Insider Trading, Stochastic Liquidity and Equilibrium Prices[☆]

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Abstract

We extend Kyle's (1985) model of insider trading to the case where noise trading volatility follows a general stochastic process. Even though the level of noise trading volatility is observable, in equilibrium, measured price impact is stochastic. If noise trading volatility is mean-reverting, then the equilibrium price follows a 'bridge' process with stochastic volatility. Thus, the model generates a form of 'excess volatility' because non-payoff relevant shocks (e.g., 'sunspots') that affect noise trading may affect the stock price volatility. More private information is revealed when noise trading volatility is higher. This is because insiders choose to optimally wait to trade more aggressively when noise trading volatility is higher and price impact is lower.

Keywords: Kyle model, insider trading, asymmetric information, liquidity, price impact, market depth, stochastic volatility, execution costs, continuous time.

1. Introduction

In his seminal contribution, Kyle (1985) derives the equilibrium price dynamics in a model where a large trader possesses long-lived private information about the value of a stock that will be revealed at some known date, and optimally trades into the stock to maximize his expected profits. Risk-neutral market makers try to infer from aggregate order flow the information possessed by the insider. Because order flow is also driven by 'noise traders,' who trade solely for liquidity purposes, prices are not fully revealing. Instead, prices respond linearly to order flow. Kyle's lambda, which measures the equilibrium price impact of order flow is constant in the model.

In this paper we generalize Kyle's model (in the continuous time formulation given by Back, 1992) to allow the volatility of noise trading to change stochastically over time. The main (economic) restriction we put on the process is that it is independent of the insider's private information and that it may not be (Granger-)caused by order flow. We ask the following questions. How does the insider adapt his optimal trading strategy to account for time-varying noise trading volatility? How are the equilibrium price dynamics affected by these shocks to noise trader volatility, which by assumption are independent of fundamentals?

Kyle's model provides the insight that the larger the noise trading volatility for a given amount of private information, the more aggressive the insider will trade, since, in equilibrium, his optimal trading rule is inversely related to Kyle's lambda (a measure of price impact). The insider makes more profits when there is more noise trading, since the market maker can recoup more profits himself from the greater volume of noise traders. In a dynamic setting where 'noise trader volume' changes stochastically, one may therefore expect price impact measures to move over time, and the insider to adjust his trading to take advantage of those moments when 'liquidity' is greater. In turn, this

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may result in more complex price dynamics than in the standard Kyle model.

Below we characterize the equilibrium fully for general stochastic volatility dynamics. We find that in equilibrium price impact is stochastic. Price impact is larger when noise-trading volatility is lower. When noise trading volatility is lower, the informed trader trades less aggressively. This leads to a negative relation between expected informed order flow and price-impact. This prediction is consistent with Collin-Dufresne and Fos (2014), who find that informed traders trade much more aggressively when measures of price impact are low.

Second, market depth (the inverse of price impact) is a martingale in equilibrium, which implies that price impact (Kyle's lambda) is a submartingale, i.e., is expected to increase on average. This is in contrast to much of the previous theoretical models (e.g., Baruch, 2002; Back and Baruch, 2004; Back and Pedersen, 1998; Admati and Pfleiderer, 1988; Caldentey and Stacchetti, 2010).[1] The prediction of our model that price impact is expected to rise on average is consistent with the empirical evidence in Madhavan et al. (1997) who find that estimated execution costs rise significantly on average over the day.

Third, when noise trading volatility is predictable, the equilibrium price follows a multi-variate 'bridge' process whose volatility is stochastic and driven by both noise trader volatility and the posterior variance of the insider's private information. This implies that non-fundamental information, if it affects the expected change in noise trader volatility, will affect equilibrium price dynamics in this model. Our model thus generates a form of 'excess volatility,' since non-payoff relevant shocks (e.g., 'sunspots')

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[1]In Baruch (2002) and Back and Baruch (2004) Kyle's lambda is actually a super-martingale. As discussed in these papers, this arises because the insider faces a random deadline or is risk-averse. See the discussion in Back and Pedersen (1998) page 387. Foster and Viswanathan (1996) and Back, Cao, and Willard (2000) may also generate an increase in λ, at least near the end of the trading horizon, because of competition among multiple informed traders (we thank a referee for pointing this out).

that affect noise trading volatility may affect the stock price volatility because the market maker rationally anticipates that in periods where noise traders are more active informed trader would be more aggressive and thus adjusts prices faster. In contrast, when noise trader volatility is a martingale (i.e., is unpredictable) or when it is constant, then the equilibrium price process is, as in the original Kyle model, independent of the level of noise trader volatility.

Mathematically, the price process resembles the Brownian Bridge process used, for example, in Back (1992) in that it converges to the liquidation value (known only to the insider) almost surely at the announcement date. However, the dynamics are more complex in that they are multi-variate and exhibit stochastic volatility. These dynamics should also be useful for other applications that use a Brownian bridge.

Finally, when noise trader volatility is predictable, then information revelation, as measured by the decrease in the posterior variance of the informed's signal, is faster when noise trading volatility is higher. This seems consistent with the evidence in Foster and Viswanathan (1993) who find a positive relation between their estimates of the adverse selection component of trading costs and volume (for actively traded stocks).

This paper is primarily related to work by Back and Pedersen (1998) ('BP' hereafter) who extend Kyle's original model to allow for deterministically changing noise trader volatility to capture intra-day patterns (clustering) of liquidity trading. They also find that the informativeness of orders and the volatility of prices follow the same pattern as the liquidity trading. However, in their setting there are no systematic patterns in the price impact of orders. This implies that 'expected execution costs of liquidity traders do not depend on the timing of their trades' (BP p. 387).[2] Instead, in our model, equilibrium price impact is a submartingale, i.e., is expected to increase on average.

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[2]Foster and Viswanathan (1990) also propose a model with discrete jumps in noise trader volume. In their framework, market depth is constant over time (as proved in Back and Pedersen, 1998).

Our paper is also related to a long list of papers investigating the impact of asymmetric information on asset prices and volatility (see Brunnermeier (2001) for a survey). For example, Admati and Pfleiderer (1988) investigate a dynamic economy, with myopic agents (essentially a sequence of one-period Kyle models), where they generate time variation in price volatility. In their model, price volatility is stochastic because the amount of private information changes from period to period, not because noise trading volatility is time varying.[3]

Section 2 introduces the general model and solves for an equilibrium. Section 3 investigates a few special cases, arbitrary deterministic noise trading volatility, and continuous time Markov Chain, to show some numerical simulations of equilibrium quantities. Section 4 concludes.

2. Informed Trading with Stochastic Liquidity Shocks

We extend Kyle's (1985) model (in the continuous time formulation given by Back, 1992) to allow for time varying volatility of noise trading . As in Kyle, we assume there is an insider trading in the stock with perfect knowledge of the terminal value v. The insider is risk-neutral and maximizes the expectation of his terminal profit.[4]

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[3]Indeed, in the standard Kyle model, price volatility only depends on the volatility of private information and not on noise trading volatility. Since Admati and Pfleiderer (1988) consider a sequence of such one period models, where informed traders have short-lived private information, all variation in price volatility arises because of variation in the private information volatility. (Watanabe, 2008, extends their work to capture GARCH features in equilibrium prices, by directly incorporating stochastic volatility in the (short-lived) private information process.) This is very different from our model where the insider has long-lived information and optimally chooses to trade when noise trading volatility is high, thus generating a link between noise trading volatility and price volatility.

[4]As in Kyle's model we assume the insider submits market orders dX_t that will be filled by the market maker at price $P_{t+dt} = P_t + dP_t$. Thus his profits are, assuming a zero risk-free rate, $\int_0^T (v - P_{t+dt})dX_t = \int_0^T (v - P_t)dX_t - \int_0^T dP_t dX_t$. When $dX_t = \theta_t dt$ is absolutely continuous (Back (1992) shows this is optimal) the second term drops out.

$$\max_{\theta_t \in \mathcal{A}} \mathrm{E}\left[\int_0^T (v - P_t)\theta_t dt \, | \mathcal{F}_t^Y, v\right], \tag{1}$$

where we denote by $\mathcal{F}_t^Y$ the information filtration generated by observing the entire past history of aggregate order flow Y (which we denote by $Y^t = \{Y_s\}_{s \le t}$).[5] In addition, the insider knows the actual value v of the stock, and, of course, his own trading. Following Back (1992) we assume that the insider chooses an absolutely continuous trading rule θ that belongs to an admissible set $\mathcal{A} = \{\theta \ s.t. \ \mathrm{E}[\int_0^T \theta_s^2 ds] < \infty\}$.[6]

The market maker is also risk-neutral, but does not observe the terminal value. Instead, he has a prior that the value v is normally distributed $N(\mu_0, \Sigma_0)$.

The market maker observes the aggregate order flow arrival which is the sum of the insider's demand and the 'noise-trader' demand:

$$dY_t = \theta_t dt + \sigma_t dZ_t, \tag{2}$$

where Z_t is a standard Brownian motion independent of v. The market maker absorbs the total order flow by trading against it at a price he sets so as to break even on average.

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[5]In the standard Kyle-Back model, assuming that the informed investor observes total order flow is innocuous, since if the insider only observes equilibrium prices, he can typically recover the total order flow (and, given his own trading, the noise trading order flow). In our setting, when uniformed order flow has stochastic volatility, this assumption is important for our equilibrium construction. Alternatively, we could assume that the informed agent observes prices *and* noise trading volatility, or prices and price impact λ_t. The point is that observing only prices may not be sufficient to recover noise trading volatility (we give some examples below, where equilibrium prices are independent of noise trading volatility, even though the insider's trades depend on it). The assumption that insiders observe noise-trader volatility may be partially justified by the fact that volume and order-book information are available in many markets.

[6]A shown in Back, it is optimal for the insider to choose an absolutely continuous trading strategy, since, in continuous time, the market maker can immediately infer from the quadratic variation of the order flow the informed component with infinite variation. Note that his proof requires that the constant noise trader volatility be common knowledge. If volatility were not known by the market maker then potentially other equilibria could arise and the trading strategy of the insider may not be absolutely continuous.

Since we assume the market maker is risk-neutral, equilibrium break even requires that the market clearing price is:

$$P_t = \mathrm{E}\left[v \,|\, \mathcal{F}_t^Y\right]. \tag{3}$$

If noise trading volatility were constant then this setup would be exactly the Kyle-Back model. Instead, we assume that the noise trading volatility, σ_t, follows a general stochastic process. Specifically, we assume there is a Brownian motion W, which is independent both of v and of Z, such that:

$$d\sigma_t = m(t, \sigma^t)dt + \nu(t, \sigma^t)dW_t, \tag{4}$$

where the drift and diffusion of σ can depend on its past history which we denote by σ^t, but not on the history of Y (or Z). Further, we assume they satisfy standard integrability requirements for the SDE to admit a unique strong solution (e.g., Liptser and Shiryaev (2001) Theorem 4.6.).

Importantly, we assume that the volatility process is uniformly bounded away from zero (to avoid the case of 'degenerate learning'). In addition, we also assume that the volatility is bounded above uniformly.[7] To summarize, we assume:

$\mathcal{B}$: {There exist two constants $\underline{\sigma}, \overline{\sigma}$ such that $0 < \underline{\sigma} \leq \sigma_t \leq \overline{\sigma} < \infty$}.

We assume that both the market maker and the insider observe the history of σ perfectly. This is natural, since by observing aggregate order flow in continuous time, its quadratic variation is perfectly observed. Thus the filtration $\mathcal{F}_t^Y$ contains both histories

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[7]These assumptions are stronger than necessary, but simplify the proof of existence of equilibrium in the general case. In the last section, we relax some of these assumptions. Specifically, we construct equilibria where W_t is not a purely continuous martingale, and where σ_t is not bounded uniformly. We also allow for some correlation between σ_t and Y_t.

of order flow (Y^t), and of volatility (σ^t).

We ask the following questions. How does the insider adapt his optimal trading strategy to account for these time-varying noise trader volatility shocks? How is the equilibrium price dynamics affected by these shocks, which are orthogonal to the private information of the insider (i.e., are not directly payoff relevant)?

At first, this problem may seem like a trivial extension of the Kyle (1985) model, as one might conjecture that one can simply 'paste together' Kyle economies with different levels of noise-trading volatility. However, this is not so. Indeed, the insider will optimally choose to trade less in the lower liquidity states than he would were these to last forever, because he anticipates the future opportunity to trade more when liquidity is better and he can reap a larger profit. Of course, in a rational expectations' equilibrium, the market maker foresees this, and adjusts prices accordingly.

At this stage it is useful to introduce a new quantity G_t which is the solution to the following recursive equation:

$$\sqrt{G_t} = \mathrm{E}\left[\int_t^T \frac{\sigma_s^2}{2\sqrt{G_s}} ds | \sigma^t\right]. \tag{5}$$

As we will show below, G_t represents the relevant quantity of expected noise trading remaining over the trading horizon that the insider should compare to his private information when deciding how aggressively to trade. In fact, for specific choices of the volatility dynamics (e.g., when it is deterministic or unpredictable) the solution to this recursive equation is precisely the expected total noise trading variance. However, when the expected change in noise trading volatility is predictable then it may differ significantly. We first establish that the solution to this recursive equation exists and some of its properties.

Lemma 1. *Under condition $\mathcal{B}$ there exists a maximal bounded solution G_t to the*

recursive equation (5). Further, that solution satisfies

$$\underline{\sigma}^2 (T - t) \leq G_t \leq \overline{\sigma}^2 (T - t). \tag{6}$$

Proof 1. *See Appendix.*

Remark 1. *The theorem in Lepeltier and San Martin (1997) used for the proof of lemma 1 does not guarantee the uniqueness of the solution to the BSDE. However, the use of the maximal solution in the construction of the equilibrium seems sensible since it achieves the highest value function for the insider, as we show below.*

The following lemma characterizes the solution to the equation for interesting special cases.

Lemma 2. *Suppose the drift m_t and diffusion ν_t of volatility are such that the following regularity conditions hold: $\int_0^T |m_s| ds$ is uniformly bounded and $\xi_t = e^{- \int_0^t \frac{\nu_s^2}{2} ds + \int_0^t \nu_s dW_s}$ is a martingale,[8] then a solution to equation (5) is given by the process:*

$$G(t) = \sigma_t^2 A(t)^2, \tag{7}$$

where $A(t)$ solves the following recursive equation:

$$A(t)^2 = \tilde{\mathrm{E}}_t \left[\int_t^T e^{\int_t^u 2m_s - \sigma_A(s)^2 ds} du \right], \tag{8}$$

where the expectation is taken with respect to the measure $\tilde{P}$ equivalent to P and defined by the Radon-Nykodim derivative $\frac{d\tilde{P}}{dP} = \xi_T$, and where σ_A is the diffusion of $\log A(t)$. It follows that:

$$G(t) \leq \mathrm{E} \left[\int_t^T \sigma_u^2 du \right]. \tag{9}$$

Proof 2. *See Appendix.*

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[8]Sufficient conditions for the latter, such as the Novikov condition, are given in LS(2001).

This lemma has interesting implications.

Corollary 1. *If the expected growth rate of noise trading volatility (m_t) is deterministic, then the solution to equation (5) is:*

$$G(t) = \sigma_t^2 \int_t^T e^{\int_t^u 2m_s ds} du = \mathrm{E}[\int_t^T \sigma_s^2 ds] \tag{10}$$

This Corollary covers the two extreme cases of a general deterministic noise trader volatility process (i.e., $\sigma(t)$ is known for all future date as in Back and Pedersen (1998)) and of general unpredictable (i.e., martingale) dynamics for σ_t. In both these cases G_t is the expected remaining noise trading variance over the trading horizon. Instead, if the growth rate of noise trading volatility is stochastic then G_t is smaller than the expected noise trading variance.

With these results established we can now proceed to characterize the equilibrium trading strategy and price process in our economy. To solve for an equilibrium, we proceed in a few steps. First, we derive the dynamics of the stock price consistent with the market maker's risk-neutral filtering rule, conditional on a conjectured trading rule followed by the insider. Then we solve the insider's optimal portfolio choice problem, given the assumed dynamics of the equilibrium price. Finally, we show that the conjectured rule by the market maker is indeed consistent with the insider's optimal choice.

The equilibrium we obtain, which constitutes the main result of our paper, is summarized in the following theorem.

Theorem 1. *Under condition $\mathcal{B}$ there exists an equilibrium where the price process has dynamics*

$$dP_t = \frac{(v - P_t)}{G_t} \sigma_t^2 dt + \sqrt{\frac{\Sigma_t}{G_t}} \sigma_t dZ_t \tag{11}$$

and the optimal strategy of the insider is:

$$\theta_t^* = \frac{1}{\lambda_t} \frac{\sigma_t^2}{G_t} (v - P_t), \tag{12}$$

where G_t solves equation (1) and Σ_t, the conditional posterior variance of v in the market makers filtration, is given by:

$$\Sigma_t = \Sigma_0 e^{-\int_0^t \frac{\sigma_u^2}{G_u} du}. \tag{13}$$

In equilibrium, price change is linear in order flow (i.e., $dP_t = \lambda_t \, dY_t$) with a price impact process given by:

$$\lambda_t = \sqrt{\frac{\Sigma_t}{G_t}}, \tag{14}$$

The optimal value function of the insider is:

$$J(t) = \frac{(v - P_t)^2 + \Sigma_t}{2\lambda_t}. \tag{15}$$

The unconditional expected profit of the insider (from the point of view of the market maker) is $\sqrt{\Sigma_0 G_0}$.

Further, P_t is a martingale with respect to the market maker's filtration that converges almost surely to v at the final date T.

Lastly, market depth $\frac{1}{\lambda_t}$ is a martingale and λ_t a submartingale with respect to both the market maker's and the insider's filtrations.

Proof 3. *See Appendix.*

We now comment on several implications of the theorem. First, the equilibrium price converges almost surely to the value v, known ex ante only to the insider, at maturity T. This guarantees that all private information will have been incorporated into equilibrium prices at maturity. This property is analogous to the result proved in Back (1992), that the equilibrium price in the continuous time Kyle model follows a standard Brownian Bridge. However, in our model the equilibrium price will typically

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display stochastic volatility if noise trading volatility is stochastic. Our model thus generates a form of 'excess volatility,' since non-payoff relevant shocks (e.g., 'sunspots') that affect noise trading volatility may affect the stock price volatility because the market maker rationally anticipates that in periods where noise traders are more active informed trader would be more aggressive and thus adjusts prices faster. We will show below 'excess volatility' only appears if noise trading volatility is stochastic and predictable.

Second, we see that the optimal trading strategy for the insider is to trade proportionally to the under-valuation of the asset $(v - P_t)$ at a rate that is inversely related to her price impact (λ_t) and to the remaining amount of noise trader risk measured by the new equilibrium quantity $(\frac{G_t}{\sigma_t^2})$. The latter quantity reduces to the remaining time horizon $T - t$ in the original Kyle model when σ_t is constant or indeed when σ_t is unpredictable. Thus, the idea that the insider trades at a rate that is inversely related to the remaining time horizon $T - t$ holds only for these specific cases.

Third, our expression for the price impact generalizes both BP's result (page 395) obtained for deterministically changing noise trader volatility and Kyle's result that price impact (the inverse of market depth) is a signal to noise ratio. The signal is measured as in the previous papers by the posterior variance of the liquidation value. But, interestingly, the relevant measure of noise, is quite different from what obtains in the deterministic case, where it is simply the remaining total variance. It solves a recursive equation (5) the solution of which is typically smaller than the expected remaining noise-trader variance.

Fourth, in our model price impact is a submartingale. This contrasts our framework from much of the previous literature. In the original Kyle model price impact is constant. In extensions of that model (Back, 1992; Back and Pedersen, 1998; Baruch, 2002; Back and Baruch, 2004), price impact is either a martingale, or a super-martingale. In these models, price impact measures have to improve (i.e, decrease) on average over time, to

incite the insider to not trade too aggressively initially.[9] Instead, with stochastic noise trading volatility price impact measures are expected to increase over time.

Fifth, in equilibrium $d\Sigma_t = -dP_t^2$, which shows that when information arrives at a higher rate, stock price volatility is high. As we show in the next section, when noise trading volatility exhibits mean-reversion, then stock price volatility is stochastic and typically increases when noise trading volatility increases. So in our model more information makes its way into prices when noise trading volatility increases. This is very different from the standard Kyle model, where private information is revealed at a constant rate that is independent of the level of noise trading volatility.

Before we consider a few specific examples of noise trader volatility process in the next section, we compute the profits to the insider and and the execution costs for noise traders. Total unconditional profits of the informed in our model can be computed by integrating the value function over the unconditional prior distribution of v, as $\mathrm{E}^v[J(0)] = \frac{\Sigma_0}{\lambda_0} = \sqrt{\Sigma_0 G_0}$. Clearly, the profits depend on how much private information remains to be released to the market, and the total expected amount of noise trading as measured by the solution to the recursive equation for G_0.

We define the aggregate execution (or slippage) costs incurred by liquidity traders at time T (defined pathwise) as:

$$\int_0^T \sigma_t dZ_t dP_t = \int_0^T \lambda_t \sigma_t^2 dt. \tag{16}$$

Intuitively, the total losses incurred between 0 and T by noise traders can be computed

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[9]Motives to trade more aggressively early on are due to risk-aversion and a random exogenous deadline. It would be interesting to combine risk-aversion or random deadline, with stochastic noise trader volatility. It is likely that price impact would be neither a sub nor a super martingale in that case.

pathwise as:

$$\int_0^T (P_{t+dt} - v)\sigma_t dZ_t = \int_0^T (P_t + dP_t - v)\sigma_t dZ_t = \int_0^T \lambda_t \sigma_t^2 dt + \int_0^T (P_t - v)\sigma_t dZ_t. \quad (17)$$

The first component is the pure execution or slippage cost due to the fact that, in Kyle's model, agents submit market orders at time t that get executed at date $t + dt$ at a price set by competitive market makers. The second component is the pure fundamental loss due to the fact that based on the price they observe at t noise traders purchase a security with fundamental value v that is unknown to them. Note that since prices are set efficiently by market makers, on average this second component has zero mean. Therefore we obtain the result that, the unconditional expected total losses incurred by noise traders are equal to the unconditional expected execution costs incurred by noise traders. Further, these are also equal the total unconditional expected profits of the insider. (Note, however, that pathwise neither quantity need be equal.[10])

In the following section we consider a few specific examples of noise trader volatility process to illustrate the above result.

3. Special Cases of the Noise Trading Volatility Dynamics

As is clear from the above proposition, much of the characterization of the equilibrium depends on the dynamics of the λ_t process, which, in turn, depends on the G_t and Σ_t

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[10]To show that unconditional expected execution costs paid by noise traders are equal to the unconditional expected profits of the insider note that the insider's unconditional expected profits are

$$\mathrm{E}^v\Big[\int_0^T \theta_t(v - P_t)dt\Big] = \mathrm{E}^v\Big[\int_0^T \frac{\sigma_t^2}{\sqrt{\Sigma_t G_t}}(v - P_t)^2 dt\Big] = \mathrm{E}^v\Big[\int_0^T \frac{\sigma_t^2}{\sqrt{\Sigma_t G_t}}\Sigma_t dt\Big] = \mathrm{E}^v\Big[\int_0^T \sigma_t^2 \lambda_t dt\Big], \quad (18)$$

where the first equality follows from the definition of θ^* and the second from the law of iterated expectations. This is the same expression obtained for the execution costs paid by noise traders. By definition this is also equal to $\mathrm{E}^v[J(0)] = \sqrt{\Sigma_0 G_0}$ where the expectation superscript emphasizes that it is taken over the unconditional distribution of v.

processes. G_t solves a backward stochastic differential equation, which can be solved for specific choices of the noise trading volatility dynamics. In this section we consider a few special cases, for which we can characterize the equilibrium further. As we show below a crucial distinction is whether the drift of noise trading volatility is stochastic or not.

3.1. Deterministic expected change in noise trading volatility

Suppose that the drift m_t of the noise trader volatility process in equation (4) is a deterministic process, but that the diffusion may be stochastic with general form $\nu(t, \sigma^t)$. Then following Corollary 1 we can derive an explicit solution for the solution G_t:[11]

$$G(t) = \sigma_t^2 B_t \tag{19}$$

$$B_t = \int_t^T e^{\int_t^u 2m_s ds} du. \tag{20}$$

With this solution in hand we can derive all the equilibrium quantities using Theorem 1. We summarize them in the following.

Theorem 2. *If the drift m_t of noise trader volatility in equation (4) follows a deterministic process, then the solution to G_t is given by (19). It follows that private information flows into prices deterministically:*

$$\frac{\Sigma_t}{\Sigma_0} = \frac{e^{\int_0^t 2m_s ds} B_t}{B_0}. \tag{21}$$

Market depth is given by:

$$\frac{1}{\lambda_t} = e^{-\int_0^t m_s ds} \sigma_t \sqrt{\frac{B_0}{\Sigma_0}}. \tag{22}$$

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[11]This can be verified by direct substitution of the solution (19) into equation (5). It also shows that assumption $\mathcal{B}$ used to derive a general existence result for G in lemma (1) is not necessary.

The trading strategy of the insider is:

$$\theta_t = \frac{e^{-\int_0^t m_s ds} \sigma_t}{B_t} \sqrt{\frac{B_0}{\Sigma_0}} (v - P_t). \tag{23}$$

The expected trading rate of the insider is given by:

$$\mathrm{E}[\theta_t \,|\, v, \mathcal{F}_0] = \frac{e^{\int_0^t 2m_s ds} \sigma_0}{\sqrt{B_0}} \frac{(v - P_0)}{\sqrt{\Sigma_0}} \tag{24}$$

Stock price dynamics are given by:

$$dP_t = \frac{(v - P_t)}{B_t} dt + e^{\int_0^t m_s ds} \sqrt{\frac{\Sigma_0}{B_0}} dZ_t. \tag{25}$$

In particular, stock price volatility is a deterministic exponentially increasing (decreasing) function of time if noise trader volatility is expected to increase (decrease). The unconditional expected profit at time zero of the insider is $T \sigma_v \sigma_0 \sqrt{\frac{B_0}{T}}$.

Proof 4. *Follows directly from theorem 1 and the expression for G_t obtained in (19) for the case of a deterministic drift. The only new result is the calculation of the expected trading rate of the insider is given by:*

$$\mathrm{E}[\theta_t \,|\, v, \mathcal{F}_0] = \mathrm{E}[\frac{(v - P_t)}{\sqrt{\Sigma_t}} \frac{\sigma_t^2}{\sqrt{G_t}}] \tag{26}$$

$$= \frac{(v - P_0)}{\sqrt{\Sigma_0}} e^{-\int_0^t \frac{1}{2B_s} ds} \frac{\sigma_0 e^{\int_0^t m_s ds}}{\sqrt{B_t}} \tag{27}$$

where we used the dynamics of h_t from lemma 5 and the expression for G_t from equation (19). The result in the theorem then follows from standard manipulations.

This result is interesting as it informs us about the equilibrium in two important special cases: when the noise trader volatility process is deterministic where we recover the model of Back and Pedersen (1998) and in the case where it is unpredictable where

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we recover a stochastic version of Kyle/Back's model. We summarize these special cases in the following.

Corollary 2. *If noise trader volatility is deterministic ($\nu_t = 0$) then the equilibrium is identical to that derived in Kyle/Back, up to a deterministic time change given by $\tau_t = \int_0^t \sigma_u^2 du$. Indeed, $G(t) = \tau_T - \tau_t$, and $\frac{\Sigma_t}{\Sigma_0} = \frac{(\tau_T - \tau_t)}{\tau_T}$. Price impact (and market depth) are constant: $\lambda = \sqrt{\frac{\Sigma_0}{\tau_T}}$. The optimal strategy of the insider is: $\theta_t^* = \frac{\sigma_t^2}{\lambda(\tau_T - \tau_t)}(v - P_t)$. The equilibrium price process follows a time-changed Brownian bridge process:*

$$dP_t = \frac{(v - P_t)}{\tau_T - \tau_t} d\tau_t + \lambda d\tilde{Z}(\tau_t). \tag{28}$$

where $\tilde{Z}$ is a standard Brownian motion with respect to $\mathcal{F}_{\tau_t}$.

This result is consistent with the analysis in Back and Pedersen (1998) and specializes to the continuous time Kyle-model also derived in Back (1992) if $\sigma(t) = \sigma$ is constant (in which case $\lambda = \frac{\sigma_v}{\sigma}$ with $\sigma_v^2 = \frac{\Sigma_0}{T}$ is the annualized variance of the market maker's prior estimate of the asset value). Note that price impact is constant in this example. However, this is no longer the case as soon as noise trader volatility is stochastic.

Corollary 3. *If noise trader volatility follows an unpredictable martingale process (i.e., $m_t = 0$) then private information is incorporated into prices linearly (independent of the level of noise trader volatility):*

$$\frac{\Sigma_t}{\Sigma_0} = \frac{(T - t)}{T}. \tag{29}$$

Market depth (the inverse of Kyle's lambda) is stochastic and proportional to noise trader volatility:

$$\frac{1}{\lambda_t} = \frac{\sigma_t}{\sigma_v}, \tag{30}$$

where $\sigma_v^2 = \frac{\Sigma_0}{T}$ is the annualized initial variance of private information. The trading strategy of the insider is:

$$\theta_t = \frac{\sigma_t}{\sigma_v(T - t)}(v - P_t). \tag{31}$$

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Equilibrium price dynamics are identical to the original Kyle/Back model:

$$dP_t = \frac{(v - P_t)}{T - t}dt + \sigma_v dZ_t. \tag{32}$$

In particular, stock price volatility is constant. The unconditional expected profit level of the insider at time zero is $T\sigma_v\sigma_0$.

This example shows that many of the features of Kyle's equilibrium survive when noise trader volatility follows *arbitrary* martingale dynamics. Indeed, we see that when noise trader volatility is not forecastable, private information gets into prices at the same rate as in the original economy (i.e., linearly). The equilibrium looks identical to the original Kyle-Back model where one substitutes a stochastic process σ_t for the constant noise trading volatility in the original model. Since in the original model, the equilibrium price process and the rate at which private information is revealed are independent of the noise trading volatility, they are unchanged in this case. However, both the trading strategy of the insider and the price impact (Kyle's lambda) change. Both become stochastic. The insider trades more aggressively when noise trading volatility is higher, but price impact moves in the exact opposite direction so that both effects cancel, leaving equilibrium prices unchanged. In equilibrium then, insiders cannot gain from timing their trades and thus their unconditional expected profit level is unchanged relative to what it would be in the Kyle-Back model with noise trader volatility set to a constant σ_0. Interestingly, even in this model however, price impact measures are stochastic and vary inversely with the level of noise trader volatility. Since the latter is a martingale, we see that on average, price impact is expected to increase in this case.

While we do recover the original Kyle/Back model in some unconditional expected sense when noise trader volatility is unpredictable, interestingly it is market depth (and not price impact) that is on average constant.

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In general when noise trading volatility is stochastic ($\nu \neq 0$) and if there is predictability ($m_t \neq 0$) then theorem 2 shows that the equilibrium differs from the standard Kyle-Back solution even unconditionally. For example, we plot in figure 1 the expected optimal trading rate of the insider normalized by the initial undervaluation ($\mathrm{E}[\theta_t \mid v, \mathcal{F}_0]/(v - P_0)$ from equation (24)) for different levels of m. As we can see, when the noise trading volatility is unpredictable ($m = 0$) then we expect the insider to trade at a constant rate as in the Kyle-Back model. Instead, if noise trading volatility is expected to increase ($m > 0$) then unconditionally we expect the insider to trade more aggressively on average in the future when more noise trading will occur.

In figure 2 we plot the corresponding $G(t)$ function normalized by the noise trader variance (i.e., $\frac{G_t}{\sigma_t^2}$, which in this case is a deterministic component of the inverse of the trading rate of the insider. When $m = 0$ his reduces to a linear function of time, leading the insider to trade inversely proportionally to time to maturity. Instead, when $m > 0$ then we see this function is convex, and for $m > 0$ concave explaining partially the shape of the unconditional expected trading rate. These different trading patterns also result in different patterns in information revelation.

Figure 3 plots the path of the posterior variance of the private information signal for three cases $m = 0.5$, $m = 0$ and $m = -0.5$. It is remarkable that private information is revealed following a deterministic path, which only depends on the expected rate of change in noise trading volatility, despite the fact that the strategy of the insider is stochastic. This is of course the result of the offsetting effect noise trading volatility has on the price impact coefficient λ_t. If the level of noise trading variance changes, the insider trades more or less aggressively, but price impact changes one for one, making price dynamics and information revelation independent of the volatility level. If variance is expected to increase, then private information gets into prices more slowly initially, and then faster when the insider trades more aggressively. So posterior variance
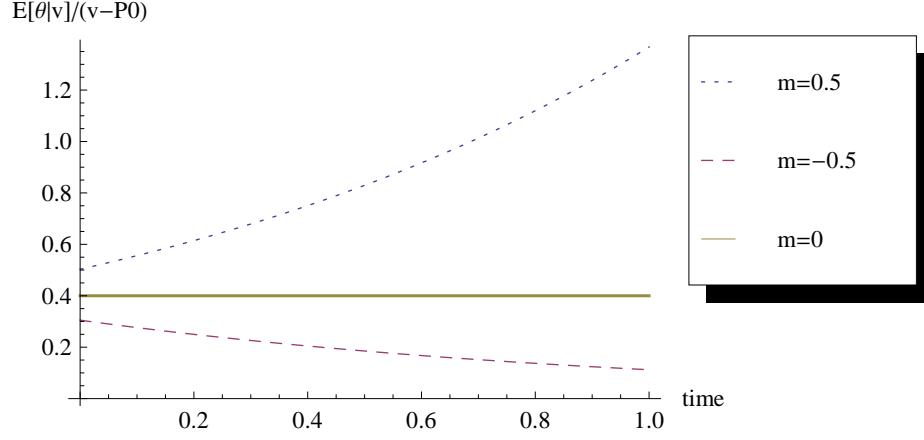
Figure 1: Trading strategy of the insider normalized by his expected profit $(\theta_t/(v - P_t))$ for a given fixed level of noise trader volatility plotted against time and for different levels of expected growth rate of noise trader volatility.
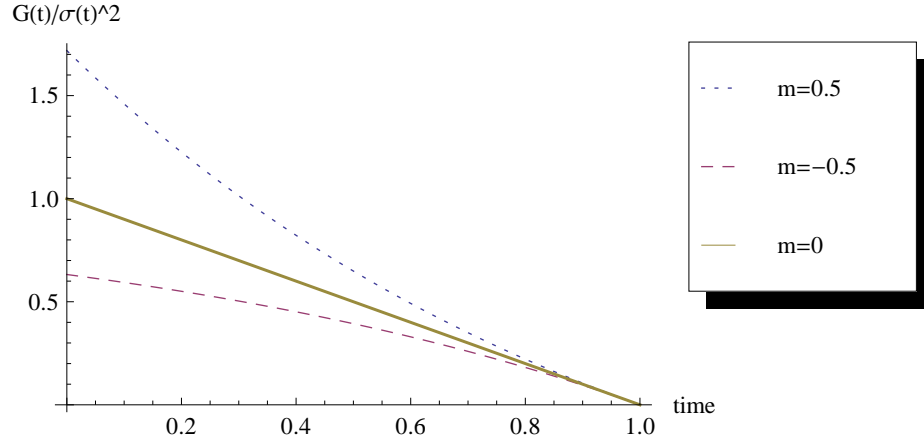

Figure 2: Expected remaining cumulative noise trading variance normalized by the noise trader variance $(G(t)/\sigma_t^2)$ plotted against time and for different levels of expected growth rate of noise trader volatility.
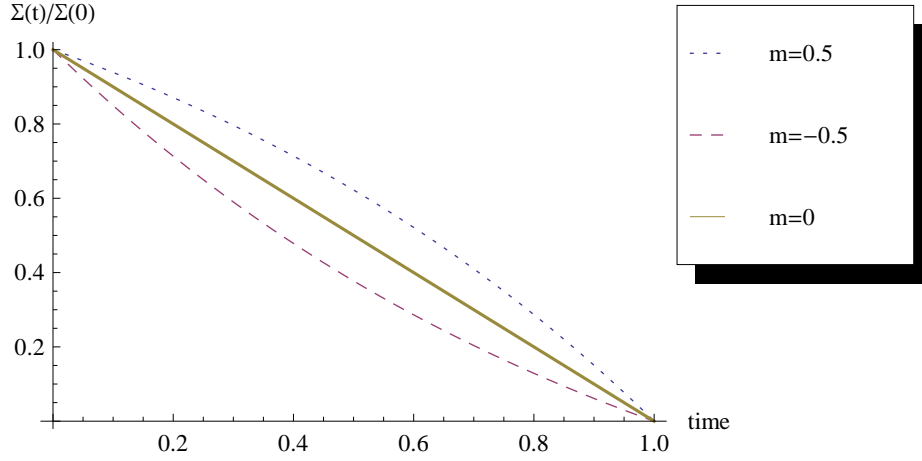
Figure 3: Path of posterior variance of the insider's private information Σ_t for various values of the expected change in noise trader volatility m.

follows a deterministic concave path if noise trader volatility is expected to increase, but a convex path if it is expected to decrease. As a result, the equilibrium price process exhibits time varying volatility. Its volatility increases (decreases) exponentially if noise trader volatility is expected to increase (decrease). Interestingly, price volatility is deterministic, despite stochastic noise trader volatility and stochastic market depth. From a mathematical point, the price process follows a one-factor-Markov bridge process with non-time homogeneous volatility (recall that P_t almost surely converges to v at time T).

In this economy, we obtain an interesting separation result. The strategy of the insider (θ_t) and the price impact measures (λ_t) expressed as a function of the level of noise trader volatility (σ_t) are independent of the volatility of the noise trader volatility (ν). As a result, the informational efficiency of prices, the price process, and the unconditional expected profits of the insider only depend on initial conditions (Σ_0, σ_0) and on the expected growth rate of noise trader volatility (m).

However, the time-series dynamics of the price impact measure and the optimal strategy of the insider is stochastic and varies with σ_t, which of course depends on ν. An implication is that we may see lots of variation in estimates of price impact measures (Kyle's lambda) in time series (i.e, at different times along one path) and in cross-section (i.e., at the same time across different 'economies' or stocks), but this is not necessarily informative about the amount of private information (Σ_t) in the market.

Theorem 2 clearly shows that for uncertainty about future noise trading volatility to affect the equilibrium price volatility and indeed for price volatility to be stochastic, the expected change in noise trading volatility m_t has to be stochastic.[12]

To further analyze how stochastic noise trading volatility can generate stochastic price volatility and how both interact with measured price impact and actual execution costs paid by noise traders, we now consider the case where noise trading volatility follows a continuous time Markov Chain. This introduces state-dependent predictability and jumps in noise trader volatility in a simple manner and leads to a tractable illustration of the dynamics of price volatility and private information revelation.

3.2. Stochastic expected change in noise trader volatility: A Markov chain example

Here we consider a case where noise trader volatility does not follow a diffusion process and show that we can still derive an equilibrium following the same approach used in the Brownian case above. We assume that noise trading volatility follows a two-state continuous Markov Chain, i.e., there are two fixed values $\sigma^L < \sigma^H$ with $\sigma_0 \in \{\sigma^H, \sigma^L\}$ and with dynamics:

$$d\sigma_t = (\sigma^H - \sigma_t)dN_L(t) - (\sigma_t - \sigma^L)dN_H(t), \tag{33}$$

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[12]In the appendix we analyze a case where volatility follows a mean-reverting diffusion process, using expansion techniques and show using an approximation to the solution for G how uncertainty about future noise trading volatility interacts with mean-reversion to generate stochastic price volatility.

where $N_i(t)$ is a standard Poisson counting process with jump intensity η_i respectively.[13]

Since the volatility process is Markov, we seek a solution to the BSDE for G of the form $G(t, \sigma_t)$ that satisfies:

$$\sqrt{G(t, \sigma_t)} = \mathrm{E}_t \left[\int_t^T \frac{\sigma_u^2}{2\sqrt{G(u, \sigma_u)}} du \right]. \tag{34}$$

We can characterize it as follows.

Theorem 3. *A solution to (34) is the function $G(t, \sigma_t) = \mathbf{1}_{\{\sigma_t = \sigma^L\}} G^L(T - t) + \mathbf{1}_{\{\sigma_t = \sigma^H\}} G^H(T - t)$, where the deterministic functions G^L, G^H satisfy the system of ODE given in equations (35)-(36) below, with boundary conditions $G^L(0) = G^H(0) = 0$.*

$$G_\tau^L(\tau) = (\sigma^L)^2 + 2\eta_L(\sqrt{G^H(\tau)G^L(\tau)} - G^L(\tau)) \tag{35}$$

$$G_\tau^H(\tau) = (\sigma^H)^2 + 2\eta_H(\sqrt{G^H(\tau)G^L(\tau)} - G^H(\tau)) \tag{36}$$

Proof 5. *Consider a pair of functions $G^L(\cdot), G^H(\cdot)$ that solve the ODE system (35)-(36) subject to the boundary condition $G^L(0) = G^H(0) = 0$, then it is straightforward to show that if we define $G(t, \sigma_t) = \mathbf{1}_{\{\sigma_t = \sigma^L\}} G^L(T - t) + \mathbf{1}_{\{\sigma_t = \sigma^H\}} G^H(T - t)$ then $J(t) = \sqrt{G(t, \sigma_t)} + \int_0^t \frac{\sigma_u^2}{2\sqrt{G(u, \sigma_u)}} du$ is a pure jump martingale (i.e., $\mathrm{E}_t[dJ(t)] = 0$). It follows that $J(t) = \mathrm{E}_t[J(T)]$ and using the definition of $J(t)$ and the boundary conditions from the ODEs that: $\sqrt{G(t, \sigma_t)} = \mathrm{E}_t[\int_t^T \frac{\sigma_u^2}{2\sqrt{G(u, \sigma_u)}} du]$.*

We note that when there is no transition between states $\eta_i = 0$ then the solution reduces to the familiar one obtained in Back (1992), i.e., $G^i(\tau) = (\sigma^i)^2 (T - t)$. In general, the system of coupled differential equations for $G^i(t)$ $i = L, H$ can be easily solved numerically. We note that as maturity approaches, as long as the switching intensities η_H, η_L are not too (i.e., unboundedly) large, the solution for the price process converges to a pure Brownian bridge as in the continuous time version of the Kyle model

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[13]For example, η_H is the intensity of moving from state H to state L.

presented in Back (1992). However, with more time to go before maturity, the possibility of transitionning from one liquidity state to another changes the optimal strategy of the insider and the price impact function.

For illustration, we choose a period length $T = 1$, $\eta_L = \eta_H = 2$ (2 transitions per period), $\sigma^L = 0.2$ and $\sigma^H = 0.5$. For these parameter values we report in figure 4 the G-function in the high and low state. As expected, close to maturity the two functions converge smoothly to the lines $(\sigma^i)^2(T - t)$ (with $i = L, H$) that would prevail, if there were no transitions between states (i.e., the state was absorbing), which also corresponds to the original Kyle model.

Figure 4 shows that, typically, when there is a switch in regime, say from the low to the high volatility regime, the measure of price impact (Kyle's lambda) will jump down, as price impact is lower in the high noise-trading volatility regime. Indeed, recall that $\lambda_t = \sqrt{\frac{\Sigma_t}{G_t}}$ and that since Σ_t is an absolutely continuous process, the immediate effect of an upward jump from G^L to G^H is to lower λ. (of course subsequently, in the high noise trading regime, information will be impounded more quickly into prices, leading to a faster drop in Σ_t than in the low volatility regime).

Using the explicit solution for the amount of private information $\Sigma(t) = \Sigma(0)e^{-\int_0^t \frac{\sigma_u^2}{G_u} du}$, we present in figure 5 four paths of $\Sigma(t)$ which depict the revelation of private information in our economy relative to the Kyle (1985) benchmark. We plot $\Sigma(t)/\Sigma(0)$ for the case where noise trading volatility switches to the high regime at date zero and stays there until maturity (high), when it starts in the low regime and stays there until maturity, and when there is a jump at $t = 0.5$ from high to low and low to high respectively.

Note that in Kyle, information always decays linearly in time, **irrespective** of the level of noise trader volatility, in the sense that $\frac{\Sigma^{kyle}(t)}{\Sigma^{kyle}(0)} = \frac{T-t}{T}$. Instead, when noise trading volatility can change stochastically, information flows into prices in a very
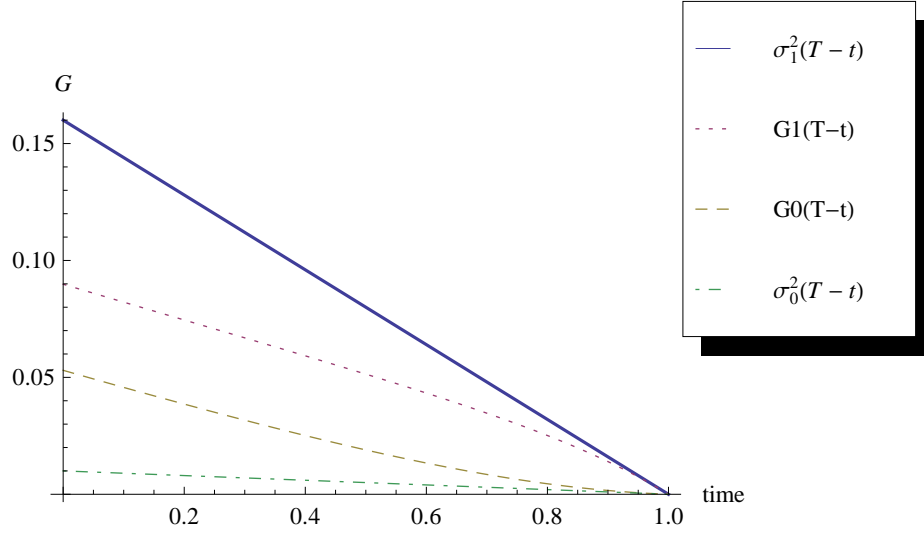
Figure 4: G function in high and low state that solve equation (34). We also plot the lines $\sigma^L(T-t)$ and $\sigma^H T - t$, which sandwich respectively $G^L(T-t)$ and $G^H(T-t)$.
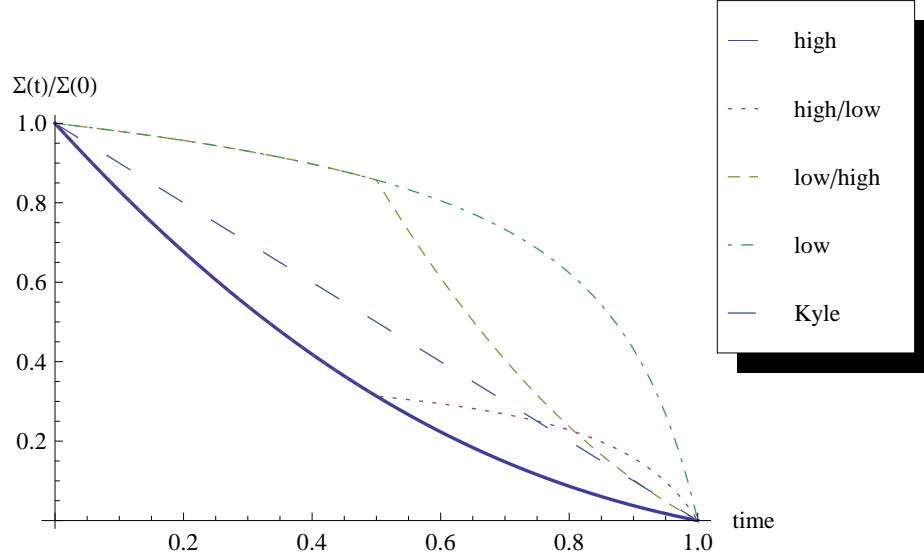

Figure 5: Four paths of the remaining amount of private information $\Sigma(t)/\Sigma(0)$ corresponding to four different noise trader volatility scenarios: (a) start and stay in the high volatility regime until T, (b) start and stay in the low volatility regime until T, (c) start in the high volatility regime and switch to low volatility at $t = 0.5$, and (d) start in low volatility regime and switch to high at $t = 0.5$. We also plot as a benchmark, the Kyle (1985) economy private information decay, which is linear and independent of the noise trader volatility level.

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different fashion. As figure 5 reveals, the posterior variance, when in the high volatility regime, is a decreasing convex function of time, but becomes decreasing concave when there is a switch to the low noise trading regime. The intuition, is that in the low noise trading regime, the insider is playing a waiting game, in the sense that he trades much less aggressively, than he would in the Kyle economy with the same level of volatility. He does so hoping for the high noise trading regime to arrive, where he trades more aggressively, leading to much faster arrival of private information. Of course, if the regime switch does not arrive then ultimately, he will have to become more aggressive so that all his information eventually makes it into prices (see the path marked as 'low' on the graph).

This suggests that all the price impact measures and execution cost measures will be path dependent. For example, we plot in figure 6, for the same four noise trading volatility scenarios, the corresponding path of the price impact $(\lambda(t))$ process. We see that if the economy starts in the high noise trading regime and stays there until maturity, then measured price impact is relatively low and decays steadily. Instead, if the economy starts in the low noise trading regime, then price impact level is at first only slightly higher than the price impact level in the high noise trading regime, but it increases exponentially as the economy approaches maturity. Similarly, if the regime switches at some point from high to low volatility, then price impact immediately jumps up a little, but subsequently, market depth worsens very rapidly as λ increases along a very convex path. This captures intuitively, the submartingale property of λ. On average, execution costs are expected to increase as the economy approaches maturity. Interestingly, note that if the economy is in the high noise trading regime, then measured price impact will be low and decrease steadily at the beginning, even though there is a lot of 'asymmetric information' in the sense that, from figure 5, we see a lot of information getting into prices. Comparing figures 5 and 6 suggests that the level of λ_t, obtained by 'regressing'

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stock price changes on order flow, does not give a valid measure of the amount of private information flowing into prices (as measured by the slope of $\Sigma(t)$ which equals $-\lambda_t^2 \sigma_t^2$).

In figure 7 we plot the volatility of the stock price (which equals $\lambda_t \sigma_t$), for the same four noise trading volatility scenarios. As we see, stock price volatility tends to be higher in the high noise-trading volatility regime. If the economy stays in that regime, then volatility drops steadily. However, if the economy jumps to a low noise trading volatility regime, then stock price volatility jumps down, a large amount, and then subsequently rises rapidly, following an exponential path. These dynamics are intuitive, given the discussion about how private information is disclosed since price volatility is higher when information is disclosed faster.

Lastly, from its definition, it is clear that execution costs paid by noise traders are closely related to the path of stock price volatility. In figure 8 we plot, for the same four scenarios, the path of realized execution costs ($\lambda_t \sigma_t^2$). The total execution costs paid by noise traders at time T is represented by the area below each curve plotted. From the graph it is clear that execution costs are lowest in the low volatility regime scenario, and much higher in the high noise trading volatility regime. We give the corresponding numbers in table 1.

This is paradoxical, since as is clear from the table, the high noise trading volatility regime is also the one where the average measured price impact (λ) is lower. However, the comparison is not appropriate since there are typically more noise trading (as measured by the quadratic variation of the order flow) in a high volatility scenario than in the low volatility scenario, and therefore it is natural that the total execution costs paid by noise traders are higher in the high volatility scenario. However, if we compare the two other scenarios (high/low to low/high), where arguably there are the same 'number' of noise traders along each path (in the sense that the cumulative quadratic variation of noise trading is the same across both paths as is confirmed in the third row of table 1), then
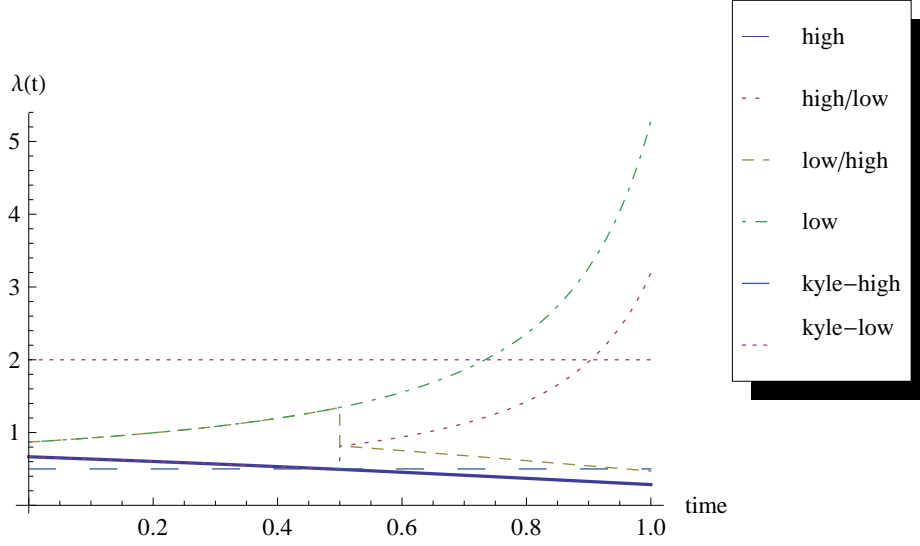
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Figure 6: Four separate paths of equilibrium price impact (lambda) dynamics corresponding to (a) start and stay in the high volatility regime until T, (b) start and stay in the low volatility regime until T, (c) start in the high volatility regime and switch to low volatility at $t = 0.5$, and (d) start in low volatility regime and switch to high at $t = 0.5$.

Figure 7: Four separate paths of stock price volatility corresponding to (a) start and stay in the high volatility regime until T, (b) start and stay in the low volatility regime until T, (c) start in the high volatility regime and switch to low volatility at $t = 0.5$, and (d) start in low volatility regime and switch to high at $t = 0.5$.
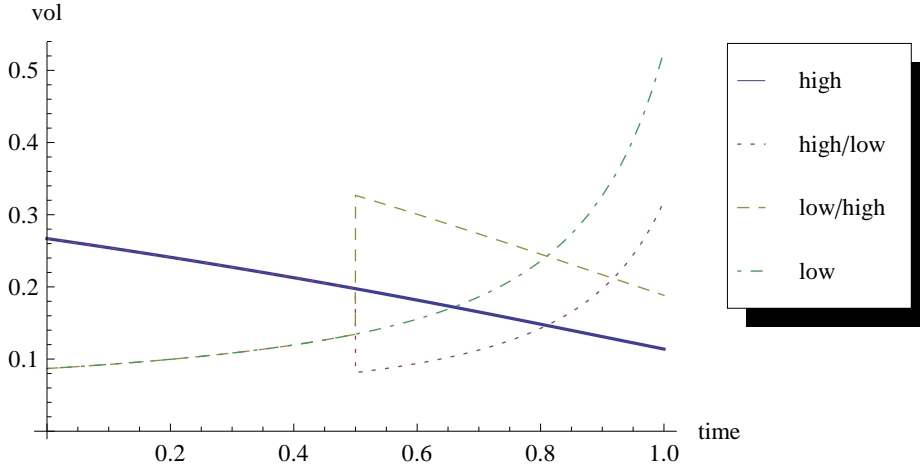
Figure 8: Four separate paths of realized execution costs $(\lambda_t \sigma_t^2)$ corresponding to (a) start and stay in the high volatility regime until T, (b) start and stay in the low volatility regime until T, (c) start in the high volatility regime and switch to low volatility at $t = 0.5$, and (d) start in l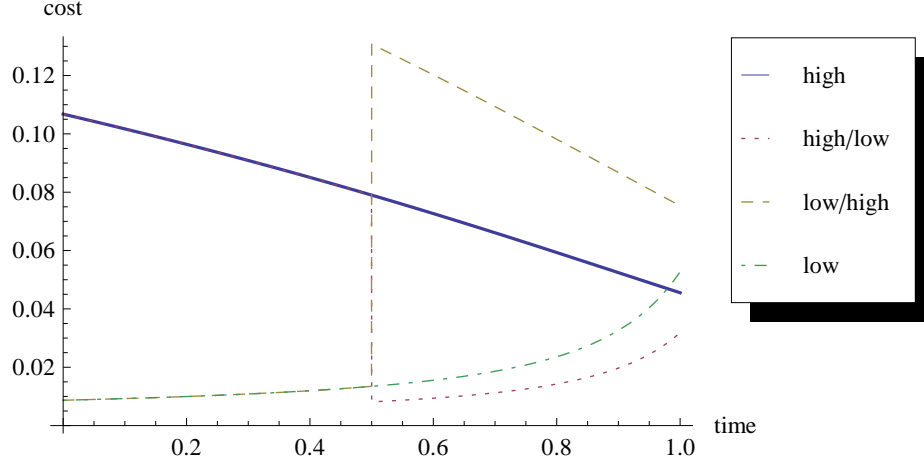ow volatility regime and switch to high at $t = 0.5$. As explained in lemma 3 the area under each path represents the execution costs incurred by noise traders.

| | Noise trading volatility paths | | | |
	high/high	low/low	high/low	low/high
Total	0.078	0.017	0.054	0.057
Execution costs $(\int_0^T \lambda_t \sigma_t^2 dt)$	0.047/0.031	0.005/0.012	0.047/0.007	0.005/0.052
Total	0.487	1.740	1.023	0.853
Average price impact $(\int_0^T \lambda_t dt)$	0.292/0.195	0.530/1.210	0.292/0.731	0.530/0.323
Total	0.16	0.01	0.085	0.085
'Number' of noise traders $(\int_0^T \sigma_t^2 dt)$	0.08/0.08	0.005/0.005	0.08/0.005	0.005/0.08
Total	0.487	1.740	0.636	0.671
Normalized execution costs $(\frac{\int_0^T \lambda_t \sigma_t^2 dt}{\int_0^T \sigma_t^2 dt})$	0.587/0.387	1/2.4	0.587/1.4	1/0.65
Total	0.195	0.174	0.190	0.182
Average stock price volatility $(\int_0^T \lambda_t \sigma_t dt)$	0.117/0.078	0.053/0.121	0.117/0.073	0.053/0.129

Table 1: This table presents the realized execution costs for noise traders depending on various scenarios of realized paths of noise trader volatility. Each path of realized noise trader volatility corresponds to a certain 'number' of uniformed traders arriving to the market. This 'number' is measured by the quadratic variation of the order flow. Normalized execution costs measure the total execution costs divided by the number of noise traders.

we see that this is not the only reason. Indeed, execution costs paid by noise traders are higher in the low/high than in the high/low scenario (see first row of table 1). Interestingly, we see that the average price impact is higher in the high/low than in the low/high regime, which indicates that focusing on the average level of price impact measures does not capture the realized execution costs. Instead, normalized execution costs, measured as 'volume' weighted price impact normalized by total noise trader volume, perhaps better captures the average execution cost paid by the average noise trader. We see in the last row of table 1 that the average price impact thus measured is indeed higher in the low/high than the high/low regime, and indeed, also higher in the low regime than the high regime, indicating the importance of normalizing price impact measures, when there can be variation in noise trading volatility.

Unlike in previous literature, aggregate execution costs are path-dependent and depend in a complex manner on the realized path of noise trader volatility (and not just on the total cumulative amount of noise trading relative to private information as in Kyle (1985) or Back and Pedersen (1998)).

4. Extensions and Empirical Implications

4.1. The Dynamics of Aggregate Order Flow

For simplicity we have assumed that aggregate order flow and noise trader volatility are conditionally uncorrelated. This assumption can be relaxed. Consider for example the more general model for total order flow:

$$d\hat{Y}_t = \theta_t dt + \sigma_t dZ_t + \eta(t, \sigma^t, \hat{Y}^t)dW_t. \tag{37}$$

In that case, it suffices to define the process Y_t by

$$dY_t = d\hat{Y}_t - \frac{\eta(t, \sigma^t, \hat{Y}^t)}{\nu(t, \sigma_t)}(d\sigma_t - m(t, \sigma^t)dt) = \theta_t dt + \sigma_t dZ_t. \qquad (38)$$

Note that the dynamics of Y_t in (38) are identical to those in equation (2). Further, since observing $(\hat{Y}_t, \sigma_t)$ is equivalent to observing (Y_t, σ_t) for all market participants, it can be shown that all our results above (and in particular Theorem 1) are unchanged with this more general model of order flow. This shows that our previous assumption that total order flow and σ_t be conditionally uncorrelated is not crucial for our results. Instead, our proofs do rely on the assumption that order flow does not Granger-cause σ_t in the sense that the history of $\hat{Y}_t$ does not affect the future dynamics of σ_t (i.e, m and ν cannot depend on $\hat{Y}^t$).

Economically, this extension is interesting as it shows that price change need not be linear in total order flow, but only in the component of the order flow that is informative about the insider's action.

4.2. Informed Trading and Adverse Selection Measures

Empirical measures of adverse selection typically rely on an estimate of the persistent price impact of trades to measure the amount of private information in trades. In their well-known survey of the micro-structure literature, Biais, Glosten, and Spatt (2005) describe the empirical relation between adverse selection and the price impact (λ) as follows: "As the informational motivation of trades becomes relatively more important, λ goes up." (page 232). Consistent with this intuition many empirical study rely on average measures of price impact to sort firms into groups with different levels of adverse selection (higher when average price impact is higher).

One of the implications of our model is that average price impact might not be a valid measure of average adverse selection costs paid by noise traders. This implication

is most evident when the Markov chain example is analyzed (see Section 3.2). In a recent empirical study, Collin-Dufresne and Fos (2014) ('CF' hereafter) report evidence consistent with this implication. Investigating a large sample of trades by informed investors[14] CF find that informed traders trade much more aggressively, when measured adverse selection is low. Their study uncovers a strong negative relation between traditional measures of adverse selection (such as estimates of Kyle's lambda obtained from high-frequency data) and trading by informed investors.They also show that these informed investors are more likely to trade when abnormal volume on the stock itself as well as other measures of liquidity (such as the abnormal volume on the S&P 500 stock index) are high, which is consistent with the economic mechanism of this paper, where informed investors wait for liquidity to be high to trade more.

4.3. Dynamics of Price volatility, bid-ask spreads, and returns

The model makes interesting predictions about the joint dynamics of price volatility, price impact (i.e, the adverse selection component of trading costs), and stock returns. For example, the model predicts that typically price volatility tends to be high when volume is high and price impact is low. This goes in an opposite direction to the relation predicted for example by inventory models of trading costs, where higher volatility would typically be positively related to trading costs. Further, our model predicts that the joint dynamics are path-dependent and will depend significantly on the realized path of noise trading. So for example, after a long period of low noise trading, price volatility and price impact can actually both rise together. This occurs in the model if the insider approaches maturity without having been able to trade much in a high noise trading environment. Eventually, he is not willing to wait anymore and trades aggressively

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[14]Exploiting an SEC disclosure requirement CF build a sample of trades by activist investors. They document that these trades are informed, based on their abnormal realized profits and analyze the price impact of these trades.

in a low noise trading environment leading to high price impact and high volatility. These predictions could be tested and used to improve our estimates of adverse selection measures.

5. Conclusion

In this paper we have extended Kyle's (1985) model of dynamic insider trading to the case where noise trading volatility can change stochastically over time. In equilibrium, we find that the insider adjusts his optimal trading strategy to trade less when noise trading volatility is lower and more when it is higher. Since market makers anticipate this, in equilibrium, measures of market depth are time varying. Market depth is a martingale and therefore its inverse, price impact, is a submartingale, indicating that on average execution costs are expected to increase over time.

Under certain conditions which we identify, the equilibrium price process exhibits stochastic 'excess volatility' in the sense that non-payoff relevant shocks that change noise trading in equilibrium also drive the price volatility. This is because rational market makers anticipate that more informed trading occurs, and thus more information is revealed, when noise trading volatility is high.

The model makes interesting predictions about the joint dynamics of price volatility, price impact (i.e, the adverse selection component of trading costs), and stock returns, which could be taken into account when estimating empirical measures of adverse selection.

The model makes many simplifying assumptions that could be relaxed to further our understanding of how information flows into prices and how volatility, price impact, and prices comove. First, we assume that the amount of private information is fixed and only noise trading volatility is time varying. Second, we assume throughout that the noise trading volatility process is common-knowledge. Third, we assume (as in Kyle) that

the presence of the insider is common-knowledge. Fourth, we assume that the trading horizon is exogenous and fixed. And lastly we assume that the insider is risk-neutral. We leave these extensions for future research.

6. Appendix

6.1. Proof of lemma 1

We note that $y_t = \sqrt{G_t}$ solves the Backward stochastic differential equation

$$dy_t = -f(t, y_t)dt - \Lambda_t dM_t$$

with $f(t, y_t) = \frac{\sigma_t^2}{2y_t}$ and with terminal condition $y_T = 0$. Now $f(t, y_t) \leq \ell(y_t) \ \forall(t, \omega)$ where we define the function $\ell(y) = \frac{\bar{\sigma}^2}{2y}$. We can thus compute $\int_0^\infty \frac{dx}{\ell(x)} = \int_{-\infty}^0 \frac{dx}{\ell(x)} = \infty$. Thus $\ell(x)$ is super-linear as shown in lemma 1 of Lepeltier and San Martin (1997). Their theorem 1 then applies, which gives us the existence of a maximal bounded solution for y_t (and therefore for G_t). In addition their Theorem 1 implies that there exist two solutions $L(t), U(t)$ that solve $L_t = -\int_t^T \ell(L_s)ds$ and $U_t = \int_t^T \ell(U_s)ds$ such that we have $L_t \leq G_t \leq U_t$. It is easy to calculate that $U_t^2 = -L_t^2 = \bar{\sigma}^2(T-t)$. This gives us the upper bound.

For the lower bound we use a comparison theorem. Consider the solution to the following Backward equation

$$dx_t = -\frac{\underline{\sigma}^2}{2x_t}dt - \tilde{\Lambda}_t dM_t,$$

with terminal condition $x_T = 0$. It can be computed straightforwardly as $x_t = \underline{\sigma}\sqrt{T-t}$ (note $\tilde{\Lambda}_t = 0$). Since $\forall(t, \omega); \ f(t, y) \geq \frac{\underline{\sigma}^2}{2y}$ we can use the comparison result Corollary 2 of Lepeltier and San Martin (1997) to obtain

$$y_t \geq x_t \ \forall(t, \omega), \tag{39}$$

which gives the lower bound on the maximal solution for G_t.

6.2. Proof of lemma 2

Define $A(t) = \frac{\sqrt{G(t)}}{\sigma_t}$. Then plugging into equation (5) we see that $A(t)$ solves:

$$A(t) = \mathrm{E}_t\Big[\int_t^T \frac{e^{\int_t^u m_s ds}}{2A(u)} \frac{\xi_u}{\xi_t} du\Big] \tag{40}$$

$$= \tilde{\mathrm{E}}_t\Big[\int_t^T \frac{e^{\int_t^u m_s ds}}{2A(u)} du\Big], \tag{41}$$

where we have used, for the first line, the fact that (for $u \geq t$):

$$\sigma_u = \sigma_t e^{\int_t^u m_s ds} \frac{\xi_u}{\xi_t}, \tag{42}$$

and, for the second line, Girsanov's theorem. Now, it follows, that $e^{\int_0^t m_s ds} A(t) + \int_0^t \frac{e^{\int_0^u m_s ds}}{2A(u)} du = \tilde{\mathrm{E}}_t\big[\int_0^T \frac{e^{\int_0^u m_s ds}}{2A(u)} du\big] =: \tilde{M}_t$ which is a continuous bounded $\tilde{P}$ martingale (given the assumed regularity conditions, the previous lemma, and the law of iterated expectations). Thus we have:

$$dA(t) + m_t A(t)dt + \frac{1}{2A(t)}dt = e^{-\int_0^t m_s ds} d\tilde{M}_t. \tag{43}$$

By Itô's formula we also have $dA(t)^2 = 2A(t)dA(t) + \sigma_A(t)^2 A(t)^2 dt$, where we define $\sigma_A(t)$ to be the diffusion of $\log A(t)$. It follows that:

$$dA(t)^2 + (2m_t - \sigma_A(t)^2)A(t)^2 dt = 2A(t)^2 e^{-\int_0^t m_s ds} d\tilde{M}_t. \tag{44}$$

Integrating, using the fact that $A(T) = 0$, and taking expectations we obtain the result.

The inequality follows immediately from the fact that $\mathrm{E}\big[\int_t^T \sigma_u^2 du\big] = \sigma_t^2 \tilde{\mathrm{E}}\big[\int_t^T e^{\int_t^u 2m_s ds} du\big]$.

6.3. Proof of main Theorem 1

The proof is in several steps.

6.3.1. Step 1: Market Maker's Updating

First, we establish that if the the market maker conjectures that the insider's trading strategy is linear in his per period profit, i.e., that:

$$\theta_t = \beta(\sigma_t, G_t, \Sigma_t)(v - P_t), \tag{45}$$

where $\beta(\cdot, \cdot, \cdot)$ measures the speed at which the insider decides to close the gap between the fundamental value v (known only to him) and the market price P_t, and where we define Σ_t as the conditional variance of the terminal payoff:

$$\Sigma_t = \mathrm{E}\left[(v - P_t)^2 \,|\, \mathcal{F}_t^Y\right], \tag{46}$$

then the equilibrium price process which results from the market maker's break-even pricing rule given in equation (3) is such that price changes are conditionally linear in order flow.

Lemma 3. *If the insider adopts a trading strategy of the form given in (45), then the stock price given by equation (3) satisfies $P_0 = \mu_0$ and:*

$$dP_t = \lambda(\sigma_t, G_t, \Sigma_t)\, dY_t, \tag{47}$$

where the price impact is a function of the conjectured trading rule:

$$\lambda(\sigma_t, G_t, \Sigma_t) = \frac{\beta(\sigma_t, G_t, \Sigma_t)\Sigma_t}{\sigma_t^2}. \tag{48}$$

Further, the dynamics of the posterior variance are given by:

$$d\Sigma_t = -\lambda(\sigma_t, G_t, \Sigma_t)^2 \sigma_t^2 dt. \tag{49}$$

Proof 6. *This follows directly from an application of theorems 12.6, 12.7 in LS 2001. We provide a simple 'heuristic' motivation of the result using standard Gaussian projection theorem below.*

$$
\begin{aligned}
P_{t+dt} &= \mathrm{E}\left[v \,|\, Y^t, Y_{t+dt}, \sigma^t, \sigma_{t+dt}\right] & (50) \\
&= \mathrm{E}\left[v \,|\, Y^t, \sigma^t\right] + \frac{Cov(v, Y_{t+dt} - Y_t \,|\, Y^t, \sigma^t)}{V(Y_{t+dt} - Y_t \,|\, Y^t, \sigma^t)}(Y_{t+dt} - Y_t - \mathrm{E}[Y_{t+dt} - Y_t \,|\, Y^t, \sigma^t]) & (51) \\
&= P_t + \frac{\beta\Sigma_t dt}{\beta^2 \Sigma_t dt^2 + \sigma_t^2 dt}(Y_{t+dt} - Y_t) & (52) \\
&\approx P_t + \frac{\beta\Sigma_t}{\sigma_t^2} dY_t. & (53)
\end{aligned}
$$

The second line uses the fact that the dynamics of σ_t is independent of the asset value distribution and of the innovation in order flow. The third line uses the fact that the expected change in order flow is zero for the conjectured policy. The last line follows from going to the continuous time limit (with $dt^2 \approx 0$). Similarly, by the projection theorem, we have:

$$Var\left[v \,|\, Y^t, Y_{t+dt}, \sigma^t, \sigma_{t+dt}\right] = Var\left[v \,|\, Y^t, \sigma^t\right] - (\frac{\beta\Sigma_t}{\sigma_t})^2 Var\left[Y_{t+dt} - Y_t \,|\, Y^t, \sigma^t\right], \tag{54}$$

which gives:

$$\Sigma_{t+dt} = \Sigma_t - \lambda_t^2 \sigma_t^2 dt. \tag{55}$$

6.3.2. Insider's Optimal strategy

Second, we establish that if price changes are linear in order flow with a specific choice of price impact process, namely:

$$dP_t = \lambda_t \, dY_t \tag{56}$$

$$\lambda_t = \sqrt{\frac{\Sigma_t}{G_t}} \tag{57}$$

with G_t, Σ_t as defined in (5) and (49), then the optimal trading strategy of the insider is indeed of the form given in equation (45).

To establish this we first need a preliminary result which establishes that the conjectured equilibrium price process converges at maturity to the liquidation value v.

Lemma 4. *Suppose price dynamics are given by equations (56), (49),(57), and (5), then the price process P_t converges almost surely to v at time T.*

Proof 7. *The conjectured equilibrium price process is:*

$$dP_t = \frac{(v - P_t)}{G_t}\sigma_t^2 dt + \sqrt{\frac{\Sigma_t}{G_t}}\sigma_t dZ_t \tag{58}$$

$$d\Sigma_t = -\frac{\Sigma_t}{G_t}\sigma_t^2 dt. \tag{59}$$

It is straightforward to solve the ODE for Σ_t and obtain equation (13). Consider the process $X(t) = P_t - v$:

$$X(t) = e^{-\int_0^t \frac{\sigma_u^2}{G_u} du} X_0 + \int_0^t e^{-\int_s^t \frac{\sigma_u^2}{G_u} du}\sqrt{\frac{\Sigma_s}{G_s}}\sigma_s \, dZ_s \tag{60}$$

$$:= I_1(t) + I_2(t), \tag{61}$$

where the second line defines the integrals I_1, I_2. Equation (6) implies that

$$\frac{\overline{\sigma}^2}{\underline{\sigma}^2} \log(\frac{T}{T-t}) \geq \int_0^t \frac{\sigma_u^2}{G_u} du \geq \frac{\underline{\sigma}^2}{\overline{\sigma}^2} \log(\frac{T}{T-t})$$

It follows immediately from this inequality that

$$\lim_{t \to T} I_1(t) = 0 \ a.s. \tag{62}$$

Further, note that $I_2(t) = e^{- \int_0^t \frac{\sigma_u^2}{G_u} du} M_t$ where we define the Brownian martingale:

$$M_t = \int_0^t e^{\int_0^s \frac{\sigma_u^2}{G_u} du} \sqrt{\frac{\Sigma_s}{G_s}} \sigma_s \, dZ_s \tag{63}$$

Note that the quadratic variation of M_t is equal to:

$$<M>_t = \int_0^t e^{\int_0^s \frac{2\sigma_u^2}{G_u} du} \frac{\Sigma_s}{G_s} \sigma_s^2 \, ds \tag{64}$$

$$= \Sigma_0 (e^{\int_0^s \frac{\sigma_u^2}{G_u} du} - 1) \tag{65}$$

where we substituted Σ_s from equation (13) to obtain the second line.

Now, from Karatzas and Shreve (1991) theorem 4.6 p. 174, we know there exists a standard Brownian motion B_t such that the continuous martingale can be seen as a time changed Brownian motion, specifically $M_t = B_{<M>_t}$. Using the Strong law of large number for Brownian Motion, which states that $\lim_{\tau \to \infty} B_\tau / \tau = 0$ a.s. (see Karatzas and Shreve (1991) page 104), we obtain:

$$\lim_{t \to T} e^{-\int_0^t \frac{\sigma_u^2}{G_u} du} M_t = \lim_{t \to T} \frac{B_{<M>_t}}{1 + \frac{<M>_t}{\Sigma_0}} = \lim_{\tau \to \infty} \frac{B_\tau / \tau}{\frac{1}{\Sigma_0} + 1/\tau} = 0 \ a.s. \tag{66}$$

This establishes that $\lim_{t \to T} I_2(t) = 0$ a.s and completes the proof.

We now establish another useful result about the limiting distribution of the standardized price process.

Lemma 5. *The process $h_t = \frac{P_t - v}{\sqrt{\Sigma_t}}$ follows a time changed Ornstein-Uhlenbeck process with the property that h_T has a normal distribution with $E[h_T] = 0$ and $\mathrm{E}[h_T^2] = 1$.*

Proof 8. *Simple calculations show that*

$$dh_t = -\frac{1}{2}\frac{\sigma_t^2}{G_t}dt + \frac{\sigma_t}{\sqrt{G_t}}dZ_t \tag{67}$$

This is a time changed Ornstein-Uhlenbeck process with stochastic time change process $\tau_t = \int_0^t \frac{\sigma_s^2}{G_s}ds$ which is independent of the filtration generated by Z_t. Straightforward calculations show that $E[h_T] = 0$ and $E[h_T^2] = 1$ and that the limiting distribution of h_T is a standard normal.

The last intermediate result we establish is that market depth is a martingale.

Lemma 6. *Market depth (which is the inverse of the price impact, i.e, Kyle's lambda) is a martingale that is orthogonal to the aggregate order flow. It follows that price impact (Kyle's lambda) is a submartingale.*

Proof 9. *Note that from its definition the G_t process satisfies:*

$$d\sqrt{G_t} + \frac{\sigma_t^2}{2\sqrt{G_t}}dt = d\mathcal{M}_t, \tag{68}$$

where $\mathcal{M}_t = \mathrm{E}[\int_0^T \frac{\sigma_t^2}{2\sqrt{G_t}}dt \mid \sigma^t]$ is a bounded martingale (adapted to the filtration generated by the noise-trader volatility process) by the law of iterated expectation. Note that from equation (6) it is straightforward to show that $\mathcal{M}_t \leq \frac{\bar{\sigma}^2}{\underline{\sigma}}\sqrt{T} \quad \forall t$.

It follows, by definition of the process σ_t, that $d\mathcal{M}_t dZ_t = 0$.

From its definition in (57) and the definition for Σ_t and G_t above we obtain:

$$d\frac{1}{\lambda(t)} = \frac{1}{\sqrt{\Sigma_t}}d\sqrt{G_t} - \frac{\sqrt{G_t}}{2(\Sigma_t)^{3/2}}d\Sigma_t \tag{69}$$

$$= \frac{1}{\sqrt{\Sigma_t}}d\mathcal{M}_t. \tag{70}$$

It also follows that $d\frac{1}{\lambda_t}dY_t = 0$.

To prove that λ is a submartingale we apply Jensen's inequality. We have: $\frac{1}{\lambda_t} = E_t[\frac{1}{\lambda_s}] \geq \frac{1}{E_t[\lambda_s]}$. It follows that $\lambda_t \leq E_t[\lambda_s]$.

We now can prove the main result for this step, namely a verification proof of optimality of the insider's trading strategy (12). Recall that the insider is optimizing the following value function:

$$J(t) = \max_{\{\theta_s\}_{s \geq t} \in \mathcal{A}} E\left[\int_t^T (v - P_s)\theta_s ds \,|\mathcal{F}_t^Y, v\right], \tag{71}$$

where the set of admissible strategies $\mathcal{A}$ is defined as the set of processes θ_t such that $E[\int_0^T |\theta_s|^2 ds] < \infty$.

Lemma 7. *Suppose price dynamics are given by equations (56), (49),(57), and (5), and that assumption $\mathcal{B}$ holds, then the optimal value function is given by:*

$$J(t) = \frac{(v - P_t)^2 + \Sigma_t}{2\lambda_t}, \tag{72}$$

and the optimal strategy is given by:

$$\theta_t^* = \frac{1}{\lambda_t}\frac{\sigma_t^2}{G_t}(v - P_t). \tag{73}$$

Proof 10. *Apply Itô's rule to the conjectured value function to get*

$$dJ(t) = \frac{(v - P_t)^2 + \Sigma_t}{2}d\frac{1}{\lambda_t} + \frac{1}{\lambda_t}\left(-(v - P_t)dP_t + \frac{1}{2}dP_t^2\right) - (v - P_t)dP_t\, d\frac{1}{\lambda_t} + \frac{1}{2\lambda_t}d\Sigma_t. \tag{74}$$

The insider takes the price impact process as given and assumes the price process follows:

$$dP_t = \lambda_t\left(\theta_t dt + \sigma_t dZ_t\right),$$

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with the λ process as in equation (57) above. Using lemma 6 and the Σ_t dynamics, and integrating the above we obtain:

$$J(T) - J(0) + \int_0^T (v - P_t)\theta_t dt = \int_0^T (P_t - v)\sigma_t dZ_t + \int_0^T \frac{(v - P_t)^2 + \Sigma_t}{2\sqrt{\Sigma_t}} d\mathcal{M}_t. \quad (75)$$

Now, since $J(T) \geq 0$ it follows by taking expectation (and using the fact that the stochastic integrals are martingales, as established in lemma 8 below), that

$$E\left[\int_0^T (v - P_t)\theta_t dt\right] \leq J(0) \quad (76)$$

for any admissible policy $\{\theta_t\}$. Further, if there exists a trading strategy θ_t consistent with the updating equations (48), such that $E[J(T)] = 0$ then the inequality holds with equality.

The candidate policy in equation (12) satisfies this.

*   Indeed note that*

$$J(T) = \frac{(v - P_T)^2 + \Sigma_T}{2\lambda_T} = \frac{(v - P_T)^2}{2\lambda_T} + \frac{\sqrt{\Sigma_T\, G_T}}{2} = \frac{(v - P_T)^2}{2\lambda_T}.$$

In turn:

$$E[\frac{(v - P_T)^2}{\lambda_T}] \leq \sqrt{E[(v - P_T)^2\, G_T]E[(\frac{v - P_T}{\Sigma_T})^2]} = 0$$

where the right hand side equality follows from lemma 4 and lemma 5.

*   We have therefore proved the optimality of the value function and of the proposed policy.*

Lemma 8. *Under assumption $\mathcal{B}$ the stochastic integrals $J_1(t) = \int_0^t (v - P_s)\sigma_s dZ_s$ and $J_2(t) = \int_0^t \frac{(v - P_s)^2 + \Sigma_s}{2\sqrt{\Sigma_s}} d\mathcal{M}_s$ are martingales for any admissible strategy.*

Proof 11. *To prove that $J_1(t)$ is a martingale it is sufficient to show that $E[\int_0^T (v - P_t)^2\sigma_t^2 dt] < \infty$. In turn becuse $\mathcal{B}$ holds it is sufficient to show that P_t has finite variance for all t. Note that $P_t = P_0 + \int_0^t \lambda_s\theta_s ds + \int_0^t \sigma_s\lambda_s dZ_s$. Thus for P_t to have finite variance*

it is sufficient that $\mathrm{E}[(\int_0^t \lambda_s \theta_s ds)^2] < \infty$ and $\mathrm{E}[\int_0^t \sigma_s^2 \lambda_s^2 ds] < \infty$. Clearly, $\mathrm{E}[\int_0^t \sigma_s^2 \lambda_s^2 ds] = \Sigma_0 - \Sigma_t < \infty$. Further, using Cauchy-Schwartz we have:

$$\mathrm{E}[(\int_0^t \lambda_s \theta_s ds)^2] \leq \mathrm{E}[\int_0^t \lambda_s^2 ds \int_0^t \theta_s^2 ds]$$

The right hand side is finite for any admissible trading strategy since $\int_0^t \lambda_s^2 ds \leq \frac{1}{\underline{\sigma}} \int_0^t \frac{\Sigma_s}{G_s} \sigma_s^2 ds = \frac{\Sigma_0 - \Sigma_t}{\underline{\sigma}} < \frac{\Sigma_0}{\underline{\sigma}}$.

Next to show that $J_2(t)$ is a martingale, since $\mathcal{M}_t$ is a bounded continuous martingale (from lemma 6) and Σ_t is a decreasing process, it is sufficient to show that $\int_0^T \frac{(v - P_t)^2}{\sqrt{\Sigma_t}} d\mathcal{M}_t$ is a martingale. To that effect we use integration by parts and write

$$\int_0^t \frac{(v - P_s)^2}{\sqrt{\Sigma_s}} d\mathcal{M}_s = \frac{(v - P_t)^2}{\sqrt{\Sigma_t}} \mathcal{M}_t - C - \int_0^t \mathcal{M}_s d\frac{(v - P_s)^2}{\sqrt{\Sigma_s}}$$

for some constant C. Thus for the stochastic integral to have finite variance it is sufficient that $\mathrm{E}[\frac{(v - P_t)^4}{\Sigma_t}] < \infty \ \forall t$ given that $\mathcal{M}_t < \frac{\bar{\sigma}^2 \sqrt{T}}{\underline{\sigma}^2} \ \forall t$ as shown in lemma 6. Since $\mathrm{E}[\frac{(v - P_t)^4}{\Sigma_t}] < \Sigma_0 \mathrm{E}[h_t^4]$, with h defined in lemma 5 we can easily verify (from the results obtained in lemma 5) that $\mathrm{E}[h_t^4] < \infty \ \forall t$.

6.4. Mean-reverting noise trading volatility

Here we consider an example where noise trader volatility follows a diffusion process with mean-reversion. Specifically, we consider the case where $x_t = \log \sigma_t$ follows a mean-reverting Ornstein-Uhlenbeck process:

$$dx_t = (-\frac{\nu^2}{2} - \kappa x_t)dt + \nu dW_t. \tag{77}$$

We parametrize the drift of x_t so that, when $\kappa = 0$, volatility is a martingale:

$$\frac{d\sigma_t}{\sigma_t} = -\kappa x_t dt + \nu dW_t. \tag{78}$$

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As a result, we can focus on the impact of mean-reversion alone, and use a series expansion in κ around the known solution when $\kappa = 0$ (derived in example 2). The following result characterizes the solution.

Theorem 4. *If the log of noise trading volatility follows a mean reverting process as given in equation (78), then the process $G(t)$ admits the solution:*

$$G(t) = \sigma_t^2 A(T - t, x_t, \kappa)^2, \tag{79}$$

where the function $A(\tau, x, \kappa)$ can be approximated by a series expansion:

$$A(\tau, x, \kappa) = \sqrt{T - t}\left(1 + \sum_{i=1}^{n}(-k\tau)^i\left(\sum_{j=0}^{i} x^j \sum_{k=0}^{i-j} c_{ijk} t^k\right) + O(\kappa^{n+1})\right), \tag{80}$$

where the c_{ijk} are positive constants that depend only on ν^2 and can be solved explicitly.[15] In that case, private information enters prices at a stochastic rate that depends on the level of noise trading volatility:

$$\frac{d\Sigma_t}{\Sigma_t} = -\frac{1}{A(T - t, x_t, \kappa)^2} dt. \tag{81}$$

Market depth is stochastic and given by:

$$\lambda_t = \frac{\sqrt{\Sigma_t}}{\sigma_t A(T - t, x_t, \kappa)}. \tag{82}$$

The trading strategy of the insider is:

$$\theta_t = \frac{\sigma_t}{\sqrt{\Sigma_t} A(T - t, x_t, \kappa)}(v - P_t). \tag{83}$$

Stock price dynamics follow a three factor (P, x, Σ) Markov process with stochastic

---

[15]We provide in the section below the fifth order solution. Higher order expansions can be obtained easily using Mathematica (program available upon request).

volatility given by:

$$dP_t = \frac{(v - P_t)}{A(T - t, x_t, \kappa)^2} dt + \frac{\sqrt{\Sigma_t}}{A(T - t, x_t, \kappa)} dZ_t. \tag{84}$$

In particular, stock price volatility is a stochastic and tends to be higher when noise trading volatility is higher. The unconditional expected profit at time zero of the insider is $T\sigma_v\sigma_0 \frac{A_0}{\sqrt{T}}$.

Proof 12. To prove this result, we observe that $m_t = -\kappa x_t$ and that x_t has following dynamics under the $\tilde{P}$ measure:

$$dx_t = (\frac{\nu^2}{2} - \kappa x_t)dt + \nu d\tilde{W}_t, \tag{85}$$

where by Girsanov's theorem we have defined $\tilde{W}_t = W_t - \nu^2 t$ a standard $\tilde{P}$-measure Brownian motion. Thus x_t is a one-factor Markov process under $\tilde{P}$. Using the Markov property for conditional expectations, we guess that the solution to equation (8) is given by a function $A(t, x_t)$.

As shown in the proof of lemma 4, this function satisfies:

$$\tilde{E}_t \left[dA(t, x_t)/dt + m_t A(t, x_t) + \frac{1}{2A(t, x_t)} \right] = 0. \tag{86}$$

Using Itô's lemma we obtain the following non-linear PDE for $A(T - t, x)$ (where we change variables to $\tau = T - t$ and drop the argument of the function for simplicity):

$$\frac{\nu^2}{2}(A_{xx} + A_x) + -\kappa x(A_x + A) - A_\tau + \frac{1}{2A} = 0 \tag{87}$$

subject to boundary conditions $A(0, x) = 0$. When $\kappa = 0$, the solution is simply $A(\tau, x; \kappa = 0) = \sqrt{\tau}$. Assuming the solution is analytic in its arguments, we seek an series expansion solution of the form given in equation (80) above. Plugging this guess into the left hand side of the PDE and Taylor expanding in κ, we find that each term

in the series expansion can be set to zero by an appropriate choice of the constants c_{ijk}.
We can thus recursively solve for these constants and obtain an approximate solution to
the PDE. In figures 9 in the appendix we plot the 0th, 1st, 2nd and 5th order expansion
solution for $\nu = 0.7$ $T = 1$, $\kappa = 0.25$ and for three values of $x_0 = \{-0.3; 0; +0.3\}$.

The first term in the series expansion of the $A(\tau, x, \kappa)$ function is instructive. Indeed, we find:

$$A(\tau, x, \kappa) = \sqrt{\tau}(1 - \frac{\kappa}{2}\tau(\frac{\nu^2\tau}{6} + x)) + O(\kappa^2). \tag{88}$$

This confirms that we need κ to be different from zero for uncertainty about future noise trading volatility to affect the trading strategy of the insider, and equilibrium prices. We see that for a given expected path of noise trading volatility (e.g., setting $x = 0$ where it is expected to stay constant), the higher the mean-reversion strength κ the lower the A function. This implies that mean-reversion tends to lower the profit of the insider for a given expected path of noise trading volatility (compare his profits to the case where $\kappa = 0$).

Further, we see that the function is decreasing in (log) noise-trading volatility if $\kappa > 0$ (we confirm this for higher order expansions). This implies that stock price volatility is stochastic and positively correlated with noise-trading volatility. Equilibrium prices follow a Bridge process with stochastic volatility that is Markovian in three state variables. Private information gets incorporated into prices faster the higher the level of noise trading volatility, as the insider trades more aggressively in these states. Note that, since the $A(\tau, \log\sigma, \kappa)$ function is decreasing and convex in volatility, the insider trades more aggressively than in the case where $\kappa = 0$ (where $A(t, \log\sigma)$ is independent of volatility). In these high volatility states, market depth also improves, but less than proportionally to volatility to account for the more aggressive insider trading.

The net effect is that the insider's strategy changes as a function of uncertainty about

future noise trading volatility, as the insider can benefit from timing market (liquidity) conditions in this context. In fact, the higher ν^2 the more aggressively does the insider choose to respond a change in noise trading volatility (as A is decreasing in ν^2).

6.5. Expansion solution

The fifth order expansion of the A function (with $v = \nu^2$).

$$
\begin{aligned}
A(\tau, x, \kappa) = \sqrt{t} \Bigg(& 1 - \kappa t \left(\frac{vt}{12} + \frac{x}{2} \right) + \kappa^2 t^2 \left(\frac{13v^2 t^2}{1440} + x \left(\frac{vt}{12} + \frac{1}{6} \right) + \frac{7vt}{96} + \frac{5x^2}{24} \right) \\
& - \kappa^3 t^3 \left(\frac{89v^3 t^3}{120960} + x \left(\frac{3v^2 t^2}{320} + \frac{323vt}{2880} + \frac{1}{24} \right) + \frac{11v^2 t^2}{640} + x^2 \left(\frac{59vt}{1440} + \frac{1}{6} \right) + \frac{3vt}{80} + \frac{x^3}{16} \right) \\
& + \kappa^4 t^4 \left(\frac{1237v^4 t^4}{29030400} + \frac{337v^3 t^3}{161280} + x^2 \left(\frac{71v^2 t^2}{16128} + \frac{2593vt}{34560} + \frac{59}{720} \right) + \frac{6827v^2 t^2}{387072} \right. \\
& \left. + x \left(\frac{17v^3 t^3}{24192} + \frac{2657v^2 t^2}{120960} + \frac{737vt}{8640} + \frac{1}{120} \right) + x^3 \left(\frac{vt}{80} + \frac{59}{720} \right) + \frac{31vt}{2160} + \frac{79x^4}{5760} \right) \\
& - \kappa^5 \left(t^5 \right) \left(\frac{6299v^5 t^5}{3832012800} + \frac{193v^4 t^4}{1244160} + \frac{51709v^3 t^3}{16588800} + x^3 \left(\frac{601v^2 t^2}{483840} + \frac{4673vt}{161280} + \frac{59}{960} \right) + \frac{18703v^2 t^2}{1451520} + \right. \\
& x^2 \left(\frac{4241v^3 t^3}{14515200} + \frac{7129v^2 t^2}{580608} + \frac{9127vt}{120960} + \frac{11}{360} \right) + x \left(\frac{287v^4 t^4}{8294400} + \frac{49439v^3 t^3}{21772800} + \frac{319777v^2 t^2}{11612160} + \frac{2293vt}{48384} + \frac{1}{720} \right) \\
& \left. + x^4 \left(\frac{431vt}{161280} + \frac{1}{40} \right) + \frac{vt}{224} + \frac{3x^5}{1280} \right) \Bigg) + O(\kappa^6)
\end{aligned}
$$

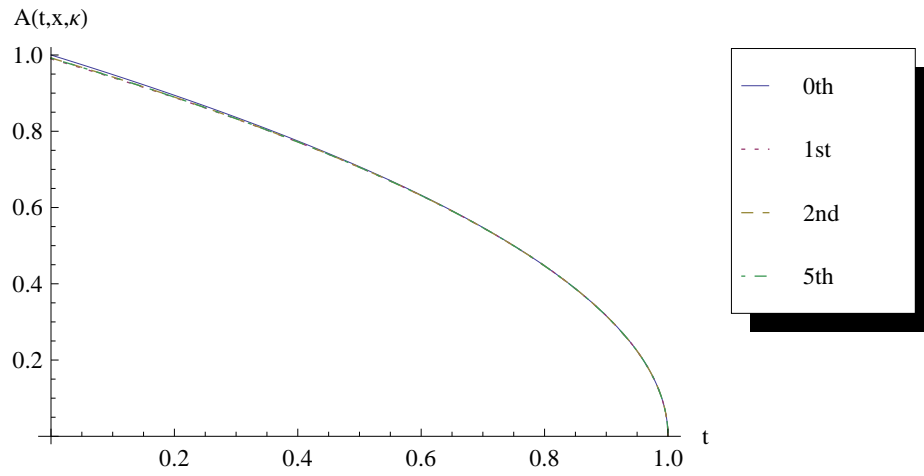We illustrate the convergence of the expansion in the following figures.

Figure 9: A function expansion solution given in equation (80) for different order (0,1,2,5) of the expansion for $x_0 = 0$. Other parameter values are $\kappa = 0.25, \nu = 0.7, T = 1$.
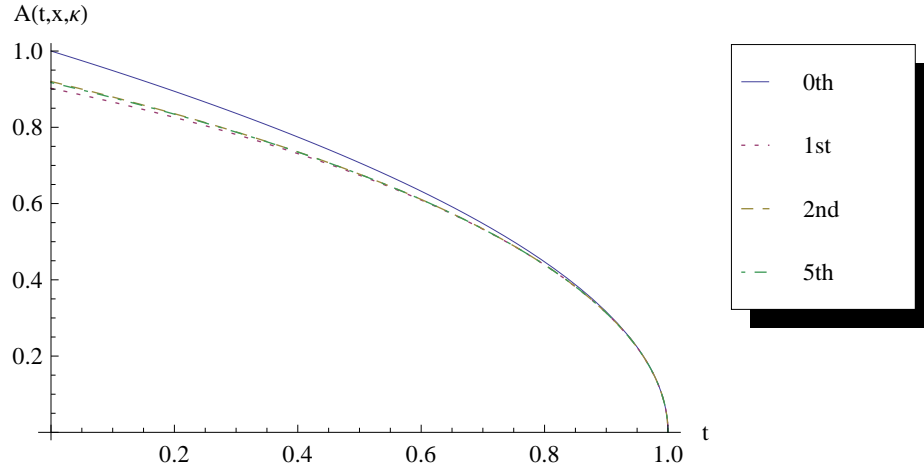
Figure 10: *A function expansion solution given in equation (80) for different order (0,1,2,5) of the expansion for $x_0 = +0.7$. Other parameter values are $\kappa = 0.25, \nu = 0.7, T = 1$.*
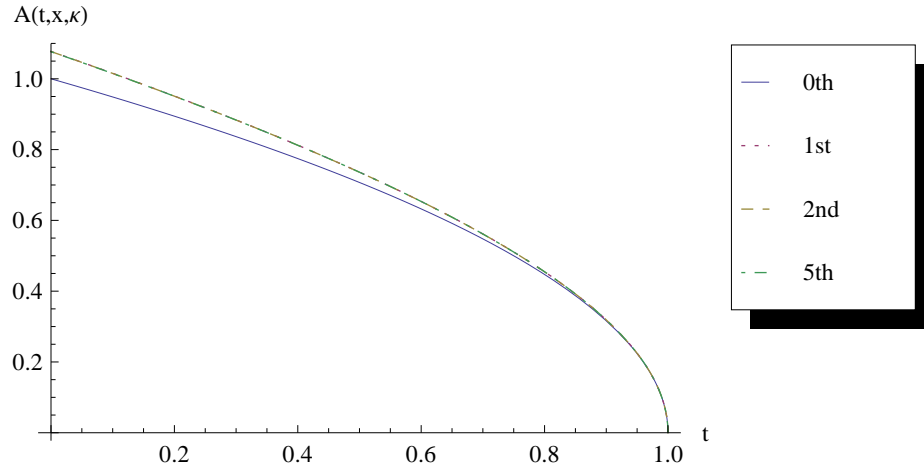

Figure 11: *A function expansion solution given in equation (80) for different order (0,1,2,5) of the expansion for $x_0 = -0.7$. Other parameter values are $\kappa = 0.25, \nu = 0.7, T = 1$.*

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