Equilibrium Collateral Constraints

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October 13, 2014

Abstract

I study a dynamic model of firms that need to raise funds to invest in risky projects whose return is private information. Firms with an investment opportunity in the future value the asset more than those without it since the asset allows them to increase the amount they can invest in productive projects. If the investment opportunities are persistent, borrowers today will value the asset more than lenders. This, together with the asymmetric information problem, implies that collateral contracts are optimal. The amount that can be borrowed against the assets is an equilibrium outcome.

1 Introduction

Collateralized debt is a widely used form of financing. Trillions of dollars are traded daily in debt collateralized by diverse financial assets, such as sale and repurchase agreements (repos) and collateralized over-the-counter derivative trades. Many financial institutions use collateralized debt to raise funds that allow them to provide intermediation services. Some of these institutions are private depository institutions, credit unions, mortgage real estate investment trusts, and security brokers and dealer. Most of these institutions are highly levered and they use the repo market as a source of financing.

The assets that are "sold" using repos are financial assets which could also be sold without a repurchase agreement in financial markets. These financial assets are not productive assets and, in principle, their intrinsic value is the same independently of the assets’ holder. Then, why do so many financial institutions choose to use these financial assets as collateral instead of selling them to raise funds?

If we assume that the borrower values the asset more than the lender, as is the case of a family heirloom, the borrower will be better off by pawning the asset than by selling it since part of the value the borrower assigns to it will not be internalized by the market. On the other hand, the lender will be happy to receive the

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†I am especially indebted to Ricardo Lagos and Tom Sargent for their advice in this project. I also thank Douglas Gale for very helpful comments and suggestions. I thank Saki Bigio, Alberto Bisin, V.V. Chari, Gian Luca Clementi, Eduardo Davila, Todd Keister, Emiliano Marambio Catan, Antoine Martin, Juan Pablo Nicolini, Michal Szkup and seminar participants at Wharton for their insightful remarks.

1See Gorton, G. and Metrick (2010a, 2010b) and Copeland, Martin, and Walker (2010).
asset as collateral since it will create incentives for the borrower to repay the debt, and in case of default the
lender keeps the collateral. However, in the case of financial assets, there is no ad-hoc reason why borrowers
would value the assets more than lenders since both borrowers and lenders can sell the assets in the same
market and receive the same dividends. Yet, we observe financial assets being collateralized.

In this paper I develop a model in which borrowers and lenders value the asset equally in autarky, but
they assign different values to it in equilibrium. As a result of this endogenous difference in valuations,
collateralized debt contracts implement the optimal funding contract. Since the contract is an equilibrium
outcome, I can also characterize the amount that can be borrowed against an asset (i.e., its debt capacity)
and its determinants.

The model is a discrete-time, infinite-horizon model. There are two types of risk neutral agents, borrowers
and lenders, and one durable asset which pays dividends each period. Borrowers can invest in risky projects
but they need external funds to do so. The return of the projects is private information of the agent who
invested in them. In order to raise funds, borrowers enter into a state contingent contract with lenders. In
equilibrium, firms value the asset more than lenders, and, therefore, choose to use collateral contracts.

In my model, borrowers have the investment opportunity before the asset’s dividends are paid and, thus,
cannot invest without external financing.\textsuperscript{2} This timing implies a maturity mismatch for the borrower
between the need of funds to invest and the availability of the dividends. Lenders do not have access to
investment opportunities. Therefore, if left in autarky, both risk neutral borrowers and lenders would value
the asset by the expected discounted sum of dividends. However, once the agents are allowed to trade, the
equilibrium features endogenous differences in valuations and, thus, collateral contracts as optimal. The main
assumptions that lead to this result are the maturity mismatch between the time in which the dividends
are paid and the investment opportunity, the persistence in the role as borrowers and lenders, and the
asymmetric information about the borrower’s ability to repay.

Suppose that an agent in this economy will have an investment opportunity tomorrow. Holding the asset
tomorrow will allow the agent to invest in the project either by selling the asset or by pledging it as collateral.
Being able to raise funds against the asset in the funding market will solve the maturity mismatch problem
between the timing of the dividend realization and the investment opportunity. In turn, this implies that
agents with investment opportunities tomorrow will value the asset more than those without them. If the
role as borrowers and lenders is persistent, as it is the case in many collateralized debt markets, it follows
that agents who are borrowers today will value the asset tomorrow more than lenders will.\textsuperscript{3} This difference
in valuations, together with the asymmetric information about the borrower’s ability to repay, implies that
collateral contracts arise optimally in equilibrium.

The extra value borrowers assign to the asset on top of the expected discounted value of dividends can
be decomposed in two premia for the borrower: a liquidity premium and a collateral premium. The liquidity

\textsuperscript{2}Collateralized debt contract can remain optimal if savings are allowed in the model.

\textsuperscript{3}This persistence is consistent with the observation that, in many collateralized debt markets, different types of institutions
specialize in borrowing or lending. For example, in the repo market, money market funds are usually lenders whereas hedge
funds and specialty lenders are usually borrowers.
premium for the borrowers arises from them being able to sell the asset and use the funds to invest in the risky projects (from solving the maturity mismatch mentioned above). The collateral premium for the borrowers is the additional value they can obtain by using the asset as collateral instead of selling it.

By solving the model in closed form, I am able to characterize the asset’s debt capacity completely. I show that when all borrowers can invest in the same type of projects, i.e., all projects have the same correlation with the asset’s future dividends, an increase in this correlation increases the asset’s debt capacity. Given the asymmetric information problem, the borrower has incentives to lie when the return of the projects is high. Therefore, an asset that has a higher value when the borrower has incentives to lie makes it easier to motivate the borrower to tell the truth, and, hence, is better collateral. I also find that when the returns of the projects are positively correlated with future dividends, a mean preserving spread of the distribution of the projects’ returns decreases the asset’s debt capacity. Finally, when borrowers can invest in different types of projects, two regimes may arise in equilibrium: one in which all borrowers use the asset as collateral and one in which some borrowers use the asset as collateral and other borrowers choose to sell the asset to raise funds. In this case, the effect of changes in the correlation structure on the overall amount intermediated in the economy depends on the distribution of project types across borrowers and on the regime of the economy.

There is a large literature that analyzes collateral contracts. My paper is closest to Lacker (2001) and Rampini (2005). Lacker (2001) shows that collateralized debt is the optimal financing contract when the borrower values the collateral good more than the lender. He shows this in a two-good, two-period model with two agents, a risk-averse borrower and a risk-neutral lender that takes the difference in valuations as given. In a similar environment with non-pecuniary costs of default, Rampini (2005) studies how default varies with aggregate income when individual income is privately observed by the agents. The optimal risk sharing contract allows for default which, given the default penalties assumed, only occur when the realization of income is low. These default penalties, which are modeled as transfers of agent-specific goods only valued by the agent who is endowed with it, can be interpreted as collateral which is only valued by the original holder.

As in these two papers, the friction that gives rise to collateral contracts in equilibrium in my model is asymmetric information between the borrower and the lender. In particular, the amount of repayment good the borrower has when he has to repay the loan is private information of the borrower. In contrast, my model is in infinite horizon, all agents are risk neutral, and, most importantly, the difference in the marginal value of the collateral good for the lender and the borrower is an equilibrium outcome: in equilibrium, the borrower values the collateral good more than the lender and, therefore, collateral contracts are optimal.

As Barro (1976) shows, enforcement frictions can also give rise to collateral contracts. Kocherlakota (2001) analyzes optimal repayment contracts in the presence of collateral when there are enforcement frictions. Collateral contracts are optimal in Kocherlakota’s model, provided that lenders are assumed to be less willing than borrowers to substitute consumption for collateral goods. In his setup, collateral is used to force the borrower to share with the lender the returns on the project in which the borrower invested.

Following Kiyotaki and Moore (1997) there is a large literature that refers to these enforcement frictions
to analyze the effects of collateral constraints on aggregate fluctuations. For example, Jermann and Quadrini (2012) find that a tightening of the firms’ collateral constraints contributed significantly to the 2008 – 2009 recession and to the downturns in 1990 – 91 and 2001. In these papers, at the time of repayment, the lender can recover only a fraction of the collateral value. This fraction is stochastic and depends on (unspecified) market conditions. Moreover, durable assets are used not only as collateral for loans, but also as inputs for production (and, thus, selling them is not an option).

In the general equilibrium literature, Geanakoplos (2003a, 2003b), and Geanakoplos and Zame (2007) study the use of durable assets as collateral. They show that when collateral contracts are assumed, the set of traded assets will be determined endogenously. Araujo, Orrillo and Pascoa (1994) and Araujo, Fajardo and Pascoa (2005) assume collateral contracts and make collateral endogenous by allowing each seller of assets to fix the level of collateral or the bundle used as collateral. In all these papers, collateral contracts are assumed.

Finally, in a search model with bilateral trading, Monnet and Narajabad (2012) show that agents prefer to conduct repurchase agreements than asset sales when they face substantial uncertainty about the value of holding the asset in the future.

In contrast to the papers mentioned above, in the model presented in this paper, the asset used as collateral is not an input in production and it can be sold to raise funds. As explained above, given the persistence in the roles as borrowers and lenders, the asset is used as collateral as an optimal response to asymmetric information about the resources available to the borrower at the time of repayment. The amount that can be borrowed against the asset is determined in equilibrium, and, thus, changes in the collateral constraints faced by borrowers reflect changes in the economy’s fundamentals. By characterizing collateral constraints in equilibrium, my model provides a link between the amount that can be borrowed against the assets and the fundamentals in the economy beyond the value of the asset. It also allows me to identify collateral and liquidity premia which affect the asset’s value and its debt capacity.

The rest of the paper is organized as follows. Section 2 presents the model. Section 3 defines and characterizes equilibrium. Comparative statics results are presented in section 4. Section 5 extends the model in section 2, allowing for heterogeneity among borrowers. Section 6 concludes.

2 Model

Time is discrete, starts at \( t = 0 \), and goes on forever. Each period \( t \) is divided in two subperiods, morning and afternoon. There are two types of consumption, non-storable good, morning and afternoon specific. There is one asset that lasts forever and which is in fixed supply \( \tilde{k} \). \( k \) units of the asset yield \( d_t \) units of (afternoon) consumption good as dividend at the end of each afternoon \( t \). The dividend \( d_t \) is known at the beginning of morning \( t \). There are two types of infinitely-lived, risk-neutral agents, borrowers and lenders. There is a measure 1 of each type and all agents have the same discount factor \( \beta \in (0,1) \). Each agent starts his life with an endowment of the asset. Each subperiod lenders are endowed with a large amount of consumption
good, $e^m_t$ in the morning and $e^a_t$ in the afternoon. Each morning $t$ borrowers receive a large endowment of (morning) consumption good, and each afternoon $t$ firms can invest in short-term projects on which they can make profits. Projects are risky: one unit of (afternoon) consumption good invested in the risky projects at the beginning of afternoon $t$ by borrower $j$ yields a random payoff $\theta^j_t$ in (afternoon) consumption good at the end of the period. $\theta^j_t \in \{\theta_L, \theta_H\}$. The return of the projects is i.i.d. across borrowers and time.

There is an underlying unobservable i.i.d. aggregate state $\omega_t \in (\omega_0, \omega_1)$ that determines the probability of success of the risky projects.\footnote{\textsuperscript{4}I assume the law of large number, and thus the fraction of successful investments in loans will also depend on the state realized.} Let $p_i^n := \Pr \left( \theta^j_t = \theta_i | \omega_t = \omega_n \right)$, and $p_i := \Pr (\omega_t = \omega_1) p_i^1 + \Pr (\omega_t = \omega_2) p_i^2 = \Pr \left( \theta^j_t = \theta_i \right)$, $i = L, H, \forall j, \forall t \geq 0$. I assume that $E \left( \theta^j_t \right) > 1 > \theta_L, \forall j, \forall t \geq 0$, i.e., that the projects are profitable in expectation but that they incur losses if the low state is realized, and that $1 > \beta \frac{E(\theta) - \theta_L}{1 - \theta_L}$. The return of the projects is private information known by the firm who invested in them, and it is potentially correlated with the dividend paid by the asset the following period.\footnote{One can think of the asset as mortgage backed security or as an asset backed security backed by loans made by banks.} The joint distribution of returns in period $t$ and dividends in period $t + 1$ is stationary, i.e.,

$$E \left( d_{t+1} | \theta^j_t = \theta_i \right) = \sum_{n=1,2} \Pr \left( \omega_t = \omega_n | \theta^j_t = \theta_i \right) E \left( d_{t+1} | \omega_t = \omega_n \right) := d_i \text{ for } i = L, H, \forall j, \forall t \geq 0,$$

where

$$\Pr \left( \omega_t = \omega_n | \theta^j_t = \theta_i \right) = \frac{p_i^n \Pr (\omega = \omega_n)}{p_i}.$$  

The dividend is paid after the investment in the projects is made. Therefore, dividends cannot be invested by the borrower. This timing implies that firms need to borrow to make risky loans, and creates a maturity mismatch between the need of funds (liquidity) and the availability of funds.\footnote{\textsuperscript{5}The results still survive if borrowers were able to save in consumption good between the morning and afternoon.} Moreover, this implies that if agents were left in autarky, they would all value the asset as the discounted sum of expected dividends.

Each morning, after the dividend level $d_t$ is known, the asset market opens. Borrowers and lenders meet randomly. Trades made in this market are bilateral and the terms of the transaction are determined through Nash bargaining. At the beginning of each afternoon, the funding market opens: each borrower is randomly paired with a lender and there is a bilateral funding contract in which all bargaining power is given to the borrower.

The timing is illustrated in figure 1.

\section{2.1 Asset Market}

At the beginning of each morning, each borrower is randomly matched with a lender in the asset market. The terms of trade, quantity and price, are determined through Nash bargaining. The bargaining power of borrowers is equal to $\gamma \in (0,1)$\footnote{\textsuperscript{6}If $\gamma = 0$ both agents value the asset the same in equilibrium and the optimal debt contract is indeterminate.}. Since matches are random both in the asset market and in the loan market, the aggregate state of the economy is $\phi = (d, F^m_L, F^m_B)$, where $F^m_L$ and $F^m_B$ are the distribution of assets in the hands of lenders in the morning, and the distribution of assets in the hands of borrowers in
the morning, respectively. I assume that borrowers have enough consumption good each morning to buy all assets from the non-producer with whom they meet. By making this assumption, the model abstracts from borrowing constraints in the asset market. Since the main focus of the paper is to understand collateralized loan contracts, considering such borrowing constraints, though interesting and realistic, makes it harder to disentangle the forces at work without adding much to the analysis.

Let \( k^T (k_B, k_L; \phi) \), and \( P (k_B, k_L; \phi) \) be the quantity and price that result from the encounter of a borrower with \( k_B \) units of the asset and a lender with \( k_L \) units of the asset.

Then, the value of a borrower in the morning, before entering the asset market, is

\[
V^m_B (k_B; \phi) = \left( \int \left[ \bar{V}^a_B (k_B + k^T (k_B, k_L; \phi) ; \phi) - P (k_B, k_L; \phi) \right] dF^m_L (k_L) \right)
\]

(1)

where \( \bar{V}^a_B (k_B; \phi) = \int V^a_B (k_B, k_L; \phi) dF_a (k_L) \), and \( V^a_B (k_B, k_L; \phi) \) is the value of a borrower with assets \( k_B \) who enters a loan contract with a lender with assets \( k_L \) in the afternoon.

The value of a lender in the morning, before entering the asset market, is

\[
V^m_L (k_L; \phi) = \int \left( P (k_B, k_L; \phi) + d \left( k_L - k^T (k_B, k_L; \phi) \right) + \beta E_\phi' \left( V^m_L (k_L - k^T (k_B, k_L; \phi); \phi') \right) \right) dF^m_B (k_B)
\]

(2)

Therefore, since the terms of trade are determined by Nash bargaining, \( P (k_B, k_L; \phi) \) and \( k^T (k_B, k_L; \phi) \) solve

\[
\max_{P_0, k_0} \left( P_0 - d k_0 - \beta E_\phi' \left( V^m_L (k_L; \phi') - V^m_L (k_L - k_0; \phi') \right) \right)^{1-\gamma} \times \left( \bar{V}^a_B (k_B + k_0; \phi) - \bar{V}^a_B (k_B; \phi) - P_0 \right)^\gamma
\]

(3)

\[\text{2.2 Funding Market}\]

Every afternoon, a bilateral funding market opens. Each borrower is matched randomly with a lender and the terms of the loan contract are determined by the borrower.\(^8\) Loan contracts are one-subperiod contracts

\(^8\)The shape of the contract is the same if all bargaining power is assigned to the lender.
and they consist of a loan amount in terms of (afternoon) consumption good, \( q \), and contingent repayments in terms of (afternoon) consumption, \( r_i \), and in terms of assets, \( t_i, i = L, H \).

Given a dividend level \( d \), a contract \((q, r_L, r_H, t_L, t_H)\) is feasible at time \( t \) if

\[
0 \leq q \leq e_L^a \\
0 \leq r_i \leq dk_{Bt} + \theta_i q \quad i = L, H \\
0 \leq t_i \leq k_{Bt} \quad i = L, H
\]

where \( k_{Bt} \) is the amount of assets held by the borrower in afternoon \( t \).

Since \( \theta_i \) is only observed by the borrower, for contracts to be credible, they must be incentive compatible.

A contract \((q, r_L, r_H, t_L, t_H)\) is incentive compatible if

\[
- r_L + \beta E_{\phi'} (V_B^m (k_B - t_L; \phi') | \theta_t = \theta_H) \leq - r_H + \beta E_{\phi'} (V_B^m (k_B - t_H; \phi') | \theta_t = \theta_H)
\]

and whenever \( r_H \leq \theta_L (dk_B + q) \)

\[
- r_L + \beta E_{\phi'} (V_B^m (k_B - t_L; \phi') | \theta_t = \theta_L) \geq - r_H + \beta E_{\phi'} (V_B^m (k_B - t_H; \phi') | \theta_t = \theta_L)
\]

where \( V_B^m (k_B - t; \bar{\alpha}, \phi') = \int V_B^m (k_B - t, k_L; \bar{\alpha}, \phi') dF^m (k_L) \).

The first constraint states that, in the high state, it is always at least as good for the borrower to tell the truth and report that the high state has occurred than to lie and report a low realization of \( \theta \). The second constraint states the analogous for the low state with the restriction that \((5)\) is only active when lying in the low state is feasible, i.e., when there are enough resources in the low state to match the contingent repayment in terms of goods in the high state, \( r_H \). I assume that the amount of the loan in equilibrium is never restricted by the amount of consumption good owned by lenders, i.e., that \( q^* < e_L^a \). \(9\)

Let \( V_B^a (k_B, k_L; \phi) \) be the value of a borrower with assets \( k_B \) who is matched with a lender with assets \( k_L \) in the loan market. Then, \( V_B^a (k_B, k_L; \phi) = \)

\[
\sup_{q, r_L, r_H, t_L, t_H} E (\theta) q - p_H r_H - p_L r_L + dk_P \\
+ \beta p_H E_{\phi'} (V_B^m (k_B - t_H; \phi') | \theta_H) + \beta p_L E_{\phi'} (V_B^m (k_B - t_L; \phi') | \theta_L)
\]

s.t.

\[
(q, r_L, r_H, t_L, t_H) \text{ is feasible and IC}
\]

\[
\beta E_{\phi'} (V_L^m (k_L; \phi')) \leq -q + p_H r_H + \beta p_H E_{\phi'} (V_L^m (k_L + t_H; \phi') | \theta_H) \\
+ p_L r_L + \beta p_L E_{\phi'} (V_L^m (k_L + t_L; \phi') | \theta_L)
\]

\[
\phi' = H (\phi; \theta)
\]

\(9\) Under certain parametric assumptions, one sub-period contracts implement long-term contracts.

\(10\) If this was not the case, the incentive compatibility constraints would not bind, the size of the loan would be efficient and debt would be riskless.
The borrower chooses a feasible and incentive compatible contract to maximize his expected utility subject to the lender’s participation constraint (7) and given the perceived law of motion for the aggregate state \( \theta \). (7) states that the lender has to be at least as good participating in the contract as he would be if he didn’t participate in it. The borrower invests all the loan amount, \( q \), in the risky technology since \( E(\theta) > 1 \), and gets an expected return \( E(\theta)q \). He expects to repay \( p_H r_H + p_L r_L \) in terms of consumption good and he gets dividends \( dk_p \) from his stock of assets at the beginning of the afternoon. Finally, his expected continuation value in the morning depends on the contingent transfers of assets \( t_L \) and \( t_H \) that are part of the contract.

By inspecting the constraints in the problem above, one can considerable simplify the borrower’s problem. First, in any equilibrium, the participation constraint for lenders (7) will hold with equality. If it did not, the borrower could increase the loan amount and increase his expected utility without violating any of the additional constraints. Similarly, as is usual in this kind of problems, the incentive compatibility constraint will bind in the high state (4) will hold with equality. Finally, in order to maximize the size of the loan, the repayment in terms of goods in the low state, \( r_L \), will be the maximum possible, i.e., \( r_L = \theta L q + dk_B \).

Lemma 1 and 10 in the appendix formalize these arguments.

**Proposition 1** The borrower’s problem can be rewritten as

\[
V_B^m (k_B, k_L; \phi) = \sup_{(t_H, t_L) \in [0, k_B]}^2 \left( \frac{E(\theta) - \theta_L}{1 - \theta_L} \right) dk_B \nabla \left( \frac{E(\theta) - \theta_L}{1 - \theta_L} \right) \beta \left( p_H E_{\phi'} (V_L^m (k_L + t_H; \phi') | \theta_H) + p_L E_{\phi'} (V_L^m (k_L + t_L; \phi') | \theta_L) - E_{\phi'} (V_L^m (k_L; \phi')) \right) \\
+ \frac{E(\theta) - 1}{1 - \theta_L} \beta p_H \left( E_{\phi'} (V_B^m (k_L - t_H; \phi') | \theta_H) - E_{\phi'} (V_B^m (k_L - t_L; \phi') | \theta_H) \right) \\
+ \beta \left( p_L E_{\phi'} (V_B^m (k_L - t_L; \phi') | \theta_L) + p_H E_{\phi'} (V_B^m (k_L - t_H; \phi') | \theta_H) \right)
\]

subject to

\[
q^* = \frac{dk_B + \beta \left( p_H E_{\phi'} (V_L^m (k_L + t_H; \phi') | \theta_H) + p_L E_{\phi'} (V_L^m (k_L + t_L; \phi') | \theta_L) \right)}{1 - \theta_L} \\
+ \beta p_H \left( E_{\phi'} (V_B^m (k_L - t_H; \phi') - V_B^m (k_L - t_L; \phi') | \theta_H) \right) - \beta E_{\phi'} (V_L^m (k_L; \phi')) \right) \\
+ \beta \left( p_L E_{\phi'} (V_B^m (k_L - t_L; \phi') - V_B^m (k_L - t_H; \phi') | \theta_H) \right) - \beta E_{\phi'} (V_L^m (k_L; \phi')) \right)
\]

The proof follows from lemmas 3 and 10 in the appendix.

### 3 Equilibrium

**Definition 2** A recursive equilibrium in this economy is a pair of value functions for borrowers, in the morning and in the afternoon, \( V_B^m (k_B; \phi) \) and \( V_B^m (k_B, k_L; \phi) \), a value function for lenders \( V_L^m (k_L; \phi) \), price
and quantity functions in the bilateral asset market, \( P(k_B, k_L; \phi) \) and \( k^T(k_B, k_L; \phi) \), a loan contract

\[
(q(k_B, k_L; \phi), r_L(k_B, k_L; \phi), r_H(k_B, k_L; \phi), t_L(k_B, k_L; \phi), t_H(k_B, k_L; \phi))
\]

and a law of motion for \( \phi \), \( H(\phi, \omega) \), such that [1], [2], [3], and [6] are satisfied and the law of motion for \( \phi \) satisfies:

\[
F^{o+}_L(k, \omega_i) = p^i_H \left( \int F^{o+}_L(k - t_H(k_B, k; d, \phi), \omega_1) dF^o_B(k_B, \omega_1) \right) \\
+ p^i_L \left( \int F^{o+}_L(k - t_L(k_B, k; d, \phi), \omega_1) dF^o_B(k_B, \omega_1) \right)
\]

\[
F^o_B(k, \omega_{-1}) = \int F^o_B(k - k^T(k, k_L; d, \phi)) dF^o_B(k_L, \omega_{-1})
\]

\[
F^{o+}_B(k, \omega_i) = p^i_H \left( \int F^{o+}_B(k + t_H(k, k_L; d, \phi), \omega_{-1}) dF^o_L(k, \omega_{-1}) \right) \\
+ p^i_L \left( \int F^{o+}_B(k + t_L(k, k_L; d, \phi), \omega_{-1}) dF^o_L(k, \omega_{-1}) \right)
\]

\[
F^o_L(k, \omega_i) = \int F^{o+}_L(k + k^T(k_B, k; d, \phi), \omega_i) dF^{o+}_B(k_B, \omega_i)
\]

of distributions where \( \omega_{-1} \) is the realized aggregate state the period before.

### 3.1 Affine Equilibrium

I will focus on recursive affine equilibria, i.e., in equilibria in which the value functions are affine in asset holdings. Within this class of equilibria, there is a unique equilibrium. I will use the guess an verify method to show that such equilibrium exists and that it is unique\(^{11}\).

Suppose that

\[
V^a_B(k_B; \phi) = V^a_B(k_B, k_L; \phi) = c_B(d) k_B + a_B(\phi)
\]

\[
V^m_L(k_L; \phi) = c_L(d) k_L + a_L(\phi)
\]

where \( c_B(d) \) and \( c_L(d) \) are affine in \( d \).

Under these assumption, the terms of trade in the asset market are the solution to

\[
\max_{P_0, k_0} (P_0 - d k_0 - \beta c_L(\bar{d}) k_0)^{1-\gamma} \times (c_B(d) k_0 - P_0)^\gamma
\]

where \( \bar{d} = E_{d'}(d') \). Then,

\[
P_0 = (1 - \gamma) c_B(d) k_0 + \gamma (d k_0 + \beta c_L(\bar{d}) k_0)
\]

and \( k^T(k_B, k_L; \phi) =

\[
\arg \max_{k_0 \in [-k_F, k_F]} (c_B(d) - d - \beta c_L(\bar{d})) k_0
\]

\(^{11}\)The mapping that characterizes an equilibrium is not a contraction mapping. Therefore I cannot show that the equilibrium is unique within a more general class of value functions.
which implies \( k^T (k_B, k_L; \phi) \in \{-k_B, k_L\} \). As it is usually the case with linear value functions, the quantity traded through Nash bargaining maximizes surplus and thus trades are efficient. Since the borrower can wait until the afternoon and sell the asset in the loan market, by setting \( t_L = t_H = k_B \), he will never choose to sell in the morning. Selling in the afternoon, allows the borrower to invest the proceeds of the sale in the risky projects which has a higher expected return than consuming the goods in the morning. Therefore, the terms of trade in the asset market are independent of the asset stock with which the borrower (buyer) enters the market, \( k_B \), and \( k^T (k_B, k_L; \phi) = k_L \), for all \( k_B, k_L \), and \( \phi \).

Given this structure, the value functions for lenders and borrowers in the morning before entering the asset market are, respectively,

\[
V_L^m (k_L; \phi) = P (k_B, k_L; \phi) = \left[ (1 - \gamma) c_B (d) + \gamma (d + \beta c_L (\bar{d})) \right] k_L
\]

and

\[
V_B^m (k_B; \phi) = \gamma \left[ c_B (d) - (d + \beta c_L (\bar{d})) \right] \int k d F_L^m (k) + c_B (d) k_B.
\]

Thus, using the guessed functional form for the value functions gives

\[
c_L (d) = (1 - \gamma) c_B (d) + \gamma (d + \beta c_L (\bar{d}))
\]

and

\[
c_L (\bar{d}) = \frac{[(1 - \gamma) c_B (\bar{d}) + \gamma \bar{d}]}{1 - \gamma \beta}.
\]

Therefore,

\[
c_L (d) = \left[ (1 - \gamma) c_B (d) + \gamma \left( d + \beta \frac{(1 - \gamma) c_B (\bar{d}) + \gamma \bar{d}}{1 - \gamma \beta} \right) \right].
\]

Finally, the value of a borrower who enters the loan market with \( k_B \) units of asset and meets a lender with \( k_L \) units of assets when the state is \( \phi \) is, using proposition 5

\[
V_B^m (k_B, k_L; \phi) = \max_{t_H, t_L \in [0, k_B]} \left( E (\theta) - 1 \right) q^* + dk_B + \beta (p_H c_L (d_H) t_H + p_H c_L (d_L) t_L) + \beta p_H c_B (d_H) (k_B - t_H) + \beta p_L c_B (d_L) (k_B - t_L) + \beta E_{\phi'} \left( \gamma [c_B (d') - (d' + \beta c_L (\bar{d}))] \right) \int k d F_L^m (k)
\]

s.t. \( q^* = \frac{dk_B + \beta (p_H c_L (d_H) t_H + p_L c_L (d_L) t_L) - p_H \beta c_B (d_H) (t_H - t_L)}{1 - \theta_L} \) \hspace{1cm} (9)

\[
q^* \geq \max \{ \beta (p_H c_L (d_H) t_H + p_L c_L (d_L) t_L) + p_L \beta c_B (d_H) (t_H - t_L) , 0 \} \hspace{1cm} (10)
\]

\[
r_H = q^* - \beta p_H c_L (d_H) t_H - \beta p_L c_L (d_L) t_L + p_L \beta c_B (d_H) (t_H - t_L)
\]

\[
r_L = q^* - \beta p_H c_L (d_H) t_H - \beta p_L c_L (d_L) t_L + p_H \beta c_B (d_H) (t_H - t_L)
\]

LOM for \( F \)

Rewriting the problem in this way shows that the borrower will choose the same contract that a planner who is subject to the asymmetric information problem and put all weight on the borrower would choose. Therefore, the equilibrium contract is constrained Pareto efficient.
Given the affine specification of the utility functions, the solution to the borrower’s problem in the afternoon will be in corner solution. If the constraint \((q^*)\) is ignored, there are four possible solutions: \(t_L = 0 = t_H, t_L = 0 \text{ and } t_H = k_B, t_L = k_B \text{ and } t_H = 0, \) and \(t_L = k_B = t_H. \) If \(t_L \geq t_H\) then the constraints on \(q\) are satisfied. If \(t_L < t_H, \) \((q^*)\) might bind. In the appendix I show that \((q^*)\) can’t bind in equilibrium. Therefore, ignoring the constraints on \(q^*, \) and using the guessed functional form for \(V_P^c (k_P, k_{NP}; \phi), \) one can match coefficients and get

\[
c_B (d) = \frac{(E (\theta) - \theta_L)}{1 - \theta_L} \left( d + \beta \left( p_H c_L (d_H) \frac{\partial t_H}{\partial k_P} + p_L c_L (d_L) \frac{\partial t_L}{\partial k_P} \right) \right)
- \frac{(E (\theta) - 1)}{1 - \theta_L} p_H \beta c_B (d_H) \left( \frac{\partial t_H}{\partial k_P} - \frac{\partial t_L}{\partial k_P} \right)
+ \beta p_H c_B (d_H) \left( 1 - \frac{\partial t_H}{\partial k_P} \right) + \beta p_L c_B (d_L) \left( 1 - \frac{\partial t_L}{\partial k_P} \right)
\]

\[
a_B (\phi) = \beta E_{\phi'} \left( \gamma \left[ c_B (d') - (d' + \beta c_L (d)) \right] \int kdF_{NP}^m (k) \right).
\]

**Proposition 3** In the only affine equilibrium, \(t_L^* = k_B \text{ and } t_H^* = 0.\)

The proof of this proposition can be found in the appendix.

### 3.2 Implementation of the Optimal Loan Contract

Using the results from the previous section, the optimal loan contract \((q^*, r_L^*, r_H^*, t_L^*, t_H^*)\) is given by

\[
q^* (d) = \frac{(d + \beta p_L c_L (d_L) + \beta p_H c_B (d_H))}{1 - \theta_L} k_B
\]

\[
r_L^* (d) = \frac{(\theta_L q^* (d) + d)}{1 - \theta_L} k_B
\]
\[
r_H^* (d) = r_L (d) + \beta c_B (d_H) k_B
\]
\[
t_L^* = k_B, t_H^* = 0
\]

The optimal loan contract only depends on the aggregate state through the current dividend level \(d.\) Borrowers are collateral constrained: the maximum amount that the borrowers are able to borrow depends linearly on the amount of assets they have. The collateral constraint is an equilibrium outcome and it depends on how much the expected holder values the asset. With probability \(p_L\) the return of the projects is low and the borrower transfers all his asset holdings to the lender whose expected discounted value of one unit of asset is \(\beta c_L (d_L).\) With probability \(p_H\) the return of the projects is high and the borrower keeps all his assets which he values \(\beta c_B (d_H)\) per unit.

The optimal loan contract can be implemented using two different debt contracts simultaneously: riskless debt and collateralized debt. This implementation is not unique. Below, I describe the implementation with the maximum amount of riskless debt, which is the one that includes riskless debt at 0 interest rate.

The maximum amount that can be repaid independently of the realized state, risklessly, is \(r_L^* (d).\) Therefore, the amount of riskless debt in this implementation is \(r_L^* (d).\) The remaining part of the loan amount
q*(d) is repaid in consumption goods only if the return of the project is high, whereas if it is low, the lender receives an asset transfer from the borrower. I will refer to this fraction of the loan amount as collateralized debt.

Then, the amount of collateralized debt in this implementation is

\[ q_c := q(d) - r_L^*(d) = \beta (p_L c_L (d_L) + p_H c_B (d_H)) k_p \]

which is independent of the dividend level. The interest rate on this loan can be computed as

\[ \frac{r_H(d) - r_L(d)}{q_c k_b} - 1 = \frac{\beta c_F (d_H) - q_c}{q_c} = \frac{\beta p_L (c_B (d_H) - c_L (d_L))}{q_c} = \frac{p_L (c_B (d_H) - c_L (d_L))}{(p_L c_L (d_L) + p_H c_B (d_H))}. \]

The expected return on collateralized debt is

\[ \frac{\beta p_L c_L (d_L) t_L^* + p_H (r_H^* - r_L^*)}{q_c} - 1 = \frac{\beta p_L c_L (d_L) + p_H \beta c_B (d_H)}{\beta (p_L c_L (d_L) + p_H c_B (d_H))} - 1 = 0. \]

The maximum amount that can be borrowed against the asset, its debt capacity, is given by

\[ D = \beta (p_L c_L (d_L) + p_H c_B (d_H)). \]

The asset’s debt capacity depends on the value of collateral for lenders when there is default (insurance) and on the value of collateral for borrowers when there isn’t default (incentives). Ceteris paribus, a higher value of collateral for lenders increases the loan amount since they can recover more when there is default, while a higher value of collateral for borrowers decreases the producers’ incentives to lie and therefore allows them to borrow more.

3.3 Premia

The difference between the agents’ valuation of the asset and the fundamental value of the asset can be decomposed in several premia. The borrower values the asset more than the expected discounted dividend stream for two reasons. The first reason is that the asset serves as a liquidity transformation device, it allows the borrower to solve the maturity mismatch problem he faces. The extra value due to this function is captured by the private liquidity premium, which I define as the difference between how much the borrower would value the asset if he chose to sell it to get funds and the fundamental value of the asset. If the borrower chose to sell the asset in the afternoon, he would get \( d + \beta c_L (d) \) per unit of asset, which is the maximum amount the lender would be willing to pay for it. With this funds, the borrower would be able to invest in projects and he would get the return on equity \( \frac{E(\theta) - \theta L}{1 - \theta L} \) on them. Therefore, the borrower would value each unit of asset \( \frac{E(\theta) - \theta L}{1 - \theta L} (d + \beta c_L (d)) \), and the private liquidity premium is defined as
The second reason why the borrower values the asset more than the lender is that he expects to use it as collateral the following period. I define the private collateral premium as the extra value a borrower gets from using the asset as collateral instead of selling it to raise funds. From the characterization of the affine equilibrium, a borrower who chooses to use the asset as collateral values it \( c_B(d) \) per unit. Then, the private collateral premium is
\[
 c_B(d) - \frac{E(\theta) - \theta_L}{1 - \theta_L} (d + \beta c_L(d)) = \frac{E(\theta) - \theta_L}{1 - \theta_L} \beta p_H (c_B(d_H) - c_L(d_H)) .
\]
This premium depends on the difference in valuations for the borrower and the lender. If both agents value the asset the same, the private collateral premium is 0. In this case, the borrower would be indifferent between selling the asset and pledging it as collateral. When the borrower values the asset more than the lender, the private collateral premium is positive and the borrower chooses to use the asset as collateral.

Finally, the lender may value the asset more than the expected discounted sum of its dividends if he has some bargaining power in the asset market. By being able to sell the asset to agents that value the asset more than themselves, lenders can extract some of this extra value whenever their bargaining power is positive, i.e., \( \gamma < 1 \). This extra value is what I call a liquidity premium and it is defined as
\[
 c_L(d) - \left( d + \beta \frac{\bar{d}}{1 - \beta} \right) = (1 - \gamma) \frac{E(\theta) - \theta_L}{1 - \theta_L} \left( \beta p_H (c_B(d_H) - c_L(d_H)) + \left( \gamma + \frac{E(\theta) - \theta_L}{1 - \theta_L} (1 - \gamma) \right) (\bar{d} - d) + \left( d + \beta \frac{\bar{d}}{1 - \beta} \right) \right) .
\]
As I mentioned above, this premium is 0 when the borrowers have all bargaining power in the asset market.

4 Comparative Statics

4.1 Different correlation structures

The debt capacity of the asset depends on the correlation between future dividends and the success of the investment made by borrowers. In this subsection, I will show that, keeping the (unconditional) expected return of the asset, \( \bar{d} \), fixed, an increase in the correlation between the success of the investment made by borrowers and the future dividends paid by the asset increases the asset’s debt capacity.
\[
\frac{\partial D}{\partial d_L|_{\hat{d}}} = \beta \frac{\partial (p_H c_B (d_H) + p_L c_L (d_L))}{\partial d_L|_{\hat{d}}} \\
\propto -\beta p_L \left( \frac{(E(\theta) - \theta_L)}{1 - \theta_L} - \frac{\partial c_L (d_L)}{\partial d_L} \right) \\
\propto -\beta p_L \left( \frac{(1 - \gamma \beta) \gamma \left( \frac{E(\theta) - \theta_L}{1 - \theta_L} \right)}{1 - \gamma \beta - \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \beta (1 - \gamma (\beta p_H + p_L))} \right)
\]

Therefore,
\[
\text{sign} \left( \frac{\partial D}{\partial d_L|_{\hat{d}}} \right) = -\text{sign} \left( 1 - \gamma \beta - \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \beta (1 - \gamma (\beta p_H + p_L)) \right).
\]

Let \( f(\gamma) := 1 - \gamma \beta - \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \beta (1 - \gamma (\beta p_H + p_L)) \). Since by assumption \( \left( 1 - \beta \frac{E(\theta) - \theta_L}{1 - \theta_L} \right) > 0, f(\gamma) > 0 \) \( \forall \gamma \in [0,1] \) and
\[
\frac{\partial D}{\partial d_L|_{\hat{d}}} < 0 \iff \frac{\partial D}{\partial d_H|_{\hat{d}}} > 0
\]

This result implies that a higher \( d_H \), increases the asset’s debt capacity and it makes it better collateral. The asset is worth more to the borrower when he has more incentives to default, in the event of a successful investment, and therefore decreases his incentives to lie.

### 4.2 Change in risk

Without loss of generality, \( \theta_L \) can be set to 0. Then, the variance of the project is given by
\[
V(\theta) = p_H (\theta_H - p_H \theta_H)^2 + p_L (p_H \theta_H)^2
\]
\[
\quad = p_H (1 - p_H) \theta_H^2
\]
\[
\quad = E(\theta) \theta_H - E(\theta)^2
\]

Therefore, one can get a mean preserving spread by increasing \( \theta_H \) and setting
\[
p_H = \frac{E(\theta)}{\theta_H}.
\]

**Proposition 4** If \( d_H \geq d_{\text{min}} \), where \( d_{\text{min}} < \hat{d} \), an increase in risk decreases the debt capacity of the asset, i.e.
\[
\frac{\partial D}{\partial p_H|_{E(\theta)}} > 0.
\]

When the return of the projects and the future dividend level are sufficiently positively correlated, an increase in the riskiness of the projects decreases the debt capacity of the asset. In this setup, an increase in risk decreases the probability of success of the project and, therefore, increases the probability of default. This shift in probabilities has a direct and an indirect effect on the debt capacity of the asset. The direct effect is that now the debt capacity of the asset puts more weight on the lender’s value of collateral in the event of default at the cost of decreasing the weight on borrower’s value of collateral in the event of no default. The sign of this direct effect depends on whether the lender values the asset more in the default
state, than does the borrower when he gets to keep it. The indirect effect comes from the change in the asset’s valuation for the lender. An increase in the probability of default decreases the lender’s valuation of the asset. When $d_H - d_L \geq 0$, both effects are negative. If $d_H - d_L < 0$, the net effect then depends on which of these effects is larger. When $d_H > d_{\min}$, the increase in the default probability is the dominating effect.

5 Multiple Project Types

In this section I extend the benchmark model and introduce heterogeneity among borrowers. Each borrower is characterized by the returns of the projects in which he is able to invest. Projects available to different borrowers differ in their correlation with the dividends paid by the asset but they share the same success probability and unconditional expected return. There are $J$ types of loans and a fraction $\mu_j$ of borrowers can invest in projects of type $j$, $j = 1, ..., J$.

In this case, in an affine equilibrium, the marginal value of assets for a lenders is $c_L (d) = (1 - \gamma) \sum_{i \in I} \mu_i c_B (d) + \gamma (d + \beta c_N p (\bar{d}))$. The borrowers’ problem remains unchanged, though now the asset value for lenders depends on the average valuation among borrowers. Depending on the parameters of the model, two different kinds of regimes might arise. In one, all borrowers choose to use the asset as collateral. This is clearly the case when $\gamma = 1$ since the multiple project type model and the benchmark model give the same contract for each agent. The existence of other types of borrowers only matters through the resale value of the asset in the asset market. If the sellers don’t get any surplus from this sale, then the price of the asset in the asset market will be equal to the expected discounted value of dividend and, thus, it would be independent of the distribution of borrower types in the economy.

If $\gamma < 1$ there might be borrowers who choose to sell the asset at the beginning of the afternoon and invest those funds rather than pledging it as collateral. Since the expected price at which lenders can sell the asset in the asset market depends on the borrowers’ average valuation of the asset, it may be the case that this average valuation is high enough to motivate some of the borrowers who value the asset the least to sell the asset to raise funds. Whether there are some borrowers that don’t use the asset as collateral or not depends on the value of $\gamma$. For high values of $\gamma$ everybody uses the asset as collateral in the only symmetric equilibrium. For lower values of $\gamma$ some borrowers may choose to sell the asset instead of using it as collateral.

**Remark 5** When $\gamma = 1$ or the asset’s dividends are uncorrelated with the projects made by borrowers (e.g. risk free assets) all borrowers choose to use the asset as collateral.

In both cases, the asset will always be used as collateral by at least one type of borrower.

**Proposition 6** The borrowers with the highest marginal valuation of the asset will always use it as collateral.

The proof of this proposition can be found in the appendix. Borrowers whose projects have the highest positive correlation with the dividend paid by the asset value the asset the most. The higher this correlation,
the larger the amount that can be borrowed against the asset since it is better at providing incentives to solve the asymmetric information problem.

5.1 Changes in the correlation structure

In this subsection I present an example to illustrate how aggregate quantities respond to changes in the correlation structure and risk when there are multiple types of projects in the economy and all borrowers choose to use the asset as collateral.

There are \( J = 2 \) types of borrowers. There is a population \( \mu_j \) of type \( j \) borrowers, \( j = 1, 2 \). Let \( \theta^j \) be the return of the projects in which a borrower of type \( j \) can invest. The unconditional probability of success is equal for both types of projects but conditional on the aggregate state, the success probabilities are given by

\[
\begin{align*}
\Pr (\theta^1 = \theta_H | \omega_1) &= \pi, \\
\Pr (\theta^2 = \theta_H | \omega_1) &= (1 - \pi), \\
\Pr (\theta^1 = \theta_H | \omega_2) &= (1 - \pi), \\
\Pr (\theta^2 = \theta_H | \omega_2) &= \pi,
\end{align*}
\]

where \( \Pr (\omega_1) = \Pr (\omega_2) = 0.5 \). Let \( d_i = E (d_{t+1} | \omega_i), i = 1, 2, \) where \( d_1 > d_2 \).

Then, the expected value of the dividend conditional on the return of project \( j \) is

\[
\begin{align*}
E (d_{t+1} | \theta^1 = \theta_H) &= \frac{\pi d_1 + (1 - \pi) d_2}{2p_H}, \\
E (d_{t+1} | \theta^2 = \theta_H) &= \frac{(1 - \pi) d_1 + \pi d_2}{2p_H}.
\end{align*}
\]

Given this correlation structure, an increase in \( \pi \), increases (decreases) the correlation between the return of type 1 (type 2) projects and the future dividend level while keeping the unconditional probability of success, \( p_H \), and the unconditional expected dividend, \( \bar{d} \), constant.

Let \( \bar{D} = \mu_1 D^1 + \mu_2 D^2 \) be the average debt capacity of the asset. This is the debt capacity a representative borrower would have in this economy.

**Proposition 7** An increase in \( \pi \) has the following effects.

- If \( \mu_1 = \mu_2 \), the economy’s average debt capacity remains unchanged, type 1 borrowers collateral constraint is relaxed and type 2 borrowers collateral constraint is tightened.

- if \( \mu_1 > \mu_2 \), the economy’s average debt capacity increases, and type 1 borrowers collateral constraint is relaxed.

- if \( \mu_1 < \mu_2 \), the economy’s average debt capacity decreases, and type 2 borrowers collateral constraint is tightened.
This proposition states that, when there are multiple types of projects, the effect of changes in the correlation structure on how much can be borrowed in the economy depends on the distribution of project types among borrowers. When there are multiple projects, an increase in the correlation between the return of the project in which a borrower invests and the future dividend has two effects. On the one hand, an increase in this correlation makes the asset more valuable for the borrower making it better collateral. On the other hand, this change in correlation decreases the value of the asset for other types of borrowers and thus affects the resale value of the asset. The net effect depends on the distribution of types of projects across borrowers.

6 Conclusion

In this paper I showed that when the roles as borrowers and lenders are persistent, and there is ex-post asymmetric information about the borrower’s ability to repay, collateralized debt is the optimal way for borrowers to raise funds. Borrowers would rather offer their asset as collateral than sell it since they value it more than lenders. If they sold it, they would get at most the valuation of lenders, whereas by offering it as collateral they keep it when there is no default.

This difference in marginal valuations of the asset between borrowers and lenders is an equilibrium outcome. In autarky, both borrowers and lenders value the asset as the expected discounted sum of the dividend stream. When the agents are able to trade, the borrower values the asset more than the lender and they both value the asset (weakly) more than its fundamental value. The borrowers’ excess valuation can be divided into two premia: a private liquidity premium and a private collateral premium. The first comes from the asset solving a maturity mismatch for the borrower: the asset pays dividends in the future but the borrower has an investment opportunity today. Being able to sell the asset provides the borrower with funds in the moment he needs them. The private collateral premium is the extra value the borrower assigns to the asset due to its role as collateral. Since the borrower will use collateral contracts in the future, the asset relaxes a borrowing constraint and thus has more value today.

Since collateral contracts are optimal in this setup, and the marginal valuations of both borrowers and lenders are endogenous, the maximum amount that can be borrowed against the asset, its debt capacity, is also an equilibrium outcome. I am able to solve the model in closed form which, in turn, allows me to compute some comparative statics with respect to the correlation structure, and the riskiness of the projects in which borrowers invest. I find that a higher correlation between the success of the projects and future dividends makes the asset better collateral since it is better at motivating the borrower to tell the truth when he has incentives to lie. If the return of the risky projects and the asset’s dividends are sufficiently positively correlated, an increase in the riskiness of the projects decreases the asset’s debt capacity.

I finally extend the model to include heterogeneity in return of the projects among borrowers. I show that changes in the correlation between the asset used as collateral and the return of the projects made by borrowers have different effects on the asset’s debt capacity depending on the distribution of types of
borrowers in the economy.

We know from previous literature that changes in financial constraints played an important role in recent crises. Having a model that characterizes these constraints as equilibrium outcomes is important both from a positive and a normative point of view. In positive terms, it is interesting to see where the financial shocks come from and how they interact with the fundamentals of the economy. From the normative side, policies that aim at stabilizing the cycle and preventing financial crises should take into account what drives changes in the financing conditions faced by financial intermediaries, firms, and households. This paper is an attempt to deliver some of the insights needed to understand collateralized debt markets better.

7 References


\[12\] Jermann and Quadrini (2009), Bianchi (2011), Perri and Quadrini (2011).


Li, Y, and Y. Li. 2012."Liquidity, Asset Prices, and Credit Constraints", mimeo.


8 Appendix

8.1 Borrower’s Problem

Lemma 8 Without loss of generality, (7) can be replaced by

\[ \beta E' \left( V_L^m (k_L; \phi') \right) = -g + \beta_p H E' \left( V_L^m (k_L + t_H; \phi') | \theta_H \right) + \beta_p L E' \left( V_L^m (k_s + t_L; \phi') | \theta_L \right). \]

in the borrower’s problem.

Proof. Let \( V^* \) be the solution to the borrower’s problem. Let \( \{V_{j}\}_{j \geq 0} \) be such that \( \lim_{j \to \infty} V_j = V^* \), where

\[ V_j = E(\theta) q_j - \beta_p H r_{H_j} - \beta_p L r_{L_j} + d_{k_B} \]

\[ + \beta_p H E' \left( V_B^m (k_B - t_{H_j}; \phi') | \theta_H \right) + \beta_p L E' \left( V_B^m (k_B - t_{L_j}; \phi') | \theta_L \right) \]

for some feasible and incentive compatible \( \{q_j, r_{H_j}, r_{H_j}, t_{L_j}, t_{H_j}\} \) that satisfies the participation constraint (7). Suppose that for some \( j \geq 0 \), \( \{q_j, r_{L_j}, r_{H_j}, t_{L_j}, t_{H_j}\} \) is such that (7) is slack. Then, one could increase \( q_j \) and increase \( V_j \) to \( V_j^0 \) still satisfying all the other constraints. Let \( \{V_{j}'\}_{j \geq 0} \) be a sequence identical to \( \{V_{j}\}_{j \geq 0} \) if at \( (q_j, r_{L_j}, r_{H_j}, t_{L_j}, t_{H_j}) \) holds with equality and \( V_j^0 \) otherwise. Then, by construction, \( V_j' \geq V_j \) and therefore,

\[ \lim_{j \to \infty} V_j' \geq \lim_{j \to \infty} V_j = V^*. \]

Therefore, we can replace (7) by (12) in the borrower’s problem. ■

Lemma 9 Without loss of generality, the incentive compatibility constraints can be replaced by

\[ -r_L + \beta E' \left( V_B^m (k_B - t_L; \phi') | \theta_L \right) = -r_H + \beta E' \left( V_B^m (k_B - t_H; \phi') | \theta_H \right). \]

in the borrower’s problem.

Proof. Let \( V^* \) be the solution to the borrower’s problem. Let \( \{V_{j}\}_{j \geq 0} \) be such that \( \lim_{j \to \infty} V_j = V^* \), where

\[ V_j = E(\theta) q_j - \beta_p H r_{H_j} - \beta_p L r_{L_j} + d_{k_B} \]

\[ + \beta_p H E' \left( V_B^m (k_B - t_{H_j}; \phi') | \theta_H \right) + \beta_p L E' \left( V_B^m (k_B - t_{L_j}; \phi') | \theta_L \right) \]

for some feasible and incentive compatible \( \{q_j, r_{L_j}, r_{H_j}, t_{L_j}, t_{H_j}\} \) that satisfies the participation constraint (12). Suppose that for some \( s \geq 0 \), no incentive compatibility constraint binds. Then, there exists \( \varepsilon_s > 0 \) such that

\[ \beta E' \left( V_B^m (k_B - t_{H_s}; \phi') - V_B^m (k_B - t_{L_s}; \phi') | \theta_L \right) \leq r_{H_s} - r_{L_s} + (\theta_H - \theta_L) \varepsilon_s \]

\[ r_{H_s} - r_{L_s} + (\theta_H - \theta_L) \varepsilon_s \leq \beta E' \left( V_B^m (k_B - t_{H_s}; \phi') - V_B^m (k_B - t_{L_s}; \phi') | \theta_H \right). \]

Replace \( \{q_s, r_{L_s}, r_{H_s}, t_{L_s}, t_{H_s}\} \) by \( \{q_s + \varepsilon_s + \varepsilon_0, r_{L_s} + \theta_L \varepsilon_s, r_{H_s} + \theta_H \varepsilon_s, t_{L_s}, t_{H_s}\} \) where \( \varepsilon_0 > 0 \) is such that the participation constraint binds. This contract still satisfies all the constraints, but it attains a value \( V_s^0 > V_s \).
If \( r_{Hs} > r_{Ls} \), \([4]\) is the only relevant incentive compatibility constraint. For all \( s \) such that \( r_{Hs} > r_{Ls} \) and \([4]\) is not binding, the previous argument applies and a value \( V^0_s > V_s \) can be attained.

Now consider those \( s \geq 0 \) such that \( r_{Hs} \leq r_{Ls} < \theta_L q_s + dk_P \). If \([5]\) binds, one could keep \( r_{Hs} - r_{Ls} \) constant by increasing both \( r_{Ls} \) and \( r_{Hs} \) and by increasing \( q_s \) to keep the participation constraint binding which would result in an increase in the objective function. Let this new value be \( V^0_s \). If for \( s \geq 0 \), \( r_{Hs} \leq r_{Ls} = \theta_L q_s + dk_P \), \([5]\) doesn’t bind unless \([4]\) binds. Suppose \([5]\) binds and \([4]\) doesn’t. Then, one could increase \( r_{Hs} \) still satisfying incentive compatibility and relaxing the participation constraint. Therefore, one could increase \( q_s \) which would increase the objective function and give a value \( V^0_s \geq 0 \). Therefore, once can construct a new sequence \( \{V^j_s\}_{j \geq 0} \), \( V^j_s \geq V_j \) such that \( V^j_s = V_j \) is the incentive compatibility constraint in the high state binds and \( V^j_s = V^0_s \) if it doesn’t. By construction,

\[
\lim_{j \to \infty} V^j_s = \lim_{j \to \infty} V_s = V^*.
\]

Therefore, without loss of generality one can concentrate on those sequences that are feasible in which \([4]\) holds with equality, i.e.,

\[
r_{Hj} - r_{Lj} = \beta E_{H} (\bar{V}_B^m (k_B - t_{Hj}; \phi') - \bar{V}_B^m (k_B - t_{Lj}; \phi') | \theta_H) \quad \text{for all } j.
\]

**Lemma 10** Without loss of generality, the feasibility constraints on contingent transfers in consumption good can be replaced by

\[
r_L = \theta_L q + dk_B \quad \text{and} \quad r_H \geq 0
\]

in the borrower’s problem.

**Proof.** By assumption \( q \leq e_L^* \) will not bind in a solution to the borrower’s problem. Using lemma \([8]\) the participation constraint can be assumed to hold with equality, and using this in the objective function one can see that the objective function is always increasing in the amount of the loan \( q \). Using lemma \([9]\) the incentive compatibility constraint holds with equality which implies that the upper bound for \( q \) is given by the maximum amount that can be repaid in the low state, i.e., by \( r_L = \theta_L q + dk_P \).

Let \( V^* \) be a solution to the borrower’s problem. Let \( \{V_j\} \) be a sequence such that \( \lim_{j \to \infty} V_j = V^* \) and

\[
V_j = (E (\theta) - 1) q_j + \beta p_H E_{\phi'} (V_L^m (k_L + t_{Hj}; \phi') | \theta^*_L = \theta_H) + \beta p_L E_{\phi'} (V_L^m (k_L + t_{Lj}; \phi') | \theta^*_L = \theta_L) + dk_P
\]

\[
+ \beta p_H E_{\phi'} (V_B^m (k_B - t_{Hj}; \phi') | \theta^*_L = \theta_H) + \beta p_L E_{\phi'} (V_B^m (k_B - t_{Lj}; \phi') | \theta^*_L = \theta_L) - \beta E_{\phi'} (V_L^m (k_{NP}; \phi'))
\]

for some feasible and incentive compatible that satisfies \([15]\), and \([7]\) with equality, that is that,

\[
r_{Lj} = q^* - \beta (p_H E_{\phi'} (V_L^m (k_L + t_{Hj}; \phi') | \theta_H)) + p_L E_{\phi'} (V_L^m (k_L + t_{Lj}; \phi') | \theta_L))
\]

\[
r_{Hj} = q^* - \beta (p_H E_{\phi'} (V_L^m (k_L + t_{Hj}; \phi') | \theta_H)) + p_L E_{\phi'} (V_L^m (k_L + t_{Lj}; \phi') | \theta_L))
\]
Then, for all \( j \), the contract can be summarized by \( \{q_j, t_{Lj}, t_{Hj}\} \). The feasibility constraints for \( r_L \) and \( r_H \) imply the following constraints

\[
q_j \leq \frac{dk_B + \beta(p_H E_{\phi'}(V_{L}^m(k + t_{Hj}; \phi') | \theta_H) + p_L E_{\phi'}(V_{L}^m(k + t_{Lj}; \phi') | \theta_L))}{1 - \theta_L} + \frac{p_H (E_{\phi'}(V_B^m(k - t_{Hj}; \phi') - V_B^m(k - t_{Lj}; \phi') | \theta_H))}{1 - \theta_L} - \frac{\beta E_{\phi'}(V_{L}^m(k_{NP}; \phi'))}{1 - \theta_L},
\]

(19)

\[
q_j \geq \frac{\beta p_H E_{\phi'}(V_{L}^m(k + t_{Hj}; \phi') | \theta_H) + \beta p_L E_{\phi'}(V_{L}^m(k + t_{Lj}; \phi') | \theta_L) + p_H E_{\phi'}(V_B^m(k - t_{Hj}; \phi') - V_B^m(k - t_{Lj}; \phi') | \theta_H)) - \beta E_{\phi'}(V_{L}^m(k_{NP}; \phi'))}{1 - \theta_L}.
\]

(20)

\[
q_j \geq \frac{dk_B + \beta p_H E_{\phi'}(V_{L}^m(k + t_{Hj}; \phi') | \theta_H) + \beta p_L E_{\phi'}(V_{L}^m(k + t_{Lj}; \phi') | \theta_L)}{1 - \theta_L} + \frac{p_L \beta E_{\phi'}(V_{L}^m(k - t_{Hj}; \phi') - V_B^m(k - t_{Lj}; \phi') | \theta_H)) - \beta E_{\phi'}(V_{L}^m(k_{NP}; \phi'))}{1 - \theta_L}.
\]

(21)

Construct the following sequence \( \{V'_j\} \) : if \( \{q_j, t_{Lj}, t_{Hj}\} \) is such that (19) holds with equality, set \( V'_j = V_j \). If \( \{q_j, t_{Lj}, t_{Hj}\} \) is such that (19) is slack let \( V'_j \) be the value attained by the contract that satisfies (19) with equality. Since the transfers in terms of consumption good are defined by (17) and (18), this contract is still incentive compatible and feasible. Moreover, \( q'_j > q_j \) and \( V'_j > V_j \). Therefore,

\[
\lim_{j \to \infty} V'_j \geq \lim_{j \to \infty} V_j = V^*.
\]

and without loss of generality one can concentrate on the sequences \( \{V_j\} \) as defined above in (16), such that the loan quantities \( \{q_j\} \) satisfy (19) with equality. Having this constraint hold with equality implies \( r_{Lj} = \theta_{Lj} q_j + dk_P \). Since \( q_j \geq 0 \) always, this implies that all contracts along this sequence satisfy \( r_L > 0 \) which is the same as satisfying (20) with strict inequality.

Suppose that for some \( j \) (21) holds with equality. This implies \( r_{Hj} = \theta_{Hj} q_j + dk_P \) and since \( r_{Lj} = \theta_{Lj} q_j + dk_P \) this would imply that participation constraint is slack, and that the producer is giving the non-producer all the gains from the project. From lemma \( \text{[10]} \) there exists a feasible and incentive compatible contract that attains a higher value that contract \( j \) and therefore, without loss of generality we can ignore sequences in which for some elements \( j \), (21) holds with equality. ■

8.2 Uniqueness

Lemma 11 \( \text{[10]} \) can’t bind in equilibrium.

Proof. Suppose \( \text{[10]} \) binds, then,

\[
dk_B + \theta_{Lj} \beta (p_H c_L (d_H) t_H + p_L c_L (d_L) t_L) = t_H - t_L.
\]

(23)

If \( r_H = 0 \) binds,

\[
q^* = (\beta p_H c_L (d_H) t_H - \beta p_L c_L (d_L) t_L) (1 - \theta_L) + p_L dk_B
\]
and using the definition of \( q^* \) together with (18) this implies
\[
q^* = \frac{p_Ldk_B + \beta (p_Hc_L(d_H) t_H + p_Lc_L(d_L)t_L)(1 - p_H\theta_L)}{1 - \theta_L}.
\]

Putting these last two equations together gives
\[
\beta (p_Hc_L(d_H) t_H + p_Lc_L(d_L)t_L)(2 - p_H - \theta_L) \theta_L = -\theta LP_Ldk_B
\]
which implies \( p_Hc_L(d_H) t_H + p_Lc_L(d_L)t_L < 0 \), a contradiction. \( \blacksquare \)

**Lemma 12** In equilibrium, \( t_L \neq 0 \)

**Proof.** Suppose \( t_L = 0 \) in equilibrium. If the objective function is decreasing in \( t_L \), then it is also decreasing in \( t_H \). Therefore, \( t_L = 0 \) implies \( t_H = 0 \). The coefficients of the value functions in an affine equilibrium would then become
\[
c_B(d) = \frac{(E(\theta) - \theta_L)}{1 - \theta_L} + \beta c_{NP}(\bar{d})
\]
\[
c_B(d) = \frac{(E(\theta) - \theta_L)}{1 - \theta_L} (d + \beta \frac{\bar{d}}{1 - \theta_L}),
\]
and
\[
c_L(d) = (1 - \gamma) c_B(d) + \gamma (d + \beta c_{NP}(\bar{d}))
\]
\[
c_L(\bar{d}) = (1 - \gamma) (\frac{(E(\theta) - \theta_L)}{1 - \theta_L} \frac{\bar{d}}{1 - \theta_L} + \gamma \bar{d})
\]
The derivative of the objective function with respect to \( t_L \) is
\[
\frac{(E(\theta) - \theta_L)}{1 - \theta_L} \beta p_Lc_L(d_L) + \frac{(E(\theta) - 1)}{1 - \theta_L} p_H\beta c_B(d_H) - \beta p_Lc_B(d_L).
\]
If \( t_L = 0 \) and \( t_H = 0 \) this becomes
\[
\frac{(E(\theta) - \theta_L)}{1 - \theta_L} \frac{\beta p_L}{1 - \gamma \beta} (1 - \gamma) \frac{(E(\theta) - 1)}{1 - \theta_L} d_L + \left( \frac{(E(\theta) - \theta_L)}{1 - \theta_L} + (\frac{(E(\theta) - \theta_L)}{1 - \theta_L}) \beta \frac{\bar{d}}{1 - \theta_L} \right)
\]
\[
+ \frac{(E(\theta) - \theta_L)(E(\theta) - 1)}{1 - \theta_L} \frac{\beta}{1 - \theta_L} p_H\beta \left( d_H + \beta \frac{d}{1 - \theta_L} \right)
\]
which is \( > 0 \) and therefore contradicts \( t_L = 0 \). \( \blacksquare \)

**Lemma 13** \( t_L = k_P = t_H \) is not an equilibrium.

**Proof.** Suppose \( t_L = k_P = t_H \) in equilibrium. Then,
\[
c_B(d) = \frac{(E(\theta) - \theta_L)}{1 - \theta_L} (d + \beta c_{NP}(\bar{d}))
\]
\[
c_L(d) = \left( 1 - \gamma \right) c_B(d) + \gamma \left( d + \beta \frac{(1 - \gamma) c_B(\bar{d}) + \gamma \bar{d}}{1 - \gamma \beta} \right)
\]
Therefore,
\[
c_L(\bar{d}) = \frac{(1 - \gamma) c_B(\bar{d}) + \gamma \bar{d}}{1 - \gamma \beta},
\]
and

\[ c_B (\tilde{d}) = \frac{(E(\theta) - \theta_L) \tilde{d}}{1 - \theta_L} \left( 1 - \beta \left( \gamma + \frac{(E(\theta) - \theta_L)}{1 - \theta_L} (1 - \gamma) \right) \right) \]

which implies

\[ c_L (\tilde{d}) = \frac{\left( \gamma + \frac{(E(\theta) - \theta_L)}{1 - \theta_L} (1 - \gamma) \right)}{1 - \beta \left( \gamma + \frac{(E(\theta) - \theta_L)}{1 - \theta_L} (1 - \gamma) \right)} \tilde{d} \]

and

\[ c_B (d) = \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \left( d + \beta \left( \gamma + \frac{(E(\theta) - \theta_L)}{1 - \theta_L} (1 - \gamma) \right) \tilde{d} \right) \cdot \]

Thus,

\[ c_L (d) - c_B (d) = \gamma \left( \frac{-(E(\theta) - 1)}{1 - \theta_L} \left( d + \beta \left( \gamma + \frac{(E(\theta) - \theta_L)}{1 - \theta_L} (1 - \gamma) \right) \tilde{d} \right) \right) < 0 \]

what would imply that the derivative of the objective function is decreasing in \( t_H \) and, thus, \( t_H = 0 \), which is a contradiction. ■

**Proposition 3.** In the only affine equilibrium, \( t_L^* = k_B \) and \( t_H^* = 0 \).

**Proof.** From the previous lemmas in this section, the only candidate left for equilibrium is \( t_L^* = k_B \) and \( t_H^* = 0 \). Assume \( t_L^* = k_B \) and \( t_H^* = 0 \). Then, (11) becomes

\[ c_B (d) = \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \left( d + \beta (p_L c_B (d_L) + p_H c_B (d_H)) \right), \]

\[ c_B (d_H) = \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \left( d_H + \beta p_L c_B (d_L) \right) \]

Therefore

\[ c_B (d) = \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \left( d + \beta \left( p_L c_L (d_L) + p_H \frac{(E(\theta) - \theta_L)}{1 - \theta_L} d_H \right) \right). \quad (24) \]

To have \( t_L^* = k_B \) and \( t_H^* = 0 \) be chosen by the producer, the objective function must be increasing in \( t_L \) and decreasing in \( t_H \), i.e.,

\[ \frac{E(\theta) - \theta_L}{1 - \theta_L} \beta p_L c_L (d_L) - \beta p_L c_B (d_L) + \frac{E(\theta) - 1}{1 - \theta_L} \beta p_H c_B (d_H) > 0 \quad (25) \]

and

\[ \frac{E(\theta) - \theta_L}{1 - \theta_L} \beta (c_L (d_H) - c_B (d_H)) < 0. \quad (26) \]

Using (24) evaluated at \( d = d_L \) and \( d = d_H \) the derivative of the objective function with respect to \( t_L \)
\[
\frac{E(\theta) - \theta_c}{1 - \sigma_c} \left( (1 - \beta) p_L c_L (d_L) - \left( 1 - \beta p_H \left( \frac{E(\theta) - \theta_L}{1 - \sigma_L} \right) \right) p_L d_L \right) \\
\frac{E(\theta) - \theta_L}{1 - \sigma_L} \left( (1 - \beta) p_L c_L (d_L) - \left( 1 - \beta p_H \left( \frac{E(\theta) - \theta_L}{1 - \sigma_L} \right) \right) p_L d_L \right) \\
\frac{E(\theta) - \theta_c}{1 - \sigma_c} \left( (1 - \beta) p_L c_L (d_L) - \left( 1 - \beta p_H \left( \frac{E(\theta) - \theta_L}{1 - \sigma_L} \right) \right) p_L d_L \right) \\
\frac{E(\theta) - \theta_L}{1 - \sigma_L} \left( (1 - \beta) p_L c_L (d_L) - \left( 1 - \beta p_H \left( \frac{E(\theta) - \theta_L}{1 - \sigma_L} \right) \right) p_L d_L \right) \\
\frac{E(\theta) - \theta_c}{1 - \sigma_c} \left( (1 - \beta) p_L c_L (d_L) - \left( 1 - \beta p_H \left( \frac{E(\theta) - \theta_L}{1 - \sigma_L} \right) \right) p_L d_L \right)
\]

Using that

\[
(1 - \beta p_L) d_H + \beta p_L d_L = (1 - \beta + \beta p_H) d_H + \beta p_L d_L = (1 - \beta) d_H + \beta \tilde{d}
\]

the expression above becomes

\[
\left( \frac{E(\theta) - \theta_c}{1 - \sigma_c} \right) \beta (1 - \beta) \left( p_L c_L (d_L) - \left( d_L + \beta \frac{\tilde{d}}{1 - \beta} \right) \right) + \left( \frac{E(\theta) - \theta_c}{1 - \sigma_c} \right) \beta (1 - \beta) \left( p_L c_L (d_L) - \left( d_L + \beta \frac{\tilde{d}}{1 - \beta} \right) \right) > 0
\]

since \( c_L (d_L) - \left( d_L + \beta \frac{\tilde{d}}{1 - \beta} \right) \geq 0 \). Therefore, the derivative of the objective function is positive under this guess which in turn implies \( t_L > 0 \).

The sign of the derivative of the objective function with respect to \( t_H \) depends on the sign of the difference in marginal valuations between the borrower (borrower) and the lender. Using the definition of \( c_L (d_L) \) and \( c_B (d) \) this difference can be rewritten as

\[
c_L (d_L) - c_B (d) = \left[ (1 - \gamma) c_B (d) + \gamma \left( d + \beta \frac{1 - \gamma c_B (d) + \gamma \tilde{d}}{1 - \gamma \beta} \right) \right] - c_B (d)
\]

\[
= \gamma \left( d + \beta \frac{1 - \gamma c_B (d) + \gamma \tilde{d}}{1 - \gamma \beta} - c_B (d) \right)
\]

\[
= \gamma \left( - \frac{E(\theta) - \theta_L}{1 - \theta_L} d + \frac{1 - \beta}{1 - \theta_L} \left( (1 - \gamma) \frac{E(\theta) - \theta_L}{1 - \theta_L} + \gamma \right) \frac{\beta \tilde{d}}{1 - \beta} - c_B (d) + \left( \frac{E(\theta) - \theta_L}{1 - \theta_L} d \right) \right)
\]

\[
\leq \gamma \left( - \frac{E(\theta) - \theta_L}{1 - \theta_L} d + \frac{1 - \beta}{1 - \theta_L} \left( \frac{E(\theta) - \theta_L}{1 - \theta_L} \frac{\beta \tilde{d}}{1 - \beta} \right) - c_B (d) \right)
\]

But

\[
c_B (\tilde{d}) > \frac{E(\theta) - \theta_L}{1 - \theta_L} \frac{\tilde{d}}{1 - \beta} \frac{1}{\frac{1 - \beta}{1 - \theta_L}}
\]

\[
p_L c_L (d_L) + p_H \left( \frac{E(\theta) - \theta_L}{1 - \theta_L} \right) d_H > \frac{\tilde{d}}{1 - \beta} \left( 1 - \beta p_H \left( \frac{E(\theta) - \theta_L}{1 - \theta_L} \right) \right)
\]

\[
p_H \left( \frac{E(\theta) - \theta_L}{1 - \theta_L} \right) \left( d_H + \frac{\beta \tilde{d}}{1 - \beta} \right) > \frac{\tilde{d}}{1 - \beta} - p_L c_L (d_L)
\]

\[
p_H \left( \frac{E(\theta) - \theta_L}{1 - \theta_L} \right) \left( d_H + \frac{\beta \tilde{d}}{1 - \beta} \right) > p_H \left( d_H + \frac{\beta \tilde{d}}{1 - \beta} \right)
\]

\[
= \frac{\tilde{d}}{1 - \beta} - p_L \left( d_L + \beta \frac{\tilde{d}}{1 - \beta} \right) \geq \frac{\tilde{d}}{1 - \beta} - p_L c_L (d_L)
\]
since \( \frac{(E(\theta) - \theta_L)}{1 - \theta_L} > 1 \).

Then,
\[
c_L(d_H) - c_B(d_H) \leq \gamma \left( -\frac{(E(\theta) - 1)}{1 - \theta_L} d_H + \frac{(1 - \beta)}{1 - \gamma \beta} \left( \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \frac{\bar{d}}{1 - \beta} - c_B(\bar{d}) \right) \right) < 0,
\]
and this implies \( t_H = 0 \). Therefore, if the contract implied by \( t^*_L = k_B \) and \( t^*_H = 0 \) is a solution to the borrower’s problem in the project market.

\[\blacksquare\]

### 8.3 Value function coefficients with one project.

From section 3, the marginal value of assets for lenders and producers are given by
\[
c_L(d) = \left[ (1 - \gamma) c_B(d) + \gamma \left( d + \beta \frac{(1 - \gamma)c_B(\bar{d}) + \gamma \bar{d}}{1 - \gamma \beta} \right) \right]
\]
\[
c_B(d) = \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \left( d + \beta \frac{p_L c_L(d_L) + p_H(E(\theta) - \theta_L) d_H}{1 - \beta p_H(E(\theta) - \theta_L)} \right)
\]
which in turn are all functions of \( c_L(d_L) \). Using the equations above, one can get
\[
c_L(d_L) = \left[ (1 - \gamma) c_B(d_L) + \gamma \left( d_L + \beta \frac{(1 - \gamma)c_B(\bar{d}) + \gamma \bar{d}}{1 - \gamma \beta} \right) \right]
\]
\[
c_L(d_L) = \left[ (1 - \gamma) \left( c_B(\bar{d}) + \frac{E(\theta) - \theta_L}{1 - \theta_L} (d_L - \bar{d}) \right) + \gamma \left( d_L + \beta \frac{(1 - \gamma)c_B(\bar{d}) + \gamma \bar{d}}{1 - \gamma \beta} \right) \right]
\]
\[
c_L(d_L) = \left[ \frac{(1 - \gamma)}{1 - \gamma \beta} c_B(\bar{d}) + \left( 1 - \gamma \right) \frac{E(\theta) - \theta_L}{1 - \theta_L} + \gamma \right] d_L + \left( \frac{\gamma^2}{1 - \gamma \beta} - (1 - \gamma) \frac{E(\theta) - \theta_L}{1 - \theta_L} \right) \bar{d}.
\]

Using the definition of \( c_B(\bar{d}) \) gives
\[
\left( (1 - \gamma) \left( 1 - \beta p_H \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \right) + (1 - \gamma) \frac{E(\theta) - \theta_L}{1 - \theta_L} \beta p_L \right) c_L(d_L)
\]
\[
= \left( (1 - \gamma) \frac{E(\theta) - \theta_L}{1 - \theta_L} + \gamma \right) \left[ 1 - \beta p_H \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \right] \gamma \beta + (1 - \gamma) \left( \frac{E(\theta) - \theta_L}{1 - \theta_L} \right)^2 \bar{d}
\]
\[
+ \left( (1 - \gamma) \frac{E(\theta) - \theta_L}{1 - \theta_L} + \gamma \right) \left[ 1 - \beta p_H \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \right] (1 - \gamma \beta) - (1 - \gamma) \left( \frac{E(\theta) - \theta_L}{1 - \theta_L} \right)^2 \beta p_L \right] d_L.
\]

### 9 Change in risk

**Proposition** If \( d_H \geq d_{\min} \), where \( d_{\min} < \bar{d} \), an increase in risk decreases the debt capacity of the asset, i.e.
\[
\frac{\partial D}{\partial p_H|_{E(\theta)}} > 0.
\]
Proof. From the definition of debt capacity

\[
\frac{\partial D}{\partial p_H|E(\theta)} = \beta \frac{\partial (p_H c_B (d_H) + p_L c_L (d_L))}{\partial p_H|E(\theta)}
\]

\[
= \beta \frac{(c_B (d_H) - c_L (d_H)) + p_L \frac{\partial c_L (d_L)}{\partial p_H|E(\theta)}}{1 - \beta p_H \frac{E(\theta) - \theta}{1 - \theta}}
\]

\[
= \frac{\beta (E(\theta) - \theta_L) (d_H - d_L) + \gamma (c_B (d_H) - (d_L + \beta c_L (\bar{d}))) + p_L \frac{\partial c_L (d_L)}{\partial p_H|E(\theta)}}{1 - \beta p_H \frac{E(\theta) - \theta}{1 - \theta}}
\]

Differentiating the expression for \( c_L (d_L) \) found in the appendix with respect to \( p_H \) keeping \( E(\theta) \) fixed, one can see that

\[
\frac{\partial c_L (d_L)}{\partial p_H|E(\theta)} > 0.
\]

Therefore, if \( d_H \geq d_{\text{min}} \)

\[
\frac{\partial D}{\partial p_H|E(\theta)} > 0,
\]

where \( d_{\text{min}} < \bar{d} \) and \( d_{\text{min}} \) solves

\[
\frac{E(\theta) - \theta_L - \bar{d} + d_{\text{min}}}{p_L} + \gamma \left( c_B \left( \frac{\bar{d} - p_H d_{\text{min}}}{p_L} \right) - \left( \frac{\bar{d} - p_H d_{\text{min}}}{p_L} + \beta c_L (\bar{d}) \right) \right) + p_L \frac{\partial c_L \left( \frac{\bar{d} - p_H d_{\text{min}}}{p_L} \right)}{\partial p_H|E(\theta)} = 0
\]

9.1 Equilibrium with multiple project types.

Proposition 14 There can’t be an equilibrium in which some type of borrowers don’t use collateral contracts and some do, nor one in which no one uses collateral.

Proof. If a borrower chooses not to use the asset as collateral, his marginal value of assets is

\[
c_j^i (d) = \frac{E(\theta) - \theta_L}{1 - \theta_L} (d + \beta c_L (d)) = c_B^{NC} (d)
\]

which is independent of the correlation structure of his own project with the dividend paid by the asset.

From the benchmark model, we know that the marginal value of a borrower with investment opportunity \( j \) that chooses to use the asset as collateral is

\[
c_j^i (d) = \frac{E(\theta) - \theta_L}{1 - \theta_L} \left( d + \beta p_L c_L \left( \frac{d_j^i}{p_L} \right) + p_H \frac{E(\theta) - \theta_L}{1 - \theta_L} \frac{d_j^H}{1 - \beta p_H \frac{E(\theta) - \theta_L}{1 - \theta_L}} \right)
\]

If some borrowers \( i \in J^{NC} \) are choosing not to use the asset as collateral, it must be the case that

\[
c_L (d_{\text{H}}) - c_B^{NC} (d_{\text{H}}) \geq 0 \text{ for all } i \in J^{NC}.
\]

This difference in marginal valuations at any \( d \) can be rewritten as

\[
c_L (d) - c_B^{NC} (d) = (1 - \gamma) \sum_{j \in J^C} \mu_j \left( c_B^j (d) - c_B^{NP^j} (d) \right) - \frac{E(\theta) - 1}{1 - \theta_L} \gamma (d + \beta c_L (\bar{d}))
\]

(27)
where $J^C$ is the set of all borrower types that use the asset as collateral. Thus, using the definition of $c^i_B (d)$ for $j \in J^C$, this becomes

$$c_L (d) - c_B^{NC} (d) = \frac{E (\theta) - \theta_L}{1 - \theta_L} (1 - \gamma) \beta \sum_{j \in J^C} \mu_j \left( \frac{p_L c_L (d_L^j) + p_H E (\theta) - \theta_L}{1 - \theta_L} d_H^j - \left( \frac{E (\theta) - \theta_L}{1 - \theta_L} \beta p_H \right) c_L (\bar{d}) \right)$$

and

$$c_L (d) - c_B^{NC} (d) = \frac{E (\theta) - \theta_L}{1 - \theta_L} (1 - \gamma) \beta p_H \sum_{j \in J^C} \mu_j \left( \frac{E (\theta) - \theta_L}{1 - \theta_L} (d_H^j + \beta c_L (\bar{d})) - c_L (d_H^j) \right)$$

Setting $d = d_H^j$ and summing over $j \in J^C$,

$$\sum_{j \in J^C} \mu_j \left( c_L (d_H^j) - c_B^{NC} (d_H^j) \right) = \sum_{j \in J^C} \mu_j \frac{E (\theta) - \theta_L}{1 - \theta_L} (1 - \gamma) \beta p_H \sum_{j \in J^C} \mu_j \left( \left( \frac{E (\theta) - \theta_L}{1 - \theta_L} \beta p_H \right) c_L (\bar{d}) \right)$$

But this implies

$$\sum_{j \in J^C} \mu_j \left( c_B^i (d) - c_B^{NP} (d) \right) = \sum_{j \in J^C} \mu_j \left( c_B^{NC} (d_H^j) - c_L (d_H^j) \right) > 0$$

and

$$\sum_{j \in J^C} \mu_j c_B^i (d) > \sum_{j \in J^C} \mu_j c_B^{NP} (d).$$

The average marginal valuation of borrowers conditional on them using the asset as collateral is higher than the marginal valuation of borrowers choosing not to use the asset as collateral, if there are any. ■

**Proposition 15** The borrowers with the highest marginal valuation of the asset always use it as collateral.

**Proof.** Let $c_B^{\text{max}}$ be the marginal valuation of the borrower who values the asset the most. Then,

$$c_L (d) - c_B^{\text{max}} (d) = (1 - \gamma) \sum_{j \in J^C} \mu_j \left( c_B^i (d) - c_B^{\text{max}} (d) \right) + \gamma \left( d + \beta c_L (\bar{d}) - c_B^{\text{max}} (d) \right)$$

By definition of $c_B^{\text{max}}$ the first term is (weakly) negative. Moreover, we know that $c_B^{\text{max}} \geq \frac{E (\theta) - \theta_L}{1 - \theta_L} (d + \beta c_L (\bar{d})) > d + \beta c_L (\bar{d})$ since the borrower can always choose to sell the asset in the afternoon and invest the proceeds.
in the risky projects. Therefore, the borrower with the maximum marginal valuation of the asset chooses 
not to set transfers of asset to 0 when the realization of the return of the projects is high.

(Need to show that derivative wrt tL > 0 always)

\[
E(\theta) - \theta_L \frac{p_{LCL} (d_L^i) - p_L (d_L^i + \beta p_{PLCL} (d_L^i) + p_{PL} \frac{E(\theta) - \theta_L}{1 - \theta_L} d_L^i)}{1 - \theta_L} + \frac{E(\theta) - \theta_L}{1 - \theta_L} p_H \left(d_H^i + \beta p_{PLCL} (d_L^i) + p_{PL} \frac{E(\theta) - \theta_L}{1 - \theta_L} d_L^i\right)
\]

\[
= \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \beta \left(p_L (1 - \beta) c_L (d_L^i) - \bar{d} + (1 - \beta p_L) \frac{E(\theta) - \theta_L}{1 - \theta_L} p_H d_H^i + \beta p_H \frac{E(\theta) - \theta_L}{1 - \theta_L} p_L d_L^i\right)
\]

\[
= \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \beta\left(p_L (1 - \beta) c_L (d_L^i) - \bar{d} + (1 - \beta p_L) \frac{E(\theta) - \theta_L}{1 - \theta_L} p_H d_H^i + \beta p_H \frac{E(\theta) - \theta_L}{1 - \theta_L} p_L d_L^i\right)
\]

\[
\geq 0
\]

9.2 Value function coefficients with multiple project types

9.2.1 All borrowers use the asset as collateral

From sections 3 and 5, the marginal value of the asset for a producer of type \( j \) is given as:

\[
c_B^j (d) = \frac{E(\theta) - \theta_L}{1 - \theta_L} \left(d + \beta \left(p_{LCL} (d_L^i) + p_{PL} c_B^j (d_H^i)\right)\right)
\]

which gives

\[
c_B^j (d_H^i) = \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \left(d_H^i + \beta p_{LCL} (d_L^i)\right)
\]

and

\[
c_B^j (d) = \frac{E(\theta) - \theta_L}{1 - \theta_L} \left[d + \beta \left(p_{LCL} (d_L^i) + p_{PL} \frac{E(\theta) - \theta_L}{1 - \theta_L} d_H^i\right)\right]
\]

Moreover, the marginal value of the asset for lenders is

\[
c_L (d) = (1 - \gamma) \sum_{j \in J} \mu_j c_B^j (d) + \gamma (\bar{d} c_{NP} (\bar{d}))
\]

Then,

\[
c_L (\bar{d}) = \frac{(1 - \gamma) \sum_{j \in J} \mu_j c_B^j (\bar{d}) + \gamma \bar{d}}{1 - \gamma \bar{d}}
\]

and

\[
c_L (d) = \frac{(1 - \gamma) \sum_{j \in J} \mu_j c_B^j (d) + (1 - \gamma \beta) \gamma d + \gamma \beta (1 - \gamma) \sum_{j \in J} \mu_j c_B^j (d) + \gamma \beta \gamma \bar{d}}{1 - \gamma \bar{d}}
\]

Using the definition of \( c_B^j \) gives

\[
(1 - \gamma \beta) c_L (d) = (1 - \gamma) \beta \frac{E(\theta) - \theta_L}{1 - \theta_L} \left(p_{LCL} \sum_{j \in J} \mu_j d_L^i + p_{PL} \frac{E(\theta) - \theta_L}{1 - \theta_L} \sum_{j \in J} \mu_j d_H^i\right)
\]

\[
+ \left((1 - \gamma) \frac{E(\theta) - \theta_L}{1 - \theta_L} + \gamma\right) \left((1 - \gamma \beta) d + \gamma \beta \bar{d}\right).
\]
Rearranging terms and setting $d = \sum_{j \in J} \mu_j d_L^j$, 

\[
\left(1 - \gamma \beta \right) \left(1 - \beta p_L \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \right) - (1 - \gamma) \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \beta p_L \left(\sum_{j \in J} \mu_j d_L^j \right) \\
= \left(1 - \beta p_H \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \right) \left(1 - \gamma \right) \frac{(E(\theta) - \theta_L)}{1 - \theta_L} + \gamma + (1 - \gamma) \left(\frac{(E(\theta) - \theta_L)}{1 - \theta_L} \right)^2 \beta \bar{d} \\
+ \left(1 - \beta p_H \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \right) \left(1 - \gamma \right) \frac{(E(\theta) - \theta_L)}{1 - \theta_L} + \gamma + (1 - \gamma) p_L \left(\frac{(E(\theta) - \theta_L)}{1 - \theta_L} \right)^2 \beta \right) \sum_{j \in J} \mu_j d_L^j.
\]

The guess is correct, the coefficients are affine in the dividend level.

### 9.2.2 Some borrowers don’t use the asset as collateral

From previous sections the marginal value of a borrower type $j \in J^C$ that uses the asset as collateral is given by

\[
c_B^j (d) = \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \left[ d + \beta p_L \frac{d_L^j + p_H \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \beta \bar{d}}{1 - \beta p_H \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \beta p_L} \right]
\]

The marginal value of a borrower type $j \in J^{NC}$ who chooses not to use the asset as collateral is

\[
c_B^j (d) = \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \left[ d + \beta c_L (\bar{d}) \right] = c_B^{NC} (d)
\]

Moreover, the marginal value of the asset for lenders is

\[
c_L (d) = \frac{(1 - \gamma \beta) (1 - \gamma) \sum_{j \in J} \mu_j c_B^j (d) + (1 - \gamma \beta) \gamma d + \gamma \beta (1 - \gamma) \sum_{j \in J} \mu_j c_B^j (\bar{d}) + \gamma \beta \gamma \bar{d}}{1 - \gamma \beta}
\]

Using the definition of $c_B^j$ gives

\[
(1 - \gamma \beta) c_L (d) = (1 - \gamma) \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \beta \left( \sum_{j \in J^C} \mu_j p_L c_L \frac{d_L^j + p_H \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \beta \bar{d}}{1 - \beta p_H \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \beta p_L} + \sum_{j \in J^{NC}} \mu_j c_L (\bar{d}) \right)
\]

\[
+ \left( \gamma + \frac{(E(\theta) - \theta_L)}{1 - \theta_L} (1 - \gamma) \right) (\gamma \beta \bar{d} + (1 - \gamma \beta) \bar{d})
\]

Using that

\[
c_L (\bar{d}) = \frac{(1 - \gamma) \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \beta \left( \sum_{j \in J^C} \mu_j p_L c_L \frac{d_L^j + p_H \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \beta \bar{d}}{1 - \beta p_H \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \beta p_L} + \sum_{j \in J^{NC}} \mu_j \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \beta \right)}{(1 - \gamma \beta) - (1 - \gamma) \sum_{j \in J^{NC}} \mu_j \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \beta}
\]

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the coefficients can be recovered using the solution to

\[
\[1 + \sum_{j \in J^NC} \mu_j p_L \left( (1 - \theta_L) \gamma \beta \frac{C}{E} \right) - \sum_{j \in J^C} \mu_j \beta \left( \gamma + (1 - \gamma) \frac{E(\theta) - \theta_L}{1 - \theta_L} \right) \left( 1 - \beta p_H \frac{E(\theta) - \theta_L}{1 - \theta_L} \sum_{j \in J^NC} \mu_j \right) \frac{1 - \beta}{1 - \beta p_H} \frac{E(\theta) - \theta_L}{1 - \theta_L} \]
\]

\[
\sum_{j \in J^C} \mu_j c_L \left( d^j_L \right) \]

\[
\left[ \begin{array}{c}
(1 - \gamma) \frac{E(\theta) - \theta_L}{1 - \theta_L} + \gamma \sum_{j \in J^C} \mu_j d^j_L + \frac{1 - \beta}{1 - \beta p_H} \frac{E(\theta) - \theta_L}{1 - \theta_L} \sum_{j \in J^C} \mu_j d^j_H
\end{array} \right]
\]

\[c_L \left( d \right) \geq c_B^{NC} \left( d \right)
\]

\[c_L \left( d \right) \geq \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \left[ d + \beta c_L \left( d \right) \right]
\]

\[c_L \left( d \right) \geq \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \left[ d + \beta c_L \left( d \right) \right]
\]

\[
DET = \left[ \begin{array}{c}
(1 - \gamma) \frac{E(\theta) - \theta_L}{1 - \theta_L} \left( 1 - \beta p_H \frac{E(\theta) - \theta_L}{1 - \theta_L} \sum_{j \in J^NC} \mu_j \right) \left( 1 - \beta \left( \gamma + (1 - \gamma) \frac{E(\theta) - \theta_L}{1 - \theta_L} \right) \right)
\end{array} \right]
\]

\[
D = \left[ \begin{array}{c}
\sum_{j \in J^C} \mu_j c_L \left( d^j_L \right) \end{array} \right] = \frac{1}{DET} \left[ \begin{array}{c}
A
\end{array} \right]
\]

\[
\left[ \begin{array}{c}
(1 - \gamma) \frac{E(\theta) - \theta_L}{1 - \theta_L} + \gamma \sum_{j \in J^C} \mu_j d^j_L + \frac{1 - \beta}{1 - \beta p_H} \frac{E(\theta) - \theta_L}{1 - \theta_L} \sum_{j \in J^C} \mu_j d^j_H
\end{array} \right]
\]

\[
\left[ \begin{array}{c}
\sum_{j \in J^C} \mu_j \beta \left( \gamma + (1 - \gamma) \frac{E(\theta) - \theta_L}{1 - \theta_L} \right) \left( 1 - \beta p_H \frac{E(\theta) - \theta_L}{1 - \theta_L} \sum_{j \in J^NC} \mu_j \right) \frac{1 - \beta}{1 - \beta p_H} \frac{E(\theta) - \theta_L}{1 - \theta_L} \sum_{j \in J^C} \mu_j d^j_H
\end{array} \right]
\]
\[ B = \frac{(1 - \gamma) \left( \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \right) \beta p_L}{1 - \beta p_H \left( \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \right)} \times \]
\[ \left( (1 - \gamma) \frac{(E(\theta) - \theta_L)}{1 - \theta_L} + \gamma \right) \sum_{j \in J_C} \mu_j d^i_L + \frac{\sum_{j \in J_C} \mu_j (1 - \gamma) \left( \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \right)^2 \beta p_H \sum_{j \in J_C} \mu_j d^j_H}{1 - \beta p_H \left( \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \right)} \]
\[ + \left( 1 + \sum_{j \in J^{NC}} \mu_j p_L (1 - \gamma) \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \right) \times \]
\[ \left( (1 - \gamma) \frac{(E(\theta) - \theta_L)}{1 - \theta_L} + \gamma \right) d + \frac{(1 - \gamma) \left( \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \right)^2 \beta p_H \sum_{j \in J_C} \mu_j d^j_H}{1 - \beta p_H \left( \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \right)} \]

9.3 Changes in correlation when there are multiple types of projects and everyone uses collateral

**Proposition** An increase in \( \pi \) has the following effects.

- If \( \mu_1 = \mu_2 \), the economy's average debt capacity remains unchanged, type 1 borrowers collateral constraint is relaxed and type 2 borrowers collateral constraint is tightened.
- If \( \mu_1 > \mu_2 \), the economy's average debt capacity increases, and type 1 borrowers collateral constraint is relaxed.
- If \( \mu_1 < \mu_2 \), the economy's average debt capacity decreases, and type 2 borrowers collateral constraint is tightened.

**Proof.** From the characterization of the value function coefficients in the appendix, for all \( j = 1, 2 \),

\[ c^j_B (d) = \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \left[ d + \beta \frac{p_L c_L \left( d^j_L \right) + p_H \left( \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \right) d^j_H}{1 - \beta \left( \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \right) p_H} \right] \]

and

\[ (1 - \gamma) c_L (d) = (1 - \gamma) \beta \frac{(E(\theta) - \theta_L)}{1 - \theta_L} p_L c_L \left( \sum_{j \in J} \mu_j d^j_L \right) + p_H \left( \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \right) \sum_{j \in J} \mu_j d^j_H \]

\[ + \left( 1 - \gamma \right) \left( \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \right) \gamma \left( (1 - \gamma) d + \gamma \beta d \right), \]

where

\[ \left( 1 - \gamma \right) \left( 1 - \beta p_H \left( \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \right) \right) = (1 - \gamma) \left( \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \right) \beta p_L \]

\[ c_L \left( \sum_{j \in J} \mu_j d^j_L \right) \]

\[ = \left( 1 - \beta p_H \left( \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \right) \right) \left( 1 - \gamma \right) \left( \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \right) + \gamma + (1 - \gamma) \left( \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \right)^2 \beta d \]

\[ + \left( 1 - \beta p_H \left( \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \right) \right) \left( 1 - \gamma \right) \left( \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \right) + \gamma \left( 1 - \gamma \right) \left( 1 - \gamma \right) p_L \left( \frac{(E(\theta) - \theta_L)}{1 - \theta_L} \right)^2 \beta \sum_{j \in J} \mu_j d^j_L. \]
The debt capacity of the asset for a borrower of type $j$ is given by

$$D^j = \beta \left( p_{HC} \left( d_H^j \right) + p_{LC} \left( d_L^j \right) \right).$$

$$\frac{\partial D^j}{\partial \pi} = \beta \left( p_{HC} \left( d_H^j \right) + p_{LC} \left( d_L^j \right) \right)$$

$$\propto \left( \frac{p_H}{1 - \theta_L} \frac{\partial d_H^j}{\partial \pi} + p_L \frac{\partial d_L^j}{\partial \pi} \right)$$

$$\propto \left( \frac{(E(\theta) - \theta_L)}{1 - \theta_L} (-1)^{j-1} \frac{d_1 - d_2}{2} + p_L \frac{\partial d_L^j}{\partial \pi} \right)$$

But,

$$(1 - \gamma \beta) c_L(d) = (1 - \gamma) \left( \frac{E(\theta) - \theta_L}{1 - \theta_L} \right) \bar{D} + \left( 1 - \gamma \right) \left( \frac{E(\theta) - \theta_L}{1 - \theta_L} + \gamma \right) \left( (1 - \gamma \beta) d + \gamma \beta \bar{d} \right),$$

where $\bar{D} = \beta \frac{p_{LC} \left( \sum_{j \in J} \mu_j d_L^j \right) + p_H \left( \frac{E(\theta) - \theta_L}{1 - \theta_L} \sum_{j \in J} \mu_j d_H^j \right)}{1 - \left( \frac{E(\theta) - \theta_L}{1 - \theta_L} \right) \beta p_H}$ is the economy’s aggregate debt capacity. Then,

$$\frac{\partial D^j}{\partial \pi} \propto (-1)^{j-1} \frac{d_1 - d_2}{2} (1 - \gamma) \left( \frac{E(\theta) - 1}{1 - \theta_L} \right) + p_L \left( 1 - \gamma \right) \left( \frac{E(\theta) - \theta_L}{1 - \theta_L} \right) \frac{\partial \bar{D}}{\partial \pi}.$$

But

$$\frac{\partial \bar{D}}{\partial \pi} = \frac{\beta p_L}{1 - \left( \frac{E(\theta) - \theta_L}{1 - \theta_L} \right) \beta p_H} \frac{\partial \left( \sum_{j \in J} \mu_j d_L^j \right)}{\partial \pi} \left( \frac{\partial \sum_{j \in J} \mu_j d_L^j}{\partial \pi} - \frac{E(\theta) - \theta_L}{1 - \theta_L} \right)$$

$$= - \frac{\beta p_L}{2p_L} \frac{(\mu_1 - \mu_2)(d_1 - d_2)}{1 - \left( \frac{E(\theta) - \theta_L}{1 - \theta_L} \right) \beta p_H} \times$$

$$\left( \frac{1}{1 - \gamma \beta} \left( \frac{E(\theta) - \theta_L}{1 - \theta_L} \right) \frac{\partial \bar{D}}{\partial \pi} \right) \left( \frac{1 - \gamma \beta - (1 - \gamma \beta) p_L \left( \frac{E(\theta) - \theta_L}{1 - \theta_L} \right) ^2 \beta - (E(\theta) - \theta_L)}{1 - \theta_L} \right)$$

$$\propto \frac{\beta (\mu_1 - \mu_2)(d_1 - d_2)}{2} \gamma \left( 1 - \beta p_H \left( \frac{E(\theta) - \theta_L}{1 - \theta_L} \right) \right) - (1 - \gamma \beta) \frac{\left( E(\theta) - \theta_L \right) \beta p_L}{1 - \theta_L}.$$

An increase in $\pi$ will have no effect whatsoever on the aggregate debt capacity if $\mu_1 = \mu_2$. It will increase it if $\mu_1 > \mu_2$, and it will decrease it if $\mu_2 > \mu_1$.

Then,

$$\frac{\partial D^j}{\partial \pi} \propto \frac{(E(\theta) - 1)}{1 - \theta_L} \frac{d_1 - d_2}{2} (1 - \gamma) \left[ (-1)^{j-1} + \frac{\gamma p_L \left( \frac{E(\theta) - \theta_L}{1 - \theta_L} \right) \beta (\mu_1 - \mu_2)}{2} \right].$$