis.

In an independent, contemporaneous, and complementary paper, Baruch and Zhang (2017) likewise study the equilibrium consequence of indexing, though from a very different perspective. They consider a multi-asset version of Grossman (1976), so that without indexing prices fully reveal agents' private signals. In this setting they show that an exogenous increase in indexing reduces the amount of information prices contain about individual assets, while the amount of information prices contain about aggregates is unaffected.

One contribution of our paper is to the understanding of welfare in economies of the type introduced by Diamond and Verrecchia (1981). As we noted, such results are surprisingly scarce. One significant algebraic complication in characterizing welfare is that, when combined with the asset price, each agent's private exposure shock contains information about expected asset payoffs. To avoid this complication, Verrecchia (1982) and Diamond (1985) both consider a sequence of economies in which the variance of each individual's exposure shock grows with the number of agents, and directly study the limit of this sequence. In the limit economy, each agent's exposure shock has infinite variance, and so expected utility prior to the realization of the exposure shock is undefined, in turn making it impossible to analyze participation decisions prior to the realization of exposure shocks.⁵

In an independent, contemporaneous, and complementary paper, Kawakami (2017) also makes progress in characterizing welfare in a setting along the lines of Diamond and Verrecchia (1981). Whereas we focus on an economy with a continuum of agents and allow for heterogeneity in the precision of signals about cash flows that agents observe, thereby allowing us to consider the effect an increase in participation by relatively uninformed agents, Kawakami instead considers a finite-agent economy with homogeneous signal precisions, in which an

⁵If instead one modeled participation decisions as being made after the exposure shock, then almost all agents would participate, since their exposure shocks are so large.

increase in the size of the market is associated with better diversification of individual exposure shocks. Analytically, we make more explicit use than Kawakami of market-clearing conditions, which allows us to incorporate heterogeneity in signal precisions in a tractable way.

Marín and Rahi (1999) obtain welfare results in a relatively specialized setting: there are two classes of agents, one class of which sees identical signals about asset payoffs and private endowments, and another class of completely uninformed agents. Moreover, the traded asset is in zero net supply. Dow and Rahi (2003) also analyze welfare, and obtain some tractability by inserting a risk-neutral market maker into the economy, which reduces the applicability of the model for analyzing aggregate financial markets.

In closely related settings, Vives and Medrano (2004) argue that “the expressions for the expected utility of a hedger . . . are complicated,” whereas Kurlat and Veldkamp (2015) write that “there is no closed-form expression for investor welfare.” The complications, common to our model as well, stem from the role of exposure shocks as signals about asset cash flows, on top of the standard risk sharing role that motivates trade.

2 Informed trading and welfare

As discussed above, we begin our analysis by characterizing welfare in an economy similar to Diamond and Verrecchia (1981). The welfare results we obtain in this section are essential to the analysis of indexing in Section 3. Although we obviously need an economy with multiple risky assets in order to study indexing, in this section we consider an economy with just one risky asset, in order to present our welfare analysis as transparently as possible.

2.1 The model

We work with a version of Diamond and Verrecchia (1981) in which there are a continuum of agents (see Ganguli and Yang (2009) and Manzano and Vives (2011)), indexed by the unit

interval, $i \in [0, 1]$. Each agent i has preferences with constant absolute risk aversion (CARA) over terminal W_i , and a coefficient of absolute risk aversion of γ .

There is one risky asset available for trading, with a normally distributed payoff X , with mean μ_X and variance $\frac{1}{\tau_X}$. The price of the asset is P , which is determined in equilibrium. Agents are small relative to the market, and act as price-takers; we characterize a competitive equilibrium of the economy.

The risky asset is in positive net supply, distributed equally among agents, with each agent having an initial endowment \bar{s} . We denote by θ_i agent i 's position in the risky asset after trade.

In addition, agents also have other sources of income (e.g., labor income) that are correlated with the cash flow of the risky asset. For simplicity, we assume the correlation is perfect, and write agent i 's income from sources other than the risky asset as $e_i X$, where e_i is (privately) known to agent i at the trading date. Differences in “exposures” e_i across agents motivate trade in the risky asset.

Hence the terminal wealth of agent i is determined by trading profits, along with the combination of asset endowment \bar{s} and other income $e_i X$:

$$W_i(\theta_i) \equiv \bar{s}P + e_i X + \theta_i(X - P) = (\bar{s} + e_i)P + (\theta_i + e_i)(X - P). \quad (1)$$

Although each agent knows his own income exposure e_i , he does not know that of other agents. Specifically, we assume $e_i = Z + u_i$, where $Z \sim \mathcal{N}\left(0, \frac{1}{\tau_Z}\right)$ and $u_i \sim \mathcal{N}\left(0, \frac{1}{\tau_u}\right)$. Note that total income thus has an aggregate component ZX , where both Z and X are unknown to individual agents, although their own income exposures e_i provide (private) signals about Z .

Prior to trading, each agent i observes a private signal of the form

$$y_i = X + \epsilon_i,$$

where $\epsilon_i \sim \mathcal{N}\left(0, \frac{1}{\tau_i}\right)$. Note that the precisions of private signals are heterogeneous across agents, so that some agents are more informed than others. Without loss, we order agents so that signal precision τ_i is decreasing in i ; and for simplicity, we assume τ_i is strictly decreasing.

An agent's information set at the time of trading is hence the triple (y_i, e_i, P) , which consists of his signal about cash flows y , his own exposure e_i , and the price P .

Finally, agents incur a cost κ of participating in financial markets, reflecting a combination of information collection and processing costs, psychic costs, expected trading costs, and the cost of potentially trading in a less than optimal way. Agents make participation decisions prior to observing any of (y_i, e_i, P) . If an agent does not participate, $\theta_i = \bar{s}$.

The equilibrium definition is a straightforward extension of that used in competitive rational expectations models (Grossman and Stiglitz (1980), Hellwig (1980)), with the participation decision incorporated:

Definition 1 *A rational expectations equilibrium is a set agents who choose to participate, $N \subset [0, 1]$, trading strategies $\{\theta_i\}_{i \in N}$, and a price function $P(X, Z)$ such that markets clear,*

$$\int_N (\theta_i - \bar{s}) di = 0, \quad (2)$$

and (taking prices as given), agent i 's trading strategy is optimal,

$$\theta_i \in \arg \max_{\tilde{\theta}_i} \mathbb{E} [u (W_i (\theta_i)) | y_i, e_i, P], \quad (3)$$

and participation decisions are optimal,

$$\mathbb{E} [u (W_i (\theta_i) - \kappa)] \geq \mathbb{E} [u ((\bar{s} + e_i) X)].$$

We emphasize that this is a canonical setting, in which risk-sharing benefits lead to gains from trade, which in turn allows for informed trading.

Throughout, we assume

$$4\gamma^2(\tau_Z^{-1} + \tau_u^{-1}) < \tau_X \quad (4)$$

$$\gamma^2 > 4\tau_0\tau_u, \quad (5)$$

where τ_0 is the precision of agent 0's information, i.e., the highest precision in the population of agents. Condition (4) ensures that expected utility is well-defined for an agent who behaves autarchically, and does not trade. Without this condition, an autarchic agent is exposed to so much risk that his expected utility is infinitely low. Condition (5) ensures that an equilibrium exists at the trading stage (see Proposition 1 below). Loosely speaking, without this condition there is too much trading based on information relative to trading based on risk-sharing; Ganguli and Yang (2009) impose essentially the same condition.⁶

2.2 Welfare benchmarks

It is useful to consider a couple of welfare benchmarks. First, in the (symmetric)⁷ unconstrained solution to the social planner's problem each agent i has terminal wealth

$$W_i = (\bar{s} + Z) X. \quad (6)$$

That is, the aggregate endowment $(\bar{s} + Z) X$ is simply split equally among agents. This is the outcome that would be obtained if agents could pool risk before knowing their exposures e_i , and if contracts could be written contingent on the realizations of e_i .

A second useful benchmark is the case in which all signals about X are public, with all other aspects the same as in the model described above (in particular, exposures e_i are

⁶The main extension in Manzano and Vives (2011) relative to Ganguli and Yang (2009) is that they allow for the error terms in the trader's signals to be correlated. Non-zero correlation eliminates the existence issues in our model. Since our focus is on welfare, we choose to study the slightly more tractable model with conditionally independent estimation errors.

⁷In non-symmetric solutions, each agent has terminal wealth $W_i = (\bar{s} + Z) X + K_i$, where K_i is a constant, and $\int K_i di = 0$.

private information, and trade occurs only after agents observe these exposures). In this case, all agents have the same posterior of X at the trading stage, and so portfolios θ_i must satisfy (see (8) below)

$$\theta_i + e_i = \bar{s} + Z.$$

So each agent's terminal wealth is

$$W_i = (\bar{s} + e_i) P + (\bar{s} + Z) (X - P) = u_i P + (\bar{s} + Z) X. \quad (7)$$

Note that in this second benchmark, each agent is exposed to an additional risk term, $u_i P$.

The comparison of these two benchmarks illustrates the challenge of characterizing how information about X affects welfare. For example, comparing (6) and (7), one can see that *ceteris paribus* agents prefer the price P to have low variance. In turn, P has low variance if it is relatively unaffected by both the realization of the cash flow X and the aggregate exposure Z . But higher-precision signals about X may increase P 's dependence on X (increasing the variance of P , and the Hirshleifer effect) but decrease P 's dependence on Z (reducing the variance of P), so that the overall effect is unclear.⁸

2.3 Basic equilibrium properties

As standard in the literature, to characterize an equilibrium we first conjecture key equilibrium characteristics, and then verify that an equilibrium with these characteristics indeed exists. More concretely, we characterize linear equilibria (i.e., the price is a linear function of the cash flow X and the aggregate endowment Z) in which all agents with sufficiently precise

⁸It is worth noting that the limiting case of perfect information about X is straightforward. In this case, the price P simply equals X , and so (7) reduces to $W_i = (\bar{s} + e_i) X$, which is the autarchy outcome. Hence welfare is minimized by perfect information about X , since in this case the financial market cannot provide any risk sharing (the Hirshleifer effect). Our analysis below concerns the more relevant non-limit case. Moreover, note that Diamond (1985) characterizes how welfare changes as the precision of public information changes, though with the mathematical compromises discussed earlier. Finally, in an online appendix we show that welfare in this benchmark case indeed monotonically declines in the precision of public information, though the proof is non-trivial, consistent with the discussion above.

signals participate. In such equilibria, there is a cutoff agent n such that all agents $i \leq n$ participate, and agents $i > n$ do not participate. In this subsection we establish some key equilibrium properties that hold in any equilibrium of this type. To maximize transparency, we establish these properties as directly as possible, making use primarily of the market clearing condition (2).

In a linear equilibrium, each agent i 's optimal portfolio has the standard mean-variance form,

$$\theta_i + e_i = \frac{1}{\gamma} \frac{\mathbb{E}[X - P|y_i, e_i, P]}{\text{var}(X - P|y_i, e_i, P)} = \frac{1}{\gamma} \frac{\mathbb{E}[X|y_i, e_i, P] - P}{\text{var}(X|y_i, e_i, P)}. \quad (8)$$

The form of the optimal portfolio (8) indicates that the reciprocal of conditional variance, $\text{var}(X|y_i, e_i, P)^{-1}$, is an important quantity. The following result uses market clearing (2) to derive a useful relation between the average reciprocal of conditional variance in the economy, and the (endogenous) covariance between returns $X - P$ and cash flows X .

Lemma 1 *In any linear equilibrium,*

$$\frac{1}{n} \int_N \frac{1}{\text{var}(X|y_i, e_i, P)} di = \frac{1}{\text{cov}(X - P, X)}. \quad (9)$$

Note that Lemma 1 nests the special case in which no agent has any information about the cash flow X , so that the price is unrelated to X , and so for any agent i , $\text{var}(X|y_i, e_i, P) = \text{var}(X) = \text{cov}(X - P, X)$.

Lemma 1 turns out to be very important in our analysis, and since its proof is short, we give it here. Differentiation of market clearing (2) with respect to X gives

$$\frac{\partial}{\partial X} \int_N \theta_i di = 0.$$

Substituting in the portfolio θ_i from (8); recalling the property of multivariate normality that conditional variances do not depend on the realizations of random variables; and noting

that $\frac{\partial P}{\partial X} = \frac{\text{cov}(P, X)}{\text{var}(X)}$, it follows that

$$\int_N \frac{\frac{\partial}{\partial X} \mathbb{E}[X|y_i, e_i, P]}{\text{var}(X|y_i, e_i, P)} di = \int_N \frac{\frac{\text{cov}(P, X)}{\text{var}(X)}}{\text{var}(X|y_i, e_i, P)} di. \quad (10)$$

Because the information set (y_i, e_i, P) consists of a set of normally distributed random variables,

$$\frac{\partial}{\partial X} \mathbb{E}[X|y_i, e_i, P] = 1 - \frac{\text{var}(X|y_i, e_i, P)}{\text{var}(X)}. \quad (11)$$

(See Lemma A-1 for a formal proof.) Substitution of (11) into (10) yields

$$\left(1 - \frac{\text{cov}(P, X)}{\text{var}(X)}\right) \int_N \frac{1}{\text{var}(X|y_i, e_i, P)} di = \int_N \frac{1}{\text{var}(X)} di,$$

which is equivalent to (9), completing the proof of Lemma 1.

Among other things, we use Lemma 1 to characterize the equilibrium risk premium $\mathbb{E}[X - P]$. Taking the unconditional expectation of (8) gives

$$E[\theta_i] = \frac{1}{\gamma} \frac{\mathbb{E}[X - P]}{\text{var}(X|y_i, e_i, P)}.$$

Combined with market clearing (2) (specifically, $\frac{1}{n} \int_N E[\theta_i] di = \bar{s}$), we obtain:

Corollary 1 *In a linear equilibrium,*

$$\mathbb{E}[X - P] = \gamma \bar{s} \text{cov}(X - P, X).$$

As for Lemma 1, it may help to note that Corollary 1 nests the special case in which no agent has any information, and so $\mathbb{E}[X - P] = \gamma \bar{s} \text{var}(X)$.

Both prices and endowment shocks play two distinct roles in determining an agent's demand: they directly affect demand, and separately, they also affect an agent's beliefs about the cash flow X , thereby indirectly affecting demand. To clarify this dual role, write $\theta_i(y_i, e_i, \tilde{e}_i, P, \tilde{P})$ for the demand of an agent who has exposure e_i and can trade at price

P , but who evaluates his conditional distribution over X using the the information set $(y_i, \tilde{e}_i, \tilde{P})$. Even though $\tilde{e}_i = e_i$ and $\tilde{P} = P$, keeping separate track of the two roles of prices and endowment shocks is conceptually useful.

To further exploit the equilibrium market clearing condition (2), note that differentiation with respect to X and Z respectively yields:

$$\frac{\partial P}{\partial X} \int_N \frac{\partial \theta_i}{\partial P} di + \frac{\partial P}{\partial X} \int_N \frac{\partial \theta_i}{\partial \tilde{P}} di + \int_N \frac{\partial \theta_i}{\partial Y_i} di = 0 \quad (12)$$

$$\frac{\partial P}{\partial Z} \int_N \frac{\partial \theta_i}{\partial P} di + \frac{\partial P}{\partial Z} \int_N \frac{\partial \theta_i}{\partial \tilde{P}} di + \int_N \frac{\partial \theta_i}{\partial e_i} di + \int_N \frac{\partial \theta_i}{\partial \tilde{e}_i} di = 0. \quad (13)$$

One immediate consequence is that, in equilibrium, $\frac{\partial P}{\partial Z} \neq 0$. To see this, note that if instead $\frac{\partial P}{\partial Z} = 0$, then Z and e_i provide no information about the cash flow X , so that $\frac{\partial \theta_i}{\partial \tilde{e}_i} = 0$ for all agents. In contrast, the non-informational effect of endowment shocks on the portfolio decision is certainly negative (formally, see Lemma 2). But then the lefthand side of (13) is strictly negative, a contradiction.

As typical for this class of models, an important equilibrium quantity is the relative sensitivity of price to X and Z , which we denote by ρ :

$$\rho \equiv -\frac{\frac{\partial P}{\partial X}}{\frac{\partial P}{\partial Z}}.$$

We refer to ρ as the price efficiency of the risky asset, since

$$\text{var}(X|P)^{-1} = \tau_X + \rho^2 \tau_Z \quad (14)$$

$$\text{var}(X|y_i, e_i, P)^{-1} = \tau_X + \rho^2(\tau_Z + \tau_u) + \tau_i. \quad (15)$$

These expressions (derived in the proof of Lemma 2) measure the ability of an outside observer and agent i , respectively, to forecast the cash flow X .

The following properties of individual demand follow only from Bayesian updating. They hold whenever price P is a linear function of X and Z , regardless of whether P is an

equilibrium price.

Lemma 2 *If P is a linear function of X and Z then the effects of non-informational factors on demand θ_i are given by*

$$\frac{\partial \theta_i}{\partial e_i} = -1; \quad \frac{\partial \theta_i}{\partial P} = -\frac{1}{\gamma} \frac{1}{\text{var}(X|y_i, e_i, P)};$$

while the effects of informational factors on demand θ_i satisfy

$$\frac{\partial \theta_i}{\partial y_i} = \frac{\tau_i}{\gamma}; \quad \frac{\partial \theta_i}{\partial \tilde{e}_i} = \frac{\rho}{\gamma} \tau_u; \quad \frac{\partial P}{\partial Z} \frac{\partial \theta_i}{\partial \tilde{P}} = -\frac{\rho}{\gamma} (\tau_Z + \tau_u); \quad \frac{\partial P}{\partial X} \frac{\partial \theta_i}{\partial \tilde{P}} = \frac{\rho^2}{\gamma} (\tau_Z + \tau_u).$$

Note that the informational effects of \tilde{e}_i and Z (as reflected in the price \tilde{P}) on demand are related by the precisions of the idiosyncratic and aggregate components of exposure shocks:

Corollary 2 *In a linear equilibrium, for any agent i ,*

$$\frac{\frac{\partial P}{\partial Z} \frac{\partial \theta_i}{\partial \tilde{P}}}{\frac{\partial \theta_i}{\partial \tilde{e}_i}} = -\frac{\tau_Z + \tau_u}{\tau_u}, \quad (16)$$

$$\frac{\partial P}{\partial Z} \frac{\partial \theta_i}{\partial \tilde{P}} + \frac{\partial \theta_i}{\partial \tilde{e}_i} = -\frac{\rho}{\gamma} \tau_Z. \quad (17)$$

In equilibrium $\frac{\partial P}{\partial Z} < 0$ and $\rho \geq 0$ (see Lemma 3 immediately below). So a higher Z is associated with lower prices, which in turn are associated with lower estimates of X . Holding u_i fixed, a higher Z also leads to a higher value of e_i , thereby raising an agent's estimate of Z , and hence (given equilibrium prices) an agent's estimate of X . Equation (17) shows that the first of these effects dominates.

We next establish the basic result that aggregate demand for the risky asset is decreasing in the price. Because of the informational content of prices, this is not completely obvious. At the same time, we show that the price is increasing in the asset's payoff, and the price is decreasing in the aggregate endowment shock. We highlight that the proof of this result makes use only of the market clearing condition (2), along with the *signs* (but not

magnitudes) established in Lemma 2.

Lemma 3 *In a linear equilibrium, the aggregate demand curve slopes down, i.e.,*

$$\int_N \frac{\partial \theta_i}{\partial P} di + \int_N \frac{\partial \theta_i}{\partial \tilde{P}} di < 0, \quad (18)$$

and the price is an increasing function of X , and a strictly decreasing function of Z ,

$$\frac{\partial P}{\partial X} \geq 0 \text{ and } \frac{\partial P}{\partial Z} < 0,$$

and so in particular $\rho \geq 0$.

A further immediate implication of Lemmas 1 and 2 is the following expression for the non-informational effect of prices on aggregate demand:

Corollary 3 *In a linear equilibrium, the non-informational effect of prices on aggregate demand, i.e., $\int_N \frac{\partial \theta_i}{\partial P} di$, satisfies*

$$\frac{1}{n} \int_N \frac{\partial \theta_i}{\partial P} di = -\frac{1}{\gamma} \frac{1}{\text{cov}(X - P, X)}. \quad (19)$$

Knowledge of individual exposure e_i contains information about X only because it helps agent i interpret the price (for example, it provides information about whether a high price is due to a high cash flow X or a low aggregate exposure Z). Because the information in exposure is subsidiary to the information in prices, it is intuitive that prices contain more information than exposures, as formalized in the following result:

Lemma 4 *In a linear equilibrium, the ratio of the informational to non-informational effect of prices on demand exceeds the ratio of the informational to non-informational effect of exposures on demand,*

$$\frac{\left| \int_N \frac{\partial \theta_i}{\partial \tilde{P}} di \right|}{\left| \int_N \frac{\partial \theta_i}{\partial P} di \right|} > \frac{\left| \int_N \frac{\partial \theta_i}{\partial e_i} di \right|}{\left| \int_N \frac{\partial \theta_i}{\partial e_i} di \right|}. \quad (20)$$

Lemma 4 turns out to be critical to the analysis of welfare below. We again highlight that its proof makes use only of basic equilibrium properties established above, along with the *signs* (but not magnitudes) established in Lemma 2 and Corollary 2.

2.4 Equilibrium at the trading stage

Ganguli and Yang (2009) and Manzano and Vives (2011) give conditions for equilibrium existence, taking the set of trading agents as given. We follow Manzano and Vives (2011) and focus on the equilibrium with lower price efficiency, since the one with higher price efficiency is unstable.⁹ The following result contains a minor extension of these papers to cover heterogeneity in information precision, and then uses results from the previous section to give an explicit expression for the risk premium $\mathbb{E}[X - P]$.

Proposition 1 *There is unique stable linear equilibrium, in which price efficiency ρ is given by*

$$\rho = \frac{\gamma}{2\tau_u} - \sqrt{\left(\frac{\gamma}{2\tau_u}\right)^2 - \frac{1}{\tau_u} \frac{1}{n} \int_N \tau_i di}. \quad (21)$$

Price efficiency ρ is decreasing in participation n . The unconditional risk premium is given by

$$\mathbb{E}[X - P] = \frac{\gamma \bar{s}}{\tau_X + \rho^2(\tau_Z + \tau_u) + \frac{1}{n} \int_N \tau_i di}. \quad (22)$$

From Proposition 1, price efficiency is determined by the average information precision of agents who actively trade. As participation increases, newly participating agents lower this average. Even though these agents bring more information to the market, they also bring more trade motivated by risk-sharing concerns, which function in the same way as noise.

⁹Manzano and Vives (2011) give a mathematical definition of stability. One way to think about stability is in terms of condition (A-7) in the proof of Proposition 1. The right hand side (RHS) describes agents' demands, which in turn depend on price efficiency (this can be seen explicitly from (A-8)). The left hand side (LHS) of (A-7) describes how prices must behave to clear the market, given agents' demands on the RHS. Equilibrium price efficiency is a fixed point of this relation. Moreover, the RHS is increasing in ρ , at least in the neighborhood of any solution. If the RHS crosses the 45° line from below, the corresponding equilibrium is unstable in the following sense: A small upwards perturbation in agents' beliefs about price efficiency affects agents' demands and shifts the RHS up (for any value of ρ). To preserve market clearing, this then pushes ρ up, and precisely because the RHS crosses the 45° line from below, the change in ρ is greater than the original perturbation in agents' beliefs about ρ , i.e., instability.

The risk premium $\mathbb{E}[X - P]$ is driven by the amount of aggregate risk, as measured by \bar{s} and τ_X , as well as the risk tolerance in the economy, γ . The risk premium also reflects the average conditional precision (see (15)) of participating agents. As participation increases, this decreases, both directly, as discussed immediately above, and also indirectly, as price efficiency falls. Consequently, agents bear more risk when they hold the asset, and the risk premium increases with participation.

2.5 Welfare and participation

We next turn to agents' participation decisions. To do so, we first characterize an agent's expected utility from participation. As noted in the introduction, a concise representation of expected utility in economies of this type has proved challenging to obtain in related work. Looking ahead, a concise representation is important for analyzing comparative statics in the participation cost κ .

Proposition 2 *In a linear equilibrium, agent i 's expected utility from participation, conditional on the realization of his exposure shock e_i , is given by*

$$\mathbb{E}[u(W_i - \kappa)|e_i] = -(d_i D)^{-\frac{1}{2}} \exp\left(-\gamma(\bar{s} + e_i)\mu_X + \frac{\gamma^2(\bar{s} + e_i)^2}{2\tau_X} - \frac{1}{2}\Lambda e_i^2 + \gamma\kappa\right) \quad (23)$$

where

$$d_i = \frac{\text{var}(X|e_i, P)}{\text{var}(X|y_i, e_i, P)}, \quad (24)$$

$$D = \frac{\text{var}(X - P|e_i)}{\text{var}(X|e_i, P)}, \quad (25)$$

$$\Lambda = \frac{\left(\frac{\text{cov}(P, e_i)}{\text{var}(e_i)} + \gamma\text{cov}(X - P, X)\right)^2}{\text{var}(X - P|e_i)}. \quad (26)$$

The key step in the proof of Proposition 2 is the substitution of Corollary 1's expression for the unconditional risk premium $\mathbb{E}[X - P]$.

To interpret Proposition 2, note that an agent's expected utility from non-participation is simply

$$\mathbb{E}[u((\bar{s} + e_i)X) | e_i] = -\exp\left(-\gamma(\bar{s} + e_i)\mu_X + \frac{\gamma^2(\bar{s} + e_i)^2}{2\tau_X}\right).$$

So an agent's expected gain from participation is reflected in the benefits D and Λ which stem from risk-sharing and are the same for all agents, no matter how precise or imprecise their private information; and the advantages stemming from more precise private information, represented by d_i . These gains must be balanced against the cost of participation, κ .

An immediate and intuitive implication of Proposition 2 is that expected utility is increasing in the precision of an agent's information, τ_i . So consistent with our initial conjecture, linear equilibria are characterized by some n such that agents $i \leq n$ participate, and agents $i > n$ do not.

Next, we show that agents' individual participation decisions exhibit strategic complementarity:

Proposition 3 *As participation n increases, each individual agent's gain from participation increases.*

Economically, the key driving force behind strategic complementarity is that, as participation n increases, price efficiency ρ drops (see Proposition 1 and related discussion). Loosely speaking, lower price efficiency increases the amount of risk-sharing that the financial market enables. Specifically, the risk sharing function of the financial market is to enable agents with high idiosyncratic exposures u_i to share cash flow risk $u_i X$ with other agents with low idiosyncratic exposures.

Lower price efficiency corresponds to agents having less information about the cash flow X , which makes risk sharing easier to sustain as in Hirshleifer (1971). However, and as discussed in the context of the public information benchmark of subsection 2.2, the risk sharing benefits of less efficient prices must be compared to the potential costs that arise from more volatile prices, since if prices are less efficient, they are relatively more exposed to

the aggregate exposure shock Z (essentially, the discount rate), and this can easily lead to greater volatility. Proposition 3 establishes that the benefits of lower price efficiency always dominate the potential costs.

Because Proposition 3 is central to our analysis, we give the structure of the argument here, focusing as much as possible on interpretable economic properties, while relegating algebraic details to the appendix.

To establish the result, we show that each of the three terms d_i , D , and Λ are increasing in participation n . We start with the term d_i , which corresponds to an agent's expected gains from the precision of his private signal y_i . Substitution of (15) delivers

$$d_i = \frac{\tau_X + \rho^2(\tau_Z + \tau_u) + \tau_i}{\tau_X + \rho^2(\tau_Z + \tau_u)}.$$

Because price efficiency ρ is decreasing in participation n , the private gains from information, d_i , are increasing in participation. Economically, when prices convey less information about cash flows X , an agent's private information about X is more valuable.

Next, we consider the risk-sharing terms Λ and D . As a first step, note that straightforward algebraic manipulation and the basic equilibrium property Corollary 3 combine to deliver

$$\Lambda = \frac{\frac{\text{var}(Z)^2}{\text{var}(e_i)^2} \left(\frac{1}{n} \frac{\partial P}{\partial Z} \int_N \frac{\partial \theta_i}{\partial P} di - \frac{\text{var}(e_i)}{\text{var}(Z)} \right)^2}{\frac{1}{\gamma^2 \text{var}(X)} + \left(\frac{1}{n} \frac{\partial P}{\partial Z} \int_N \frac{\partial \theta_i}{\partial P} di \right)^2 \text{var}(Z|e_i)} \quad (27)$$

$$D = 1 + \frac{(-\rho + \gamma \text{var}(Z|e_i) \frac{1}{n} \frac{\partial P}{\partial Z} \int_N \frac{\partial \theta_i}{\partial P} di)^2}{\gamma^2 \text{var}(Z|e_i) \text{var}(X) \left(\frac{1}{n} \int_N \frac{\partial \theta_i}{\partial P} di \right)^2}. \quad (28)$$

Given these expressions, it is clearly important to understand the term $\frac{1}{n} \frac{\partial P}{\partial Z} \int \frac{\partial \theta_i}{\partial P} di$. On the one hand, from Lemma 2 and (15),

$$\frac{1}{n} \int_N \frac{\partial \theta_i}{\partial P} di = -\frac{1}{\gamma} \frac{1}{n} \int_N \frac{1}{\text{var}(X|y_i, e_i, P)} di = -\frac{\tau_X + \rho^2(\tau_Z + \tau_u) + \frac{1}{n} \int_N \tau_i di}{\gamma}.$$

So as participation n increases, $\left| \frac{1}{n} \int_N \frac{\partial \theta_i}{\partial P} di \right|$ declines, both because price efficiency declines, as discussed above; and because the average signal precision of participating agents, $\frac{1}{n} \int_N \tau_i di$, declines. Economically, both forces mean that the average participating agent is exposed to more risk when they trade, and so $\left| \frac{\partial \theta_i}{\partial P} \right|$ increases. On the other hand, greater price efficiency is typically associated with prices that depend less on Z , corresponding to a reduction in $\left| \frac{\partial P}{\partial Z} \right|$. However, substitution of (17) of Corollary 2 into the market clearing condition (13), along with the very basic property that the non-informational effect of exposure shocks on demand is $\frac{\partial \theta_i}{\partial e_i} = -1$, yields

$$\frac{1}{n} \frac{\partial P}{\partial Z} \int_N \frac{\partial \theta_i}{\partial P} di = 1 + \frac{\rho}{\gamma} \tau_Z. \quad (29)$$

Hence the former effect is the dominant one, and $\frac{1}{n} \frac{\partial P}{\partial Z} \int_N \frac{\partial \theta_i}{\partial P} di$ is increasing in price efficiency ρ , and hence (by Proposition 1) is decreasing in participation n .

It then clear from (27) that Λ is indeed increasing in participation n if

$$\frac{1}{n} \frac{\partial P}{\partial Z} \int_N \frac{\partial \theta_i}{\partial P} di - \frac{\text{var}(e_i)}{\text{var}(Z)} < 0. \quad (30)$$

This is indeed the case, since (16) of Corollary 2 and $\frac{\partial \theta_i}{\partial e_i} = -1$ imply that (30) is equivalent to fact that the prices contain more information than endowments in the sense of (20), as established in Lemma 4.

Turning to the determinant term D , it is immediate that the denominator $\text{var}(X|e_i, P)$ is decreasing price efficiency ρ . Loosely speaking, one would also expect the numerator $\text{var}(X - P|e_i)$ to fall: as prices become more efficient, P is more closely related to X , and so $X - P$ is less volatile. Below, we establish that the numerator indeed falls, and moreover, that effect dominates the fall in $\text{var}(X|e_i, P)$.

Substituting for $\frac{\partial P}{\partial Z} \int_N \frac{\partial \theta_i}{\partial P} di$ in (28) using (29), and noting that $\text{var}(Z|e_i) = \frac{1}{\tau_Z + \tau_u}$, we obtain

$$D = 1 + \frac{(\gamma - \rho \tau_u)^2 \text{var}(Z|e_i)}{\gamma^2 \text{var}(X) \left(\frac{1}{n} \int_N \frac{\partial \theta_i}{\partial P} di \right)^2}. \quad (31)$$

As discussed above, both price efficiency ρ and the sensitivity of demand to price, $\left| \frac{1}{n} \int_N \frac{\partial \theta_i}{\partial P} di \right|$, are decreasing in participation n . So D is indeed decreasing in participation n if $\gamma - \rho \tau_u > 0$, which using the equality of (28) and (31) and the fact that $\frac{1}{n} \frac{\partial P}{\partial Z} \int \frac{\partial \theta_i}{\partial P} di = -\frac{\rho}{\gamma \text{var}(Z|e_i)}$ (see Lemma 2) is equivalent to the aggregate demand curve sloping down, i.e., inequality (18), as is established in Lemma 3.

Finally, and for use below, note that the above arguments establish that the benefits to agent i of financial market participation depend on the participation decisions of other agents, summarized by n , only via the the average precision of participating agents, $\frac{1}{n} \int_0^n \tau_i di$, and the associated price efficiency ρ (see Proposition 1). Accordingly, define

$$\begin{aligned} T(n) &\equiv \frac{1}{n} \int_0^n \tau_i di \\ f(n) &\equiv \frac{\gamma}{2\tau_u} - \sqrt{\left(\frac{\gamma}{2\tau_u}\right)^2 - \frac{T(n)}{\tau_u}}. \end{aligned}$$

2.6 Equilibrium and the effects of changes in participation costs

To establish both equilibrium existence, and to evaluate comparative statics with respect to participation costs, we define the function

$$\begin{aligned} g(n, \kappa) &= \sup \{i \in [0, 1] : \text{agent } i \text{ participates in the financial market if} \\ &\quad \text{the average signal precision of participating agents is } T(n) \\ &\quad \text{and price efficiency is } f(n)\}. \end{aligned}$$

That is, $g(n, \kappa)$ gives agents' participation decisions in response to an average signal precision in the market of $T(n)$, and price efficiency $\rho = f(n)$. By Proposition 3, $g(n, \kappa)$ is increasing in the argument n , while clearly it is decreasing in the cost argument κ .

An equilibrium participation level is then simply a solution to the fixed point relation $g(n, \kappa) = n$. Hence existence is immediate from Tarski's fixed point theorem, and compara-

tive statics are immediate from Milgrom and Roberts (1994):

Proposition 4 *An equilibrium exists. As the participation cost κ increases, participation declines, price efficiency increases, and the welfare of participating agents declines.¹⁰*

Given Propositions 1 and 3, the economics behind Proposition 4 are relatively straightforward. As participation costs decline, there is a direct increase in the utility of any agent who participates in the financial market. This direct increase in utility in turn leads to more participation, which by Proposition 1 reduces price efficiency, and by Proposition 3 further increases the utility of participation, in turn leading to further increases in participation. The ultimate equilibrium effect of the initial reduction in participation costs is characterized by the fixed point equilibrium condition $g(n, \kappa) = n$.

Agents with information of all precision levels benefit from the reduction in participation costs, though for slightly different reasons. Agents with low precisions of private information benefit both directly (lower participation costs), and indirectly, because greater participation improves risk sharing. It is worth reiterating that the better risk sharing is not a direct consequence of more agents in the market—after all, each agent brings more risk that needs to be shared, as well as more risk-sharing capacity. Instead, better risk sharing is a consequence of the fall in equilibrium price efficiency.

Agents with high levels of precision of private information *additionally* benefit from more profitable trading opportunities, again a consequence of lower equilibrium price efficiency. The following result sheds light on the trading behavior and profits of agents with different precisions of private information.

Proposition 5 *Fix an equilibrium with participation level n , along with an arbitrary $\tilde{n} \in (0, n)$. Then for any realization of X and Z , the average position of “smart” agents $[0, \tilde{n}]$*

¹⁰Given strategic complementarity of participation decisions, it is possible that there are multiple solutions to the equilibrium condition $g(n, \kappa) = n$. In such cases, Proposition 3 should be understood in the sense of Milgrom and Roberts (1994), i.e., as applying to the equilibria with minimum and maximum participation.

differs from the average position of “dumb” agents $[\tilde{n}, n]$ by an amount

$$\frac{1}{\gamma} \left(\frac{1}{\tilde{n}} \int_0^{\tilde{n}} \tau_i di - \frac{1}{n - \tilde{n}} \int_{\tilde{n}}^n \tau_i di \right) (X - P). \quad (32)$$

Moreover, $\mathbb{E}[X - P|P]$ is decreasing in P . Hence the price P is positively correlated with holdings of the asset by “dumb” agents in $[\tilde{n}, n]$, and negatively correlated with subsequent returns.

Proposition 5 follows straightforwardly from earlier analysis. Equation (32) says that better informed (“smart”) agents buy more of the asset than worse informed (“dumb”) agents precisely when the asset is undervalued ($X > P$). This follows immediately from the explicit evaluation of (8) (formally, see (A-2) in the proof of Lemma 2). The fact that $\mathbb{E}[X - P|P]$ is decreasing in P follows from the fact that, by market-clearing (2),

$$\frac{1}{\gamma} \mathbb{E}[X - P|P] \frac{1}{n} \int_0^n \frac{1}{\text{var}(X|y_i, e_i, P)} = \bar{s} + \mathbb{E}[Z|P], \quad (33)$$

along with the fact that $\mathbb{E}[Z|P]$ is decreasing in P (since $\frac{\partial P}{\partial Z} < 0$ by Lemma 3).¹¹

The prediction of Proposition 5 that assets are reallocated from informed to uninformed investors as the prices rise, and that subsequent returns are then low, is consistent with the empirical evidence in Ben-Rephael, Kandel, and Wohl (2012) and Jiang, Verbeek, and Wan (2017).¹²

¹¹Consequently, if an investor observes a low price for an asset, and has no exposure to economic shocks, he should take a long position in the asset, since its conditional expected return is high. That is, an investor can profit from buying “value” stocks. Although this point is often overlooked, it is nonetheless a standard implication of models of the type we consider here (see, e.g., Biais, Bossaerts, and Spatt, 2010).

¹²There is a sizable empirical literature studying the relation between flows between different types of investors and subsequent returns. Results depend significantly on how flows are measured.

3 The equilibrium consequences of indexing

In order to explicitly analyze the equilibrium consequences of indexing, we now extend our basic model to allow for multiple (specifically, two) risky assets. We model a rise in indexing as stemming from a fall in participation costs for agents who trade assets only in the same proportions as they are present in the market. We note that results of this section all follow straightforwardly from our earlier analysis.

3.1 The model

There are now two risky assets available for trading, with payoffs X_1 and X_2 , where we assume that both assets are normally distributed.¹³ The assets are symmetric, and both have mean μ_X and variance $\frac{1}{\tau_X}$. We denote the price of asset $j = 1, 2$ by denoted P_j . The two assets are again in positive net supply, initially distributed equally among agents, so that each agent i starts with an endowment \bar{s} of each of assets 1, 2. Let θ_{ij} denote agent i 's position in asset j after trade.

Similar to before, agents also have other sources of income, now correlated with the cash flows of both assets X_1 and X_2 . Specifically, agent i has income $e_{i1}X_1 + e_{i2}X_2$, where e_{i1} and e_{i2} are privately known at the time of trading. For example, cash flows X_1 and X_2 may correspond to two sectors of the economy, or two aggregate sources of risk (e.g., retail versus technology, or value versus growth stocks), and agents differ in the extent to which their non-trading income is exposed to these two sources of risk. Similar to before, $e_{ij} = Z_j + u_{ij}$ for $j = 1, 2$, where $Z_j \sim \mathcal{N}\left(0, \frac{1}{\tau_Z}\right)$ and $u_{ij} \sim \mathcal{N}\left(0, \frac{1}{\tau_u}\right)$. Hence agent i 's wealth after trading (and excluding information acquisition costs, described below) is

$$\sum_{j=1,2} (\bar{s}P_j + e_{ij}X_j + \theta_{ij}(X_j - P_j)).$$

¹³In other words, we extend Diamond and Verrecchia (1981) to multiple assets, just as Admati (1985) extends Hellwig (1980). Note that in Admati (1985) and other analyses of multiple-asset economies, “liquidity” trades are entirely exogenous.

As in the single-asset economy, each agent i observes private signals about the future cash flows of the two assets, $y_{ij} = X_j + \epsilon_{ij}$, for $j = 1, 2$. The precision of these signals again varies across agents, with $\text{var}(\epsilon_{ij}) = \frac{1}{\tau_i}$ for assets $j = 1, 2$.

The most significant change relative to the single-asset economy is that each agent now faces a choice between full participation in trading both assets 1 and 2; limited participation, meaning that an agent is restricted to “indexing” strategies; and no participation.

By indexing, we mean a portfolio strategy in which each asset’s weight in an agent i ’s portfolio matches the asset’s weight in market as a whole, i.e., a portfolio θ_{i1}, θ_{i2} such that

$$\frac{\theta_{i1}P_1}{\theta_{i1}P_1 + \theta_{i2}P_2} = \frac{\bar{s}P_1}{\bar{s}P_1 + \bar{s}P_2}.$$

So given our symmetry assumptions, an indexing strategy is simply one with $\theta_{i1} = \theta_{i2}$.

Participation that is limited to indexing strategies has a lower cost than full participation. Specifically, the costs of full participation, indexing participation, and no participation are respectively κ , κ_m , and 0, where $\kappa_m \in (0, \kappa)$. The lower participation cost of indexing reflects lower trading costs (because of the availability of low cost index mutual funds and exchange traded funds (ETFs)); lower cognitive demands and attention costs; and lower information costs, since as we will see shortly below, a sufficient statistic for an agent’s private information if he is indexing is the sum of the two signals, $y_{i1} + y_{i2} = (X_1 + X_2) + (\epsilon_{i1} + \epsilon_{i2})$, which can be interpreted as agent i simply paying attention to broad economic aggregates, instead of individual stocks.

3.2 Disentangling assets 1 and 2

Although the fundamentals of assets 1 and 2 are independent in all dimensions, the potential presence of indexing agents introduces a link between the prices of the two assets, since, for example, an indexing agent’s exposure e_{i1} to cash flow risk X_1 affects the agent’s demand for asset 2 as well as for asset 1.

Because of this entanglement that indexing produces between assets 1 and 2, analytically it is very convenient to effectively change basis and study the economy in terms of a “market” asset that pays $X_m \equiv \frac{1}{\sqrt{2}}(X_1 + X_2)$, and a “spread” asset that pays $X_s \equiv \frac{1}{\sqrt{2}}(X_1 - X_2)$.¹⁴ Importantly, note that the cash flows of these assets are uncorrelated, i.e.,

$$\text{cov}(X_m, X_s) = 0. \quad (34)$$

Indexing is then simply a restriction that an agent can trade only the market asset, and not the spread asset. We write P_m and P_s for the price of the market and spread assets.

Note that $\text{var}(X_m) = \text{var}(X_s) = \frac{1}{\tau_X}$, $\mathbb{E}[X_m] = \sqrt{2}\mu_X$ and $\mathbb{E}[X_s] = 0$. Each agent has an endowment $\bar{s}_m \equiv \sqrt{2}\bar{s}$ of the market asset, and no endowment of the spread asset (i.e., $\bar{s}_s \equiv 0$).

Define also $y_{im} \equiv \frac{1}{\sqrt{2}}(y_{i1} + y_{i2})$, $y_{is} \equiv \frac{1}{\sqrt{2}}(y_{i1} - y_{i2})$, $Z_m \equiv \frac{1}{\sqrt{2}}(Z_1 + Z_2)$, $Z_s \equiv \frac{1}{\sqrt{2}}(Z_1 - Z_2)$, $e_{im} \equiv \frac{1}{\sqrt{2}}(e_{i1} + e_{i2})$, and $e_{is} \equiv \frac{1}{\sqrt{2}}(e_{i1} - e_{i2})$. Note that $\text{cov}(y_{im}, y_{is}) = \text{cov}(Z_m, Z_s) = \text{cov}(e_{im}, e_{is}) = 0$; $\text{var}(Z_m) = \text{var}(Z_s) = \frac{1}{\tau_Z}$; $\text{var}(e_{im}) = \text{var}(e_{is}) = \frac{1}{\tau_u} + \frac{1}{\tau_d}$; and $\mathbb{E}[Z_m] = \mathbb{E}[Z_s] = \mathbb{E}[e_{im}] = \mathbb{E}[e_{is}] = 0$.

Write θ_{im} and θ_{is} for agent i ’s positions in the market and spread asset. Then agent i ’s terminal wealth W_i is given by an analogous expression to (1). As before, W_i is determined by trading profits combined with initial asset endowments and other income:

$$W_i(\theta_{im}, \theta_{is}) = \sum_{j=m,s} ((\bar{s}_j + e_{ij}) P_j + (\theta_{ij} + e_{ij})(X_j - P_j)). \quad (35)$$

The equilibrium definition for the two-asset economy exactly parallels the equilibrium definition for the single-asset economy (see Definition 1).

Because the market and spread assets trade independently, even in the presence of indexing agents, there is an equilibrium of the trading stage in which the price of the market

¹⁴The multiplicative constant $\frac{1}{\sqrt{2}}$ in these expressions is simply a normalization that leads to simpler algebraic expressions.

asset P_m is a linear function of X_m and Z_m only, while the price of the spread asset P_s is a linear function of X_s and Z_s only.

Note also that it is straightforward to recover prices of the underlying assets 1 and 2 from the prices of the market and spread asset. Specifically, since $X_1 = \frac{1}{\sqrt{2}}(X_m + X_s)$ and $X_2 = \frac{1}{\sqrt{2}}(X_m - X_s)$, it follows that

$$(P_1, P_2) = \left(\frac{1}{\sqrt{2}}(P_m + P_s), \frac{1}{\sqrt{2}}(P_m - P_s) \right). \quad (36)$$

3.3 Equilibrium

Given a linear equilibrium at the trading stage, expected utility takes the same form as in Proposition 2, but with separate expressions capturing the gains from trading the market and spread assets. Specifically, the expected utility of an agent who chose to restrict participation to indexing strategies is

$$\begin{aligned} \mathbb{E}[u(W_i - \kappa_m) | e_{im}, e_{is}] &= - (d_{im} D_m)^{-\frac{1}{2}} \exp \left(-\gamma(\bar{s}_m + e_{im})\mu_X + \frac{\gamma^2(\bar{s}_m + e_{im})^2}{2\tau_X} - \frac{1}{2}\Lambda_m e_{im}^2 + \gamma\kappa_m \right) \\ &\quad \times \exp \left(\frac{\gamma^2 e_{is}^2}{2\tau_X} \right), \end{aligned}$$

where d_{im} , D_m and Λ_m are all defined in exactly an analogous way as in Proposition 2.

Similarly, the expected utility of an agent who chose full participation is

$$\begin{aligned} \mathbb{E}[u(W_i - \kappa) | e_{im}, e_{is}] &= - (d_{im} D_m)^{-\frac{1}{2}} \exp \left(-\gamma(\bar{s}_m + e_{im})\mu_X + \frac{\gamma^2(\bar{s}_m + e_{im})^2}{2\tau_X} - \frac{1}{2}\Lambda_m e_{im}^2 + \gamma\kappa_m \right) \\ &\quad \times (d_{is} D_s)^{-\frac{1}{2}} \exp \left(\frac{\gamma^2 e_{is}^2}{2\tau_X} - \frac{1}{2}\Lambda_s e_{is}^2 + \gamma(\kappa - \kappa_m) \right), \end{aligned}$$

where d_{is} , D_s and Λ_s are again all defined in exactly an analogous way as in Proposition 2.

Note that if agent i prefers full participation to the alternatives of limited indexing participation, and no participation, then the same is true for any other agent $\tilde{i} < i$. Hence an equilibrium is characterized by an $n \in [0, 1]$ and an $n_m \in [n, 1]$ such that agents $[0, n]$

fully participate, agents $(n, n_m]$ index, and agents $(n_m, 1]$ do not participate at all.

A useful property to note is that an equilibrium can be characterized by first solving for total participation in the market asset, n_m , given participation cost κ_m ; and then solving for participation in the spread asset, n , given participation cost $\kappa - \kappa_m$. Consequently, an equilibrium straightforwardly exists, by analogous arguments to those in the single-asset economy.

3.4 The rise of indexing

Our main comparative static is to consider the effects of a fall in κ_m the participation cost associated with indexing. This corresponds to falling fees, greater availability, and greater awareness of products such as low-cost index funds and ETFs. It may also reflect an increase in public awareness of the standard advice given by finance academics.

We consider the effects of a drop in κ_m while holding the cost of full participation, κ , fixed. Although trading costs for individual assets have also fallen in recent decades, holding κ fixed allows us to focus on the market change most directly responsible for a rise in indexing. However, we would obtain similar results if κ fell also, provided it falls more slowly than κ_m .

The following result, which is immediate given our prior analysis, summarizes the effects of a fall in κ_m :

Proposition 6 *As the participation cost of indexing κ_m falls:*

- (i) *Participation n_m in the market asset increases.*
- (ii) *Participation n in the spread asset falls.*
- (iii) *Indexing $n_m - n$ increases.*
- (iv) *Price efficiency for the market as a whole, ρ_m , falls.*
- (v) *Relative price efficiency ρ_s increases.*
- (vi) *Expected utility for existing indexers increases, while the effect for agents who fully participate is ambiguous.*

The increase in indexing has two distinct components. First, given the lower cost κ_m , some agents who previously did not participate at all in financial markets now index. Second, as the cost of the indexing κ_m falls, the marginal cost $\kappa - \kappa_m$ of full participation increases, leading some agents who previously fully participated in financial markets to switch to indexing.

As with the single-asset economy, participation decisions are strategic complements, so these direct effects in turn generate additional changes: indexing increases still more, while full participation drops still further.

For agents with precise private signals, who fully participate in financial markets, there are offsetting effects in terms of expected utility. On the one hand, the utility derived from trading the market asset increases. But on the other hand, the utility derived from trading the spread asset decreases, because the price efficiency of the spread asset is now higher. As discussed, higher price efficiency both leads to worse risk-sharing, and to lower profits for relatively informed traders.

With the caveat that expected utilities are distinct objects from expected profits, this prediction is consistent with evidence that the share of profits for hedge funds (typically viewed as agents with precise information) stemming from aggregate trading strategies has increased (see discussion in introduction).

Similarly, the increase in the price efficiency of the market as a whole is consistent with empirical evidence in Israeli, Lee, and Sridharan (2017).

In Proposition 5 we noted that the expected return is decreasing in the price P , i.e., reversal. The steepness of the negative slope of $\mathbb{E}[X - P|P]$ captures the strength of reversal, i.e., when this relation is strongly negative, the expected returns following high prices are much lower than following low prices. Our next result shows that, in equilibrium, this relation becomes stronger as κ_m falls (and hence indexing increases).

Lemma 5 $\frac{\partial}{\partial P_m} \mathbb{E}[X_m - P_m|P_m]$ is negative, and decreasing (i.e., becomes further from 0) as κ_m falls.

This is consistent with the empirical findings of Baltussen, van Bekkum, and Da (2017).

4 Discussion

In Section 3 we analyze what we believe is the most direct impact of a decline in the costs of indexing, namely those that stem from the entry of new participants into financial markets in response to lower costs, along with substitution of other traders away from full participation to index-only strategies.

Nonetheless, our analysis inevitably omits other potentially important forces. In particular, we have held constant the precision of agents' private signals (beyond the basic question of whether or not to acquire information at all, which is one possible interpretation of participation decisions). In an earlier draft of this paper we fully analyzed a model that includes this force. In brief, consider the consequences of an exogenous increase in indexing among agents with low-precision private signals. The direct effect is an increase in the price efficiency of the spread asset, with no effect on price efficiency of the market asset (because, by assumption, the increase in indexing simply consists of agents with low-precision signals stopping trading the spread asset, with no increase in trade of the market asset). Allowing agents to reoptimize the precision of their private signals, these changes in price efficiency in turn induce agents to acquire less precise signals about the spread asset, since this information is now less valuable; and in turn to substitute their information collection activities towards acquiring information about economic aggregates. The net effect is price efficiency increases in both the market and spread assets. The same economic forces as in our current analysis then lead to a reduction in expected utility, since (exactly as in Proposition 2) agents prefer to participate in financial markets where price efficiency is low.

So to summarize: by themselves, information acquisition decisions in response to an exogenous rise in indexing end up reducing rather than increasing the welfare of agents with low-precision private signals, who can be interpreted as retail investors.

More generally, our analysis highlights that the welfare consequences of shifts in indexing—or, indeed, or other changes to financial markets—depend critically on how such shifts affect price efficiency. At least when the gains from trade that underpin financial markets are driven by the benefits from risk sharing, as is the case of many standard models, agents generally prefer low levels of price efficiency.

(Of course, price efficiency may be desirable for other reasons, such as for the efficient allocation of capital in primary financial markets, or because of information conveyed by markets that provides incentives and guides “real” productive decisions).¹⁵

5 Conclusion

We develop a benchmark model to study the equilibrium consequences of indexing in a standard rational expectations setting (Grossman and Stiglitz (1980); Hellwig (1980); Diamond and Verrecchia (1981)). Individuals must incur costs to participate in financial markets, and these costs are lower for individuals who restrict themselves to indexing strategies. Individuals’ participation decisions exhibit strategic complementarity. As indexing becomes cheaper (1) indexing increases, while individual stock trading decreases; (2) aggregate price efficiency falls, while relative price efficiency increases; (3) the welfare of relatively uninformed traders increases; (4) for well-informed traders, the share of trading gains stemming from market timing increases, and the share of gains from stock selection decreases; (5) market-wide reversals become more pronounced. We discuss empirical evidence for these predictions.

¹⁵See Bond, Goldstein, and Edmans (2012) for a survey.

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Appendix

Results omitted from main text

Lemma A-1 Suppose that the information set \mathcal{F}_i consists of a set of normally distributed random variables. Then

$$\frac{\partial}{\partial X} \mathbb{E} [\tilde{X} | \mathcal{F}_i] = 1 - \frac{\text{var}(\tilde{X} | \mathcal{F}_i)}{\text{var}(\tilde{X})}.$$

Proof of Lemma A-1 Let Σ_{22} be the variance matrix of the random variables in \mathcal{F}_i ; and Σ_{12} be the row vector of covariances between X and the random variables in \mathcal{F}_i . By the properties of multivariate normality

$$\begin{aligned} \frac{\partial}{\partial X} \mathbb{E} [\tilde{X} | \mathcal{F}_i] &= \Sigma_{12} \Sigma_{22}^{-1} \frac{\Sigma'_{12}}{\text{var}(\tilde{X})} \\ \text{var}(\tilde{X} | \mathcal{F}_i) &= \text{var}[\tilde{X}] - \Sigma_{12} \Sigma_{22}^{-1} \Sigma'_{12}. \end{aligned}$$

Combining these two equations yields the result, completing the proof.

Lemma A-2 Let $\xi \in \mathbb{R}^n$ be a normally distributed random vector with mean μ and variance-covariance matrix Σ . Let $b \in \mathbb{R}^n$ be a given vector, and $A \in \mathbb{R}^{n \times n}$ a symmetric matrix. If $I - 2\Sigma A$ is positive definite, then $\mathbb{E} [\exp(b^\top \xi + \xi^\top A \xi)]$ is well defined, and given by:

$$\mathbb{E} [\exp((b^\top \xi + \xi^\top A \xi))] = |I - 2\Sigma A|^{-1/2} \exp \left(b^\top \mu + \mu^\top A \mu + \frac{1}{2} (b + 2A\mu)^\top (I - 2\Sigma A)^{-1} \Sigma (b + 2A\mu) \right). \quad (\text{A-1})$$

Proof of Lemma A-2: Standard result.

Proofs of results stated in main text

Proof of Lemma 2: Consider first the case in which $\frac{\partial P}{\partial X} \neq 0$. The information content of (y_i, e_i, P) is the same as the information content of

$$\left(y_i, \frac{P - \mathbb{E}[P]}{\frac{\partial P}{\partial X}} + \mu_X, \frac{P - \mathbb{E}[P]}{\frac{\partial P}{\partial X}} + \mu_X + \rho^{-1} e_i \right) = (X + \epsilon_i, X - \rho^{-1} Z, X + \rho^{-1} u_i).$$

Since ϵ_i , Z , and u_i are independent, the conditional variance expressions (14) and (15) follow by standard normal-normal updating. Using $\frac{\partial P}{\partial X} = -\rho \frac{\partial P}{\partial Z}$, the corresponding conditional

expectation $\mathbb{E}[X|y_i, e_i, P]$ is given by

$$\frac{\mathbb{E}[X|y_i, e_i, P]}{\text{var}(X|y_i, e_i, P)} = \tau_X \mu_X - \rho(\tau_Z + \tau_u) \left(\frac{P - \mathbb{E}[P]}{\frac{\partial P}{\partial Z}} + \mu_X \right) + \rho \tau_u e_i + \tau_i y_i. \quad (\text{A-2})$$

Finally, if $\frac{\partial P}{\partial X} = 0$ then neither the price nor the endowment e_i contains any information about X ; and $\rho = 0$; so (14), (15), and (A-2) are all immediate.

The expressions in Lemma 2 are then immediate from the demand equation (8), completing the proof.

Proof of Lemma 3: From Lemma 2, $\frac{\partial \theta_i}{\partial P} < 0$ for all agents.

If $\frac{\partial P}{\partial X} = 0$, then P contains no information about X , so $\int_N \frac{\partial \theta_i}{\partial P} di = 0$, and (18) is then immediate.

If instead $\frac{\partial P}{\partial X} \neq 0$, then $\frac{\partial P}{\partial X} \int_N \frac{\partial \theta_i}{\partial P} di > 0$ by Lemma 2. By (12) and Lemma 2,

$$\frac{\partial P}{\partial X} \int_N \frac{\partial \theta_i}{\partial P} di + \frac{\partial P}{\partial X} \int_N \frac{\partial \theta_i}{\partial \tilde{P}} di = - \int_N \frac{\partial \theta_i}{\partial Y_i} di < 0. \quad (\text{A-3})$$

Hence $\frac{\partial P}{\partial X} > 0$, which (again using (A-3)) implies (18).

Note that the above arguments also establish that $\frac{\partial P}{\partial X} \geq 0$.

The main text establishes that $\frac{\partial P}{\partial Z} \neq 0$. So to establish $\frac{\partial P}{\partial Z} < 0$, suppose to the contrary that $\frac{\partial P}{\partial Z} > 0$. Then $\rho \leq 0$, and Lemma 2 implies $\int_N \frac{\partial \theta_i}{\partial e_i} di < 0$ and $\int_N \frac{\partial \theta_i}{\partial \tilde{e}_i} di \leq 0$. Combined with (18), this in turn implies that the lefthand side of (13) is strictly negative. The contradiction completes the proof.

Proof of Lemma 4: At various points in the proof, we make use of $\rho < 0$ and $\frac{\partial P}{\partial Z} < 0$ (by Lemma 3), and $\frac{\partial \theta_i}{\partial P} > 0$ (by Lemma 2).

Re-arranging market-clearing (13) gives

$$0 = \frac{\partial P}{\partial Z} \int_N \frac{\partial \theta_i}{\partial P} di + \int_N \frac{\partial \theta_i}{\partial e_i} di + \left(\frac{\frac{\partial P}{\partial Z} \int_N \frac{\partial \theta_i}{\partial \tilde{P}} di}{\int_N \frac{\partial \theta_i}{\partial \tilde{e}_i} di} + 1 \right) \int_N \frac{\partial \theta_i}{\partial \tilde{e}_i} di. \quad (\text{A-4})$$

By Lemma 3 and market-clearing (13), we know

$$\int_N \frac{\partial \theta_i}{\partial e_i} di + \int_N \frac{\partial \theta_i}{\partial \tilde{e}_i} di < 0. \quad (\text{A-5})$$

From (17) of Corollary 2, $\frac{\partial P}{\partial Z} \frac{\partial \theta_i}{\partial P} + \frac{\partial \theta_i}{\partial \tilde{e}_i} < 0$, which together with $\frac{\partial \theta_i}{\partial \tilde{e}_i} > 0$ implies

$$\frac{\frac{\partial P}{\partial Z} \int_N \frac{\partial \theta_i}{\partial P} di}{\int_N \frac{\partial \theta_i}{\partial \tilde{e}_i} di} + 1 < 0.$$

So substituting (A-5) into (A-4) gives

$$\begin{aligned} 0 &> \frac{\partial P}{\partial Z} \int_N \frac{\partial \theta_i}{\partial P} di + \int_N \frac{\partial \theta_i}{\partial e_i} di - \left(\frac{\frac{\partial P}{\partial Z} \int_N \frac{\partial \theta_i}{\partial P} di}{\int_N \frac{\partial \theta_i}{\partial \tilde{e}_i} di} + 1 \right) \int_N \frac{\partial \theta_i}{\partial e_i} di \\ &= \frac{\partial P}{\partial Z} \int_N \frac{\partial \theta_i}{\partial P} di - \frac{\partial P}{\partial Z} \int_N \frac{\partial \theta_i}{\partial \tilde{P}} di \frac{\int_N \frac{\partial \theta_i}{\partial e_i} di}{\int_N \frac{\partial \theta_i}{\partial \tilde{e}_i} di}. \end{aligned} \quad (\text{A-6})$$

Using $\frac{\partial P}{\partial Z} < 0$ and the signs established in Lemma 2, inequality (A-6) is equivalent to

$$\frac{\int_N \frac{\partial \theta_i}{\partial P} di}{\int_N \frac{\partial \theta_i}{\partial \tilde{P}} di} > \frac{\int_N \frac{\partial \theta_i}{\partial e_i} di}{\int_N \frac{\partial \theta_i}{\partial \tilde{e}_i} di},$$

which is in turn equivalent to (20), completing the proof.

Proof of Proposition 1: From (12) and (13),

$$-\frac{\frac{\partial P}{\partial X}}{\frac{\partial P}{\partial Z}} = -\frac{\int_N \frac{\partial \theta_i}{\partial Y_i} di}{\int_N \frac{\partial \theta_i}{\partial e_i} di + \int_N \frac{\partial \theta_i}{\partial \tilde{e}_i} di}. \quad (\text{A-7})$$

Substituting in from Lemma 2,

$$\rho = \frac{\frac{1}{\gamma} \int_N \tau_i di}{\int_N \left(1 - \frac{\rho}{\gamma} \tau_u\right) di}, \quad (\text{A-8})$$

and so

$$\rho^2 \tau_u - \gamma \rho + \frac{1}{n} \int_N \tau_i di = 0,$$

leading to (21). The comparative statics of price efficiency is immediate from the fact that $\frac{1}{n} \int_N \tau_i di$ is decreasing in n . To derive (22), note that Corollaries 1 and 3 combine to give

$$\mathbb{E}[X - P] = -\frac{\bar{s}}{\frac{1}{n} \int \frac{\partial \theta_i}{\partial P} di},$$

and so by Lemma 2,

$$\mathbb{E}[X - P] = \frac{\gamma \bar{s}}{\frac{1}{n} \int \frac{1}{\text{var}(X|y_i, e_i, P)} di}.$$

Substituting in (15) completes the proof.

Proof of Proposition 2: The final wealth of agent i , given optimal trading (8), is

$$W_i = (\bar{s} + e_i) P + \frac{\mathbb{E}[X - P|y_i, e_i, P] (X - P)}{\gamma \text{var}(X|y_i, e_i, P)}.$$

So by the standard expression for the expected utility of an agent with CARA utility facing normally shocks, combined with simple manipulation, agent i 's expected utility at the time of trading is

$$\mathbb{E}[u(W_i - \kappa) | y_i, e_i, P] = -\exp\left(-\gamma\left((\bar{s} + e_i) P + \frac{1}{2} \frac{\mathbb{E}[X - P|y_i, e_i, P]^2}{\gamma \text{var}(X|y_i, e_i, P)} - \kappa\right)\right). \quad (\text{A-9})$$

To obtain (23), we use (A-9), and proceed in two stages. First, we integrate out possible realizations of the private signal y_i . Second, we integrate out possible realizations of the price P . Note that the first stage is relatively standard, and closely related algebraic arguments can be found in the related literature. Readers familiar with these arguments should proceed directly to the second stage.

For the first stage, define $\xi_i = \mathbb{E}[X - P|y_i, e_i, P]$ and $A_i = -1/(2\text{var}(X|y_i, e_i, P))$. Minor algebraic manipulation of Lemma A-2 implies

$$\mathbb{E}[\exp(\xi_i^2 A_i) | e_i, P] = (1 - 2A_i \text{var}(\xi_i | e_i, P))^{-\frac{1}{2}} \exp\left(\frac{A_i}{1 - 2A_i \text{var}(\xi_i | e_i, P)} \mathbb{E}[\xi_i | e_i, P]^2\right). \quad (\text{A-10})$$

By the law of total variance,

$$\text{var}(X - P | e_i, P) = \text{var}(\mathbb{E}[X - P | y_i, e_i, P] | e_i, P) + \mathbb{E}[\text{var}(X - P | y_i, e_i, P) | e_i, P]$$

which implies

$$\text{var}(\xi_i | e_i, P) = \text{var}(X | e_i, P) - \text{var}(X | y_i, e_i, P)$$

and so

$$1 - 2A_i \text{var}(\xi_i | e_i, P) = \frac{\text{var}(X | e_i, P)}{\text{var}(X | y_i, e_i, P)} = d_i,$$

where d_i is as defined in (24). Substitution and straightforward manipulation implies that expression (A-10) equals

$$d_i^{-\frac{1}{2}} \exp\left(-\frac{1}{2} \frac{\mathbb{E}[X - P | e_i, P]^2}{\text{var}(X | e_i, P)}\right),$$

and so

$$\mathbb{E}[u(W_i - \kappa) | e_i, P] = -d_i^{-\frac{1}{2}} \exp \left(-\gamma(\bar{s} + e_i)P - \frac{1}{2} \frac{\mathbb{E}[X - P | e_i, P]^2}{\text{var}(X | e_i, P)} + \gamma\kappa \right), \quad (\text{A-11})$$

completing the first stage.

In the second stage, we integrate out possible realizations of P . Since $P = (P - \mathbb{E}[P | e_i]) + \mathbb{E}[X | e_i] - \mathbb{E}[X - P | e_i]$, the expression in the exponent of (A-11) equals

$$-\frac{1}{2} \frac{\mathbb{E}[X - P | e_i, P]^2}{\text{var}(X | e_i, P)} - \gamma(\bar{s} + e_i)(P - \mathbb{E}[P | e_i]) - \gamma(\bar{s} + e_i)\mathbb{E}[X | e_i] + \gamma(\bar{s} + e_i)\mathbb{E}[X - P | e_i] | e_i + \gamma\kappa. \quad (\text{A-12})$$

Denote the expected return $X - P$ given exposure e_i by α_e , i.e.,

$$\alpha_e \equiv \mathbb{E}[X - P | e_i] = \mathbb{E}[X - P] - \frac{\text{cov}(P, e_i)}{\text{var}(e_i)} e_i, \quad (\text{A-13})$$

and note that

$$\mathbb{E}[X - P | e_i, P] = \frac{\text{cov}(X - P, P | e_i)}{\text{var}(P | e_i)} (P - \mathbb{E}[P | e_i]) + \alpha_e. \quad (\text{A-14})$$

By substitution and Lemma A-2, (A-11) equals

$$\begin{aligned} & -d_i^{-\frac{1}{2}} D^{\frac{1}{2}} \exp(-\gamma(\bar{s} + e_i)(\mu_X - \alpha_e) + \gamma\kappa) \\ & \times \exp \left(-\frac{1}{2} \frac{\alpha_e^2}{\text{var}(X | e_i, P)} + \frac{1}{2} \left(\frac{\alpha_e \text{cov}(X - P, P | e_i)}{\text{var}(P | e_i) \text{var}(X | e_i, P)} + \gamma(\bar{s} + e_i) \right)^2 \frac{\text{var}(P | e_i)}{D} \right) \end{aligned} \quad (\text{A-15})$$

where

$$D = 1 + \frac{\text{cov}(X - P, P | e_i)^2}{\text{var}(X | e_i, P) \text{var}(P | e_i)}. \quad (\text{A-16})$$

The law of total variance and (A-14) together yield

$$\text{var}(\mathbb{E}[X - P | e_i, P] | e_i) = \text{var}(X - P | e_i) - \text{var}(X | e_i, P) = \frac{\text{cov}(X - P, P | e_i)^2}{\text{var}(P | e_i)}, \quad (\text{A-17})$$

and substitution into (A-16) delivers (25).

For use below, note also that (A-17) implies that

$$\begin{aligned}
\frac{\text{var}(P|e_i)}{D} &= \frac{\text{var}(P|e_i) \text{var}(X|e_i, P)}{\text{Dvar}(X|e_i, P)} = \frac{\text{var}(P|e_i) \text{var}(X - P|e_i) - \text{cov}(X - P, P|e_i)^2}{\text{Dvar}(X|e_i, P)} \\
&= \frac{\text{var}(X|e_i) \text{var}(X - P|e_i) - \text{cov}(X - P, X|e_i)^2}{\text{Dvar}(X|e_i, P)} \\
&= \text{var}(X|e_i) - \frac{\text{cov}(X - P, X|e_i)^2}{\text{Dvar}(X|e_i, P)}
\end{aligned} \tag{A-18}$$

where the penultimate equality follows from the fact that for any random variables r_1 and r_2 ,

$$\text{cov}(r_1 - r_2, r_1)^2 - \text{cov}(r_1 - r_2, r_2)^2 = \text{var}(r_1 - r_2)(\text{var}(r_1) - \text{var}(r_2)),$$

and the final equality follows from (25).

By expanding and then re-arranging the quadratic, (A-15) equals

$$\begin{aligned}
& - (d_i D)^{-\frac{1}{2}} \exp \left(-\gamma(\bar{s} + e_i)\mu_X + \gamma\kappa + \frac{1}{2} \frac{\alpha_e^2}{\text{var}(X|P, e_i)} \left(\frac{\text{cov}(X - P, P|e_i)^2}{\text{Dvar}(P|e_i)\text{var}(X|e_i, P)} - 1 \right) \right) \\
& \times \exp \left(\gamma(\bar{s} + e_i)\alpha_e \left(\frac{\text{cov}(X - P, P|e_i)}{\text{Dvar}(X|e_i, P)} + 1 \right) + \frac{1}{2}\gamma^2(\bar{s} + e_i)^2 \frac{\text{var}(P|e_i)}{D} \right) \\
&= - (d_i D)^{-\frac{1}{2}} \exp \left(-\gamma(\bar{s} + e_i)\mu_X + \gamma\kappa - \frac{1}{2} \frac{\alpha_e^2}{\text{Dvar}(X|e_i, P)} \right) \\
& \times \exp \left(\gamma(\bar{s} + e_i)\alpha_e \frac{\text{cov}(X - P, X|e_i)}{\text{Dvar}(X|e_i, P)} + \frac{1}{2}\gamma^2(\bar{s} + e_i)^2 \left(\text{var}(X|e_i) - \frac{\text{cov}(X - P, X|e_i)^2}{\text{Dvar}(X|e_i, P)} \right) \right) \\
&= - (d_i D)^{-\frac{1}{2}} \exp \left(-\gamma(\bar{s} + e_i)\mu_X + \frac{\gamma^2(\bar{s} + e_i)^2}{2\tau_X} + \gamma\kappa \right) \\
& \times \exp \left(-\frac{1}{2} \frac{(\alpha_e - \text{cov}(X - P, X)\gamma(\bar{s} + e_i))^2}{\text{Dvar}(X|e_i, P)} \right). \tag{A-19}
\end{aligned}$$

where the first equality follows from (A-16), (25), and (A-18). Substituting Corollary 1's expression for $\mathbb{E}[X - P]$ into (A-13)'s expression for α_e and then in turn substituting into (A-19) yields (26). This completes the proof.

Proof of Proposition 3: The main steps are given in the main text. Here, we give the details for a few of the steps. For use below, recall that $\frac{\partial P}{\partial X} = \frac{\text{cov}(X, P)}{\text{var}(X)}$, $\frac{\partial P}{\partial Z} = \frac{\text{cov}(Z, P)}{\text{var}(Z)}$, and

note that

$$\text{var}(P|e_i) = \frac{\text{cov}(X, P)^2}{\text{var}(X)^2} \text{var}(X) + \frac{\text{cov}(Z, P)^2}{\text{var}(Z)^2} \text{var}(Z|e_i), \quad (\text{A-20})$$

$$\text{var}(X - P|e_i) = \left(\frac{\text{cov}(X - P, X)}{\text{var}(X)} \right)^2 \text{var}(X) + \left(\frac{\text{cov}(P, Z)}{\text{var}(Z)} \right)^2 \text{var}(Z|e_i). \quad (\text{A-21})$$

Details for the argument that Λ is decreasing in price efficiency ρ : Substituting (A-21) into (26) gives

$$\begin{aligned} \Lambda &= \frac{\frac{\text{var}(Z)^2}{\text{var}(e_i)^2} \left(-\frac{\text{cov}(P, e_i)}{\text{var}(Z)} - \gamma \text{cov}(X - P, X) \frac{\text{var}(e_i)}{\text{var}(Z)} \right)^2}{\left(\frac{\text{cov}(X - P, X)}{\text{var}(X)} \right)^2 \text{var}(X) + \left(\frac{\text{cov}(P, Z)}{\text{var}(Z)} \right)^2 \text{var}(Z|e_i)} \\ &= \frac{\frac{\text{var}(Z)^2}{\text{var}(e_i)^2} \left(-\frac{\text{cov}(P, Z)}{\text{var}(Z)} \frac{1}{\gamma \text{cov}(X - P, X)} - \frac{\text{var}(e_i)}{\text{var}(Z)} \right)^2}{\frac{1}{\gamma^2} \frac{1}{\text{var}(X)} + \left(-\frac{\text{cov}(P, Z)}{\text{var}(Z)} \frac{1}{\gamma \text{cov}(X - P, X)} \right)^2 \text{var}(Z|e_i)}. \end{aligned} \quad (\text{A-22})$$

Substitution (19) of Corollary 3 into (A-22) yields (27).

By Corollary 2 and $\frac{1}{n} \int_N \frac{\partial \theta_i}{\partial e_i} di = -1$,

$$\frac{\text{var}(e_i)}{\text{var}(Z)} = \frac{\tau_Z + \tau_u}{\tau_u} = -\frac{\frac{\partial P}{\partial Z} \int_N \frac{\partial \theta_i}{\partial P} di}{\int_N \frac{\partial \theta_i}{\partial e_i} di} = \frac{1}{n} \frac{\partial P}{\partial Z} \int_N \frac{\partial \theta_i}{\partial P} di \frac{\int_N \frac{\partial \theta_i}{\partial e_i} di}{\int_N \frac{\partial \theta_i}{\partial e_i} di}. \quad (\text{A-23})$$

Hence (30) is equivalent to (A-6) in the proof of Lemma 4, which itself is equivalent to (20).
Details for the argument that D is decreasing in price efficiency ρ : Using the law of total variance,

$$\begin{aligned} \text{var}(X|e_i, P) \text{var}(P|e_i) &= \left(\text{var}(X|e_i) - \frac{\text{cov}(X, P|e_i)^2}{\text{var}(P|e_i)} \right) \text{var}(P|e_i) \\ &= \text{var}(X|e_i) \text{var}(P|e_i) - \text{cov}(X, P|e_i)^2. \end{aligned}$$

Substituting in (A-20) gives

$$\text{var}(X|e_i, P) \text{var}(P|e_i) = \frac{\text{cov}(Z, P)^2}{\text{var}(Z)^2} \text{var}(Z|e_i) \text{var}(X). \quad (\text{A-24})$$

Also by (A-20), and making use of (19) of Corollary 3,

$$\begin{aligned}
\text{cov}(X - P, P|e_i) &= \text{cov}(X, P|e_i) - \text{var}(P|e_i) \\
&= \frac{\text{cov}(X, P)}{\text{var}(X)} (\text{var}(X) - \text{cov}(X, P)) - \frac{\text{cov}(Z, P)^2}{\text{var}(Z)^2} \text{var}(Z|e_i) \quad (\text{A-25}) \\
&= \left(\frac{\frac{\text{cov}(X, P)}{\text{var}(X)}}{\frac{\text{cov}(Z, P)}{\text{var}(Z)}} - \frac{\frac{\text{cov}(Z, P)}{\text{var}(Z)} \text{var}(Z|e_i)}{\text{cov}(X - P, X)} \right) \frac{\text{cov}(Z, P)}{\text{var}(Z)} \text{cov}(X - P, X) \\
&= \left(-\rho + \gamma \text{var}(Z|e_i) \frac{1}{n} \frac{\partial P}{\partial Z} \int_N \frac{\partial \theta_i}{\partial P} di \right) \frac{\text{cov}(Z, P)}{\text{var}(Z)} \left(-\frac{1}{\gamma \frac{1}{n} \int_N \frac{\partial \theta_i}{\partial P} di} \right) \quad (\text{A-26})
\end{aligned}$$

Substitution of (A-24) and (A-26) into (A-16) yields (28).

To establish the equivalence of $\gamma\rho - \tau_u > 0$ with (18), note that the equality of (28) and (31) and $\frac{1}{n} \frac{\partial P}{\partial Z} \int_N \frac{\partial \theta_i}{\partial P} di = -\frac{\rho}{\gamma \text{var}(Z|e_i)}$ imply that $\gamma\rho - \tau_u > 0$ is equivalent to

$$1 + \frac{\frac{\partial P}{\partial Z} \int_N \frac{\partial \theta_i}{\partial P} di}{\frac{\partial P}{\partial Z} \int_N \frac{\partial \theta_i}{\partial P} di} < 0,$$

which given $\frac{\partial \theta_i}{\partial P} > 0$ is equivalent to (18).

Proof of Lemma 5: For transparency of notation, we establish the result for a single-asset economy. From (33), Lemma 2, and (29),

$$\mathbb{E}[X - P|P] = \frac{\bar{s} + \mathbb{E}[Z|P]}{\frac{1}{\gamma} \frac{1}{n} \int_0^n \frac{1}{\text{var}(X|y_i, e_i, P)}} = -\frac{\bar{s} + \mathbb{E}[Z|P]}{\frac{1}{n} \int_N \frac{\partial \theta_i}{\partial P} di} = -\frac{\bar{s} + \mathbb{E}[Z|P]}{1 + \frac{\rho}{\gamma} \tau_Z} \frac{\partial P}{\partial Z}.$$

Note that

$$\frac{\partial}{\partial P} \mathbb{E}[Z|P] = \frac{\text{cov}(Z, P)}{\text{var}(P)} = \frac{\text{cov}(Z, P)}{\text{var}(Z)} \frac{\text{var}(Z)}{\text{var}(P)} = \frac{\partial P}{\partial Z} \frac{\text{var}(Z)}{\left(\frac{\partial P}{\partial X}\right)^2 \text{var}(X) + \left(\frac{\partial P}{\partial Z}\right)^2 \text{var}(Z)}.$$

Hence (and using the fact that $\frac{\partial P}{\partial Z}$ is independent of Z)

$$\frac{\partial}{\partial P} \mathbb{E}[X - P|P] = -\frac{1}{\left(1 + \frac{\rho}{\gamma} \tau_Z\right) \left(\rho^2 \frac{\text{var}(X)}{\text{var}(Z)} + 1\right)}.$$

We know that as κ_m falls, ρ_m likewise falls, completing the proof.

Online appendix

Proposition A-1 Consider the benchmark economy described in subsection 2.2, in which agents do not possess any private information about the asset's cash flow X , but instead all observe a public signal of the form $Y = X + \epsilon$, where $\epsilon \sim \mathcal{N}(0, \tau_\epsilon^{-1})$. For generality, agents potentially differ in their asset endowments, with agent i possessing s_i units of the asset before trade. Let $\bar{s} = \int s_i di$. In such a setting, each agent's expected utility is decreasing in the precision of the public signal, τ_ϵ .

Proof: Agent i 's terminal wealth is

$$W_i = (s_i + e_i)P + (\theta_i + e_i)(X - P),$$

and he optimally chooses the portfolio

$$\theta_i + e_i = \frac{1}{\gamma} \frac{\mathbb{E}[X|Y] - P}{\text{var}(X|Y)}.$$

So agent i 's expected utility at the trading stage is

$$\begin{aligned} \mathbb{E}[-\exp(-\gamma W_i)|Y] &= \mathbb{E}\left[-\exp\left(-\gamma(s_i + e_i)P - \frac{\mathbb{E}[X|Y] - P}{\text{var}(X|Y)}(X - P)\right)|Y\right] \\ &= -\exp\left(-\gamma(s_i + e_i)P - \frac{(\mathbb{E}[X|Y] - P)^2}{\text{var}(X|Y)} + \frac{1}{2} \frac{(\mathbb{E}[X|Y] - P)^2}{\text{var}(X|Y)}\right) \\ &= -\exp\left(-\gamma(s_i + e_i)P - \frac{1}{2} \frac{(\mathbb{E}[X|Y] - P)^2}{\text{var}(X|Y)}\right). \end{aligned}$$

We evaluate

$$\mathbb{E}\left[-\exp\left(-\gamma(s_i + e_i)P - \frac{1}{2} \frac{(\mathbb{E}[X|Y] - P)^2}{\text{var}(X|Y)}\right)|e_i\right]. \quad (\text{A-1})$$

Expanding, this expression equals

$$\mathbb{E}\left[-\exp\left(-\gamma(s_i + e_i)\mathbb{E}[X|Y] + \gamma(s_i + e_i)(\mathbb{E}[X|Y] - P) - \frac{1}{2} \frac{(\mathbb{E}[X|Y] - P)^2}{\text{var}(X|Y)}\right)|e_i\right].$$

By market clearing,

$$\frac{1}{\gamma} \frac{\mathbb{E}[X|Y] - P}{\text{var}(X|Y)} = \bar{s} + Z,$$

i.e.,

$$\mathbb{E}[X|Y] - P = \gamma \text{var}(X|Y) (\bar{s} + Z),$$

and so (A-1) equals

$$\mathbb{E} \left[-\exp \left(-\gamma (s + e_i) \mathbb{E}[X|Y] + \frac{\gamma^2 \text{var}(X|Y)}{2} (2(\bar{s} + Z)(s_i + e_i) - (\bar{s} + Z)^2) \right) | e_i \right].$$

Moreover,

$$\mathbb{E}[X|Y] = \frac{\tau_X \mathbb{E}[X] + \tau_\epsilon Y}{\tau_X + \tau_\epsilon} = \frac{\tau_\epsilon^{-1} \mathbb{E}[X] + \tau_X^{-1} Y}{\tau_X^{-1} + \tau_\epsilon^{-1}} = \frac{(\text{var}(Y) - \text{var}(X)) \mathbb{E}[X] + \text{var}(X) Y}{\text{var}(Y)}.$$

Hence (A-1) equals

$$\begin{aligned} & -\exp \left(-\gamma (s_i + e_i) \mathbb{E}[X] + \frac{\gamma^2 (s_i + e_i)^2}{2} \frac{\text{var}(X)^2}{\text{var}(Y)} \right) \\ & \times \mathbb{E} \left[\exp \left(\frac{\gamma^2 \text{var}(X|Y)}{2} (2(\bar{s} + Z)(s_i + e_i) - (\bar{s} + Z)^2) \right) | e_i \right]. \end{aligned}$$

By the law of total variance,

$$\text{var}(X) = \text{var}(X|Y) + \text{var}(E[X|Y]) = \text{var}(X|Y) + \frac{\text{var}(X)^2}{\text{var}(Y)}.$$

So (A-1) equals

$$\begin{aligned} & -\exp \left(-\gamma (s_i + e_i) \mathbb{E}[X] + \frac{\gamma^2 (s_i + e_i)^2}{2} \text{var}(X) \right) \\ & \times \mathbb{E} \left[\exp \left(\frac{\gamma^2 \text{var}(X|Y)}{2} (2(\bar{s} + Z)(s_i + e_i) - (\bar{s} + Z)^2 - (s_i + e_i)^2) \right) | e_i \right] \\ & = \mathbb{E} \left[-\exp \left(-\gamma (s_i + e_i) X - \frac{\gamma^2 \text{var}(X|Y)}{2} ((s_i + e_i) - (\bar{s} + Z))^2 \right) | e_i \right]. \end{aligned}$$

This expression is increasing in $\text{var}(X|Y)$, completing the proof.