Using Elasticities to Derive Optimal Bankruptcy Exemptions

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Abstract

This paper characterizes the optimal bankruptcy exemption for risk averse borrowers who use unsecured contracts but have the possibility of defaulting. It provides a novel general formula — which holds in a wide variety of environments — for the optimal exemption as a function of a few observable sufficient statistics. Borrowers' leverage, the sensitivity of the equilibrium credit spread with respect to the level of bankruptcy exemption, the probability of default and the change in consumption by bankrupt borrowers turn out to be the key determinants of the optimal bankruptcy exemption. When calibrated to US data, the optimal bankruptcy exemption implied by the model ($100,000) is slightly larger than the average exemption in the US ($70,000), but of the same order of magnitude.

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1 Introduction

Motivation Should we make bankruptcy procedures harsher or more lenient for borrowers who decide not to repay their debts? What is the socially optimal level of bankruptcy exemptions? These are important policy questions — see, for instance, White (2011), who recently documents the large increase in consumer bankruptcy filings in the US in the last decades — that are still subject to debate. This paper shows that a few observable variables are sufficient to answer such questions under a broad array of circumstances.

This paper addresses the problem of optimal bankruptcy design by analytically characterizing the welfare maximizing bankruptcy exemption for risk averse borrowers who use arbitrary unsecured contracts but can choose to default. The welfare maximizing bankruptcy exemption optimally trades off reduced access to credit ex-ante with improved insurance ex-post.

Main results I derive the main results of the paper in a baseline two period model with price-taking borrowers who only have access to a contract which promises a flat repayment; I subsequently extend the results in many dimensions. Throughout the paper, the bankruptcy exemption m is defined as the dollar amount that a borrower who declares bankruptcy is allowed to keep.

The main contribution of this paper is to characterize a) the welfare change induced by a marginal change in m and b) the optimal bankruptcy exemption m^*, as a function of sufficient statistics. This characterization provides a clear interpretation of the forces that determine the optimal exemption for a broad range of primitives and directly links the theoretical tradeoffs to observable variables. Most of the intuition behind the results can be seen in the case with logarithmic utility borrowers whose income always exceeds the bankruptcy exemption. In that case, the optimal exemption m^* (measured in dollars) must satisfy the following equation:

$$m^* = \frac{\beta\pi}{\Lambda\varepsilon_{\tilde{r},m}}, \tag{1}$$

where β is the borrowers' discount factor, π is the probability of default in equilibrium, Λ is a measure of borrowers' leverage and $\varepsilon_{\tilde{r},m}$ is the equilibrium derivative of the credit spread charged to the borrower with respect to the bankruptcy exemption measured in dollars. Importantly, all four variables in the formula for m^* have direct empirical counterparts.[1]

Equation (1) captures the key tradeoff regarding the optimal determination of the bankruptcy exemption. On the one hand, if borrowing rates rise quickly with the level of the bankruptcy exemption (high $\varepsilon_{\tilde{r},m}$), it is optimal to set a low exemption level, especially when borrowers' leverage is high (high Λ): a low exemption facilitates the access to credit ex-ante in that case. On the other hand, if default is very frequent in equilibrium (high π), especially when borrowers value consumption relatively more in the terminal period (high β), it is optimal to set a high exemption level, allowing borrowers to consume more when bankrupt. Equation (1) trades off these forces optimally.[2]

Note that the decisions of how much to borrow and when to default do not affect the assessment of marginal interventions nor the formula for m^* directly: the fact that borrowers borrow and default optimally, through the

[1] When borrowers' utility is not logarithmic, the term $\beta\pi$ in equation (1) becomes the borrowers' valuation, using their own stochastic discount factor, of a claim to their own consumption in default states as a fraction of initial consumption — a price-consumption ratio. With CRRA utility, this term corresponds to a power (determined by borrowers' risk aversion) of the ratio of consumption in bankrupt states to consumption at the time of borrowing. In the log utility case, when, for some default states, borrowers' income is lower than the bankruptcy exemption, π then corresponds to the probability of bankruptcy while claiming the full bankruptcy exemption.

[2] All variables in equation (1) are endogenous. The logic behind this characterization is identical to the one behind the CAPM, in which the beta of an asset becomes a sufficient statistic to determine expected returns, or consumption based asset pricing models, in which the consumption process, independently of how it is generated, is a sufficient statistic for pricing assets.

envelope theorem, guarantees that the effect on these variables induced by a change in the bankruptcy exemption vanishes from the optimal exemption formula. This fact greatly simplifies the characterization of the optimal exemption.

This paper does not restrict the shape of the contracts/assets available for trading between borrowers and lenders but it assumes that they are exogenously determined: this is the key friction that the optimal bankruptcy design seeks to overcome. However, not modeling why borrowers do not use the first-best full insurance contract entails no loss of generality if the shape of the contracts traded do not change with the level of exemptions. For instance, we observe that debt contracts are traded in economies with very large and very small exemptions, which suggests that the level of exemptions need not modify the shape of the contract. Because all contingencies are observable and verifiable by the parties, the first-best allocation, which implies perfect insurance for the risk averse borrowers, could be achieved with an unrestricted contracting set. Hence, the planning problem solved in this paper is a second-best problem.

Extensions I explore several extensions to the baseline model. First, allowing for additional margins of adjustment, like labor supply or effort choices, only alters the formula for the optimal exemption if these decisions enter non-separably into the utility function. Second, allowing for bankruptcy costs or non-pecuniary losses associated with defaulting does not alter the optimal exemption formulas. Third, allowing borrowers to internalize the effect of their borrowing choice in the rate offered by lenders introduces a new term in the optimal exemption formulas corresponding to the wedge in the borrowers' Euler equation. Fourth, allowing for more general utility specifications for consumption — like Epstein-Zin preferences or state dependent utility — only affects the optimal exemption formula through changes in the borrowers' stochastic discount factor. Fifth, allowing for multiple traded assets/contracts with arbitrary payoffs is straightforward: a leverage-weighted average of credit spread sensitivities captures then the marginal cost of increased leniency. Sixth, when borrowers are ex-ante heterogeneous but exemptions cannot be individual specific, the optimal exemption becomes a function of a weighted average across borrowers of the marginal effects present in the baseline case. Seventh, similarly, in a dynamic context, the optimal exemption is given by an average across periods/states of benefits and costs of the bankruptcy exemption, all weighted according to the borrowers' own stochastic discount factor.

These extensions yield two robust insights. First, the precise determination of the region(s) in which borrowers decide to default does not modify the formula for the optimal exemption, because borrowers default optimally. Hence, evaluating the welfare implications of changing bankruptcy exemptions does not require explicit modeling of the large number of factors that may influence default decisions.[3] Second, to assess the welfare benefits of increasing the bankruptcy exemption, the planner only needs to measure the consumption of bankrupt borrowers. Preference parameters, like risk aversion, will determine how to translate measures of consumption into welfare, but the responses of labor supply and other endogenous choice variables are irrelevant once borrowers' consumption is known, as long as utility is separable.

Calibration Finally, I show the applicability of the theoretical results by calibrating the formula for the optimal exemption to US bankruptcy data and assessing the magnitude of the welfare gains generated by adjusting the bankruptcy exemption. The preferred calibration to US data implies that the optimal bankruptcy exemption should be in the range of $100,000$ dollars, an amount slightly larger than the average exemption in the US (of approximately $70,000$ dollars). For the same calibration, implementing the optimal exemption achieves welfare gains on the order

[3] For instance, default is jointly determined with other endogenous choice variables. Similarly, forward looking borrowers internalize the option value of waiting for further uncertainty to be realized before defaulting. Both these considerations, which prevent a simple characterization of default regions, only affect the optimal exemption through the sufficient statistics.

of approximately 0.03% of ex-ante consumption. I should emphasize that this calibration exercise is not meant to be conclusive. Further empirical work will help us refine the best estimate for the optimal exemption.

Related Literature

This paper touches on several literatures in economics.[4] First, this paper is directly related to the literature on **general equilibrium with incomplete markets**, which has studied the possibility of default in very general competitive environments in which markets are exogenously or endogenously incomplete. Zame (1993) and Dubey, Geanakoplos and Shubik (2005) are the first to theoretically analyze the core tradeoff present in this paper. They show that default may be welfare improving in a model with incomplete markets, since it enhances insurance opportunities by introducing new contingencies into contracts. These papers take default penalties as exogenous and do not characterize optimal penalties.[5] It is well known that default is only beneficial when markets are initially incomplete. Allowing for default when agents can write fully state contingent contracts, as in Kehoe and Levine (1993), Kocherlakota (1996), Alvarez and Jermann (2000), or Chien and Lustig (2010), only restricts the contracting space, reducing welfare unequivocally.

Second, this paper is also related to the **quantitative literature** on default and bankruptcy. On the one hand, the papers by Athreya (2002, 2006), Chatterjee et al. (2007), Livshits, MacGee and Tertilt (2007), Athreya, Tam and Young (2009, 2012), Krebs, Kuhn and Wright (2013), and Mitman (2012), among several others, provide a careful quantitative structural analysis of unsecured credit and default from a macroeconomic perspective. Due to their dynamic nature and their rich general equilibrium features, these papers must rely on numerical methods to evaluate the welfare implications of bankruptcy policies. This paper, by contrast, focuses on providing analytical insights into this question. The work by Gropp, Scholz and White (1997), Gross and Souleles (2002), Fay, Hurst and White (2002), Fan and White (2003), Iverson (2013), Dobbie and Song (2013), and Severino, Brown and Coates (2013), among others, uses empirical microeconomic methodology to understand the implications of actual bankruptcy policies. See White (2007, 2011) for recent surveys of this body of work.

Third, this paper presents a novel application of the **sufficient statistic** approach to the problem of bankruptcy and security design. As stated by Chetty (2009), the central concept of the sufficient statistic approach is to derive formulas for the welfare consequences of policies that are functions of high-level elasticities rather than deep primitives. The optimal policies characterized under this approach are robust to a broad range of environments. Some recent successful applications of this approach are Diamond (1998) and Saez (2001) – hence the title analogy – on income taxation, Shimer and Werning (2007) and Chetty (2008) on unemployment insurance, Arkolakis, Costinot and Rodríguez-Clare (2012) on the welfare gains from trade liberalizations and Basu et al. (2013) on productivity and capital accumulation.

Fourth, this paper is related to the literature on **optimal contracting** and corporate finance, which studies borrower-lender relationships, bankruptcy, and default.[6] Costly state verification, limited commitment or enforcement, moral hazard, adverse selection, and secret cash-flow manipulation are features studied in static and dynamic environments within this extensive literature. A full review of these papers would take me too far afield. No other paper has solved for optimal exemptions or characterized the welfare effect of varying exemptions as a function

[4] I do not refer to the vast legal literature on bankruptcy; see Hermalin, Katz and Craswell (2007) for a survey.

[5] It is surprising that, despite acknowledging — see section 7 of their paper — that the optimal penalty associated with default is neither zero nor infinite, Dubey, Geanakoplos and Shubik (2005) do not characterize it formally.

[6] Townsend (1979), Diamond (1984), Gale and Hellwig (1985), Bolton and Scharfstein (1990), Hart and Moore (1994, 1998); Hart (1995), Bizer and DeMarzo (1999), Krasa and Villamil (2000), Albuquerque and Hopenhayn (2004), Jappelli, Pagano and Bianco (2005), Rampini (2005), Bisin and Gottardi (2006), Hopenhayn and Werning (2008), Grochulski (2010), Gale and Gottardi (2011) are examples of this vast literature. Bolton and Dewatripont (2005), Freixas and Rochet (2008) and Tirole (2005) are textbook references.

of sufficient statistics.

Fifth, the results of this paper also relate to the work in **security design** motivated by risk sharing, compactly summarized in Allen and Gale (1994) and Duffie and Rahi (1995). This literature starts by allowing for some form of market incompleteness and then asks the question of which securities should be introduced in the market to improve risk sharing and welfare among agents. The optimal bankruptcy exemption in this paper solves a restricted optimal security design problem, under a set of particular behavioral (e.g., risk averse borrowers and risk neutral lenders) and environmental (e.g., restricted set of contracts traded) constraints.

Finally, a dynamic version of the environment used in this paper — following Eaton and Gersovitz (1981) —, has become the workhorse model to understand **sovereign default**. If the ex-post penalties imposed on sovereigns that default were enforceable, the results of this paper would also apply to the international context with minor modifications. Formally closer to this paper, Bolton and Jeanne (2007, 2009) use a two-period model of defaultable credit to study the optimal number of creditors and the ability to refinance. See Aguiar and Amador (2013) for a recent survey of the sovereign default literature.

Outline Section 2 lays out the baseline model and analyzes its positive implications and section 3 executes the normative analysis, solving for the optimal bankruptcy exemption. Section 4 analyzes multiple extensions and section 5 calibrates the theoretical formulas to US data. Section 6 concludes. All proofs and derivations are in the appendix.

2 Baseline model

This paper studies the problem of a social planner who wants to determine optimally the leniency of the bankruptcy system. This problem can be solved backwards. First, I characterize the equilibrium of the economy taking as given the level of the exemption level. Subsequently, I characterize the welfare maximizing bankruptcy exemption. This section presents a simplified model which illustrates the main results. Section 4 extends the results in many dimensions.

2.1 Environment

Time is discrete, there are two dates $t = \{0, 1\}$ and there is a unit measure of borrowers and a unit measure of lenders. I will shortly impose conditions on fundamentals so that borrowers always decide to borrow — instead of saving — in equilibrium. Because borrowers are risk averse, the results of this paper are more applicable to household borrowing rather than corporate borrowing.[7]

Borrowers Borrowers are risk averse and maximize expected utility of consumption with discount factor $\beta > 0$. Their flow utility $U(C)$ satisfies standard regularity conditions: $U'(\cdot) > 0$, $U''(\cdot) < 0$ and $\lim_{C \to 0} U'(C) = \infty$. Unlike many papers in the optimal contracting literature, which assume risk neutrality to simplify the design process, the results of this paper crucially rely on interior first order conditions.

There is a single consumption good (dollar) in this economy, which serves as numeraire.[8] Every borrower is endowed with y_0 units of the consumption good at $t = 0$. At $t = 1$, every borrower receives a stochastic endowment of y_1 units of the consumption good, whose distribution follows a cdf $F(\cdot)$ with support in $\left[\underline{y_1}, \overline{y_1}\right]$. I assume that $\underline{y_1} > 0$,

[7] There is scope to adapt the results of this paper to the corporate context, in which firms — often assumed risk neutral — engage in risk management and become endogenously risk averse, along the lines of Froot, Scharfstein and Stein (1993) and Rampini and Viswanathan (2010).

[8] Extending all findings to economies with positive inflation is straightforward – those results are available under request.

so borrowers always have some positive income before facing any repayment; $\overline{y_1}$ could be infinite. The realizations of y_1 are iid among borrowers. Therefore, under a law of large numbers, there is no aggregate risk in this economy. All variables are observable and verifiable by all the parties.

Contract space/Bankruptcy procedure Borrowers can trade a single noncontingent contract, i.e., a debt contract.[9] That is, they receive $q_0 B_0$ units of the consumption good from lenders at $t = 0$ and they promise to repay B_0 units at $t = 1$. Hence, the gross interest rate paid by borrowers can be defined as $1 + r \equiv \frac{1}{q_0}$.[10] When needed, I denote by $\tilde{r}$ the logarithmic interest rate, i.e., $\tilde{r} \equiv \log(1 + r)$. Importantly, in the baseline model, borrowers are price takers: they form their loan demand B_0 taking q_0 as given.

At $t = 1$, once y_1 is realized, borrowers can decide to repay the amount owed B_0 or to default;[11] hence their commitment is limited. If they default, they consume $C_1^D = \min\{y_1, m\}$, that is, they keep $m > 0$ units of the consumption good, unless m is larger than y_1, in which case they only keep y_1 units. Any positive remainder $y_1 - m$ is seized from borrowers and transferred costlessly to lenders. Note that, because bankruptcy entails no resource costs and the economy ends at $t = 1$, there is no scope for ex-post renegotiation of the terms of the contract.

I refer to m as the exemption level. Since I assume that the bankruptcy procedure cannot rely on external funds, m is constrained to be in the closed interval $[0, \overline{y_1}]$. This paper assumes that m is the choice variable of a third party, the planner, and focuses on its optimal determination. Alternatively, the bankruptcy exemption determined in this paper can be seen as the optimal commitment device chosen by borrowers to enforce ex-post repayments, which allows them to borrow ex-ante. Both views are equivalent.

Lenders Lenders are risk neutral and commitment is irrelevant on their side, since they do not promise to deliver anything at $t = 1$. They are perfectly competitive and require an effective rate of return $1 + r^*$, determined exogenously. Lenders determine the supply of credit through a price schedule q_0^l per each level of B_0, which accounts for the possibility of default.

The following remarks highlight the two key features of the economic environment.

Remark 1. *(Exogenous contracts/assets)* The key assumption made so far is that the set of contracts/assets traded is exogenous: borrowers cannot choose the shape of the traded contract optimally. This paper does not take a stand on why borrowers do not use such a contract, which would deliver the first-best outcome. Hysteresis in contracting, informational frictions or bounded rationality are plausible explanations. The rich literature on financial contracting, referenced in the literature section, contains more elaborated justifications for why debt contracts are prevalent. Taking as given the traded contract, this paper determines the optimal degree of leniency in bankruptcy. The possibility of bankruptcy introduces new contingencies into the original contract, a mechanism originally pointed out by Zame (1993).

Remark 2. *(Constant exemption level)* Adopting a constant bankruptcy exemption level is optimal even when the planner can choose a nonlinear bankruptcy scheme depending on y_1, as long as lenders are risk neutral. In this environment, the optimal contract would imply that risk neutral lenders fully insure risk averse borrowers. A constant exemption level is the optimal way for the planner to replicate such contract.

[9] I use throughout the word contract to facilitate the exposition. The word asset could be used instead, given that agents do not optimize the shape of the contract and prices are determined competitively.

[10] A model in which borrowers borrow B_0 units at $t = 0$ and repay $(1 + r) B_0$ units at $t = 1$ is identical to this one.

[11] I use the words bankruptcy and default as synonyms in this paper. See White (2011) for a discussion of how borrowers may default on their obligations without entering in bankruptcy and section 4.9 for how to deal with that possibility within the framework of this paper.

2.2 Equilibrium definition/regularity conditions

The definition of equilibrium is standard: an equilibrium is defined as a set of consumption allocations $C_0, \{C_1\}_{y_1}$, default decisions $\{\xi\}_{y_1}$, amount of credit B_0 and price q_0 such that borrowers maximize utility given prices and lenders break even.

The following three assumptions guarantee that the problem solved by borrowers is well behaved. These assumptions provide sufficient conditions, so the conclusions of the paper may apply even when they do not hold.

Assumption 1. (Borrowers borrow) *Borrowers' discount factor β, initial endowment y_0 and distribution of future endowments $F(\cdot)$ are such that borrowers borrow in equilibrium, that is, $B_0 > 0$.*

The exact condition is given in the appendix. Intuitively, sufficiently impatient agents are natural borrowers in equilibrium. A combination of low initial endowments with high expected future endowments can also guarantee that borrowers borrow in equilibrium.

Assumption 2. (Bounded credit choice)

a) The choice of B_0 is constrained to be in a closed interval $\left[\underline{B_0}, \overline{B_0}\right]$.

b) The upper bound $\overline{B_0}$ is chosen so that it never binds in equilibrium.

Given assumption 1, we can set $\underline{B_0} = 0$ without loss of generality. I precisely describe the determination of the upper bound in the appendix. Intuitively, because borrowers are price-takers their problem can be non-convex, yielding sometimes two local optima. The first local optimum is characterized by an interior first-order condition. The second second local optimum entails borrowing as much as possible. When the feasible set of B_0 is unbounded above, borrowers credit demand is infinite at any given rate, preventing the existence of an equilibrium. A judicious restriction on the set of feasible credit demands ensures the existence of equilibrium by guaranteeing that the interior local optimum is also global. Eaton and Gersovitz (1981) use a similar approach to guarantee an interior solution in a related environment.

Assumption 3. (Regularity conditions on $F(\cdot)$ and ψ)

a) The distribution of future endowment $F(\cdot)$ is such that the second order condition for (3) is strictly negative in the interval $\left[\underline{B_0}, \overline{B_0}\right]$. The appendix provides the precise condition, which requires a sufficiently flat pdf $f(\cdot)$.

b) The elasticity of intertemporal substitution (EIS) ψ, defined as $\frac{1}{\psi} \equiv -C_0 \frac{U''(C_0)}{U'(C_0)}$, is larger or equal than unity, that is, $\psi \geq 1$.[12]

Assumption 3a makes the borrowers' problem convex, guaranteeing a single interior optimum. Intuitively, a marginal increase in B_0 reduces the marginal cost of borrowing by increasing the default region, in which debt does not have to repaid at all. If this effect is too strong, there may be no interior optimum at all.[13]

Assumption 3b provides a sufficient condition that guarantees that borrowers' loan demand is upward sloping, which makes the equilibrium unique and comparative statics straightforward. Intuitively, varying q_0 creates opposing income and substitution effects on B_0. For instance, cheaper credit (high q_0) induces borrowers to shift consumption towards the initial period by borrowing more: this is a standard substitution effect. However, cheaper credit makes investors more willing to consume at both periods, creating a force to reduce the amount owed at $t = 1$, B_0 and

[12] In this model, there is no distinction between the coefficient of relative risk aversion and the elasticity of intertemporal substitution. The analysis with Epstein-Zin preferences in section 4 shows that the regularity conditions must be imposed on the EIS.

[13] This argument is unrelated to the fact that $B_0 = \infty$ is the optimal solution for price-taking borrowers when their choice set is unrestricted, as just described when discussing assumption 2.

therefore to increase C_1: this is the income effect, which is stronger when borrowers' leverage is high. Hence, if $\psi > \Lambda$, where $\Lambda \equiv \frac{q_0 B_0}{C_0}$ is a measure of leverage, the substitution effect dominates and borrowers demand B_0^d is increasing in q_0. Consequently, because $\Lambda < 1$, the condition $\psi \geq 1$ is sufficient for the loan demand to be upward sloping.[14]

2.3 Equilibrium characterization

I first solve borrowers' and lenders' individual problems. Subsequently, I characterize the equilibrium level of credit and interest rates.

Borrowers' problem

The solution to the borrowers' problem characterizes the loan demand. Each borrower solves:

$$\max_{C_0, \{C_1\}_{y_1}, B_0^b, \{\xi\}_{y_1}} U(C_0) + \beta \mathbb{E}[V(C_1)],$$

where $V(C_1) = \max_{\xi \in \{0,1\}} \left\{ \xi U\left(C_1^D\right) + (1-\xi) U\left(C_1^{ND}\right) \right\}$ and ξ is an indicator for default for every realization y_1. If a borrower decides to repay, $\xi = 0$, and, if he decides to default, $\xi = 1$.

The budget constraint at $t = 0$ for a borrower is:

$$C_0 = y_0 + q_0^b B_0^b,$$

where I use the subscript b to denote that q_0^b is the per unit price perceived by a borrower, and that B_0^b is his demand for credit. For a given realization of income y_1, the budget constraints at $t = 1$ when a borrower chooses to repay and when it chooses to default are, respectively:

$$\begin{aligned} C_1^{ND} &= y_1 - B_0^b \\ C_1^D &= \min\{y_1, m\} \end{aligned}$$

The problem of a borrower can be decomposed into a) his default decision and b) his choice of B_0^b, and can be solved backwards. First, given an amount B_0^b borrowed at $t = 0$, he chooses optimally when to default ex-post. Second, taking as given his default choice, he chooses B_0^b optimally.

Default decision At $t = 1$, a borrower solves the problem $\max_{\xi \in \{0,1\}} \left\{ \xi U\left(C_1^D\right) + (1-\xi) U\left(C_1^{ND}\right) \right\}$, given B_0^b. Since flow utility is strictly monotonic, this problem is equivalent to $\max_{\{D,ND\}} \left\{ C_1^D, C_1^{ND} \right\}$. The optimal default decision is characterized as a threshold on the realization of y_1. When y_1 is high, it is optimal not to default, but when y_1 is sufficiently low, it is preferable to default than to repay the loan.

Figure (1) shows graphically the default problem at $t = 1$. The upper envelope of the default and repayment options determines the optimal consumption choice given B_0^b. The 45^o degree line is shown for reference. To simplify notation, I implicitly assume for the rest of the analysis that $\underline{y_1} < m$.

Formally, the optimal default decision is:

$$\begin{cases} \text{if } y_1 < m + B_0^b & \text{Default} \quad \xi = 1 \\ \text{if } y_1 \geq m + B_0^b & \text{No Default} \quad \xi = 0 \end{cases} \tag{2}$$

[14] There exists a large empirical literature that estimates the EIS parameter in different environments. Calibrations with $\psi > 1$ have become standard; see Campbell (2003) and references therein.

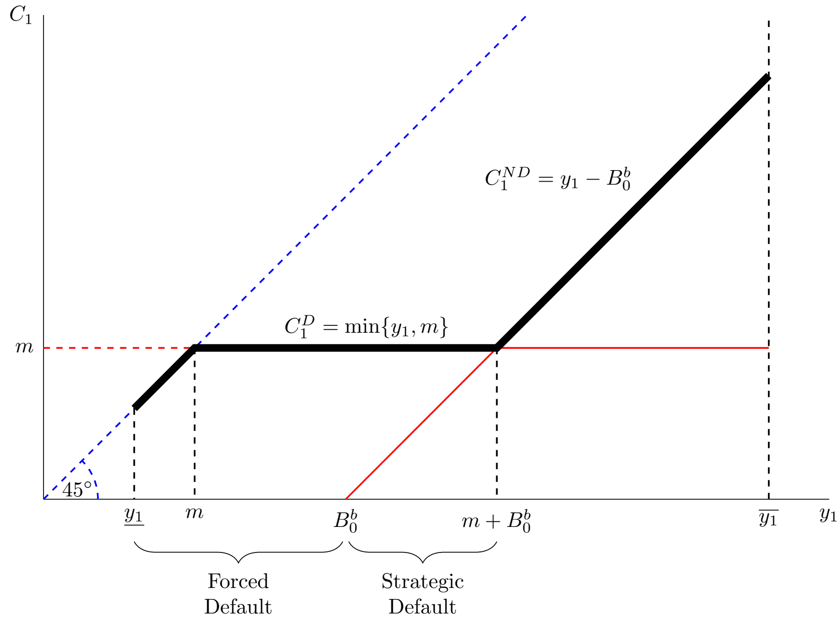

Figure 1: Optimal default decision given B_0^b

The default threshold is determined by the indifference condition between the amount to be repaid B_0^b and the amount transferred to lenders $y_1 - m$. I assume that an indifferent borrower decides not to default. Given the default decision, the fraction of borrowers that defaults in equilibrium is deterministic and given by $F\left(m + B_0^b\right)$. Note that when $\underline{y_1} > m + B_0^b$ there is no default at all in equilibrium.

This model incorporates both "forced default" and "strategic default". Forced default occurs when a borrower does not have enough resources to fully pay back its debt, that is, $B_0^b > y_1$. Strategic default happens when borrowers have enough resources to fully repay but they decide not to do it; this occurs when $m + B_0^b > y_1 > B_0^b$. The probability that borrowers default strategically in this model is given by $F\left(m + B_0^b\right) - F\left(B_0^b\right)$. Neither strategic nor forced default occur under the standard Arrow-Debreu assumptions.

Choice of B_0^b After characterizing the default decision given a value of B_0^b, I now solve for the optimal demand for B_0^b. We must incorporate the optimal default decisions when evaluating $\mathbb{E}\left[V\left(C_1\right)\right]$. The problem to solve is then:

$$\max_{B_0^b} U\left(y_0 + q_0^b B_0^b\right) + \beta\left[\int_{\underline{y_1}}^{m} U\left(y_1\right) dF\left(y_1\right) + \int_{m}^{m+B_0^b} U\left(m\right) dF\left(y_1\right) + \int_{m+B_0^b}^{\overline{y_1}} U\left(y_1 - B_0^b\right) dF\left(y_1\right)\right], \tag{3}$$

Under assumptions 2 and 3, the problem of a borrower is convex, so the following first-order condition fully characterizes his choice of B_0^b, which is positive under assumption 1:

$$q_0^b U'\left(y_0 + q_0^b B_0^b\right) = \beta \int_{m+B_0^b}^{\overline{y_1}} U'\left(y_1 - B_0^b\right) dF\left(y_1\right), \tag{4}$$

where I have used the envelope condition for the default decision at $t = 1$. I refer to equation (4) as the loan demand. Intuitively, the left hand side of (4) represents the benefit of issuing an additional unit of B_0^b at $t = 1$ — it is given by the amount raised per unit at $t = 0$, q_0^b, valued at the marginal utility $U'\left(C_0\right)$. The right hand side of (4) represents the marginal cost of repaying the debt — given by the marginal utility when the payment is due. This cost is only paid in those states in which borrowers do not default; debt imposes no effective costs in those states in which it doesn't have to be repaid.

The following proposition summarizes the properties of loan demand.

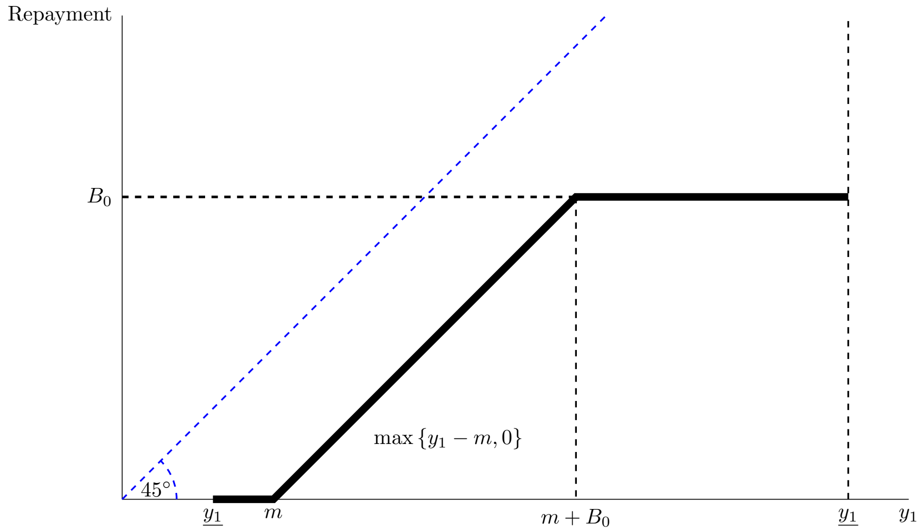

Figure 2: Repayment to lenders given B_0

Proposition 1. (Loan demand properties) *Under assumptions 1 to 3, the properties of loan demand are:*

a) Borrowers' loan demand is strictly positive, i.e., $B_0^b > 0$.

b) Loan demand is upward sloping, i.e., $\frac{\partial q_0^b}{\partial B_0^b} > 0$.

c) Loan demand shifts downwards with m, i.e., $\frac{\partial q_0^b}{\partial m} < 0$.

Assumptions 1 and 3 guarantee that loan demand is always positive and upward sloping. The statement in proposition 1c is also intuitive. When m is larger, borrowers default less frequently, reducing the effective marginal cost of an additional unit of debt, which has to be repaid less often. However, this does not change its marginal benefit. It is then straightforward to conclude that loan demand has to be higher at a given price q_0^b (or, alternatively, a lower price q_0^b is required to borrow the same amount B_0^b).

Lenders' problem

Lenders are competitive and they must break even in equilibrium, so the return on the expected loan must equal $1+r^*$. Because they are risk neutral, by arbitrage, the loan supply price per unit of debt q_0^l, which directly determines the credit spread, is pinned down by the following expression:

$$q_0^l B_0^l = \frac{\int_m^{m+B_0^l} (y_1 - m)\, dF(y_1) + B_0^l \int_{m+B_0^l}^{\overline{y_1}} dF(y_1)}{1+r^*} \tag{5}$$

I refer to equation (5) as the loan supply. When there is default, lenders are repaid the residual amount $\max\{y_1 - m, 0\}$ after borrowers keep m. Figure (2) graphically shows the repayment to lenders. The upper envelope between $\max\{y_1 - m, 0\}$ and B_0^l represents the effective repayment to lenders.

The following proposition summarizes the properties of loan supply.

Proposition 2. (Loan supply properties) *The properties of loan supply are:*

a) The credit spread is positive, i.e., $r > r^*$.

b) Loan supply is downward sloping, i.e., $\frac{\partial q_0^l}{\partial B_0^l} < 0$.

c) Loan supply shifts downwards with m, i.e., $\frac{\partial q_0^l}{\partial m} < 0$.

Proposition 2a merely states that lenders require a spread to account for the possibility of default. Proposition 2b implies that the required spread by lenders increases in the amount of credit. This occurs because the amount recovered per unit in the case of default is lower. Finally, 2c shows that, for given level of borrowing B_0^l, a lender is willing to lend at a lower interest rate (a higher level of q_0^l) when m is lower. Intuitively, the recovery rate is larger in default states, so in equilibrium lenders are willing to lend at cheaper rates.

Equilibrium borrowing and rates

Given a value for m and the other exogenous variables of the model, the equilibrium values of credit and interest rates are determined by the intersection of loan supply and loan demand. I use q_0 and B_0 to denote the equilibrium values of price and credit demand. Unfortunately, even in the simplest environment, it is impossible to solve for q_0 and B_0 in closed form. Figure (3) represents the equilibrium graphically for a given level of m — see the appendix for the exact parametrization used to draw such figure.

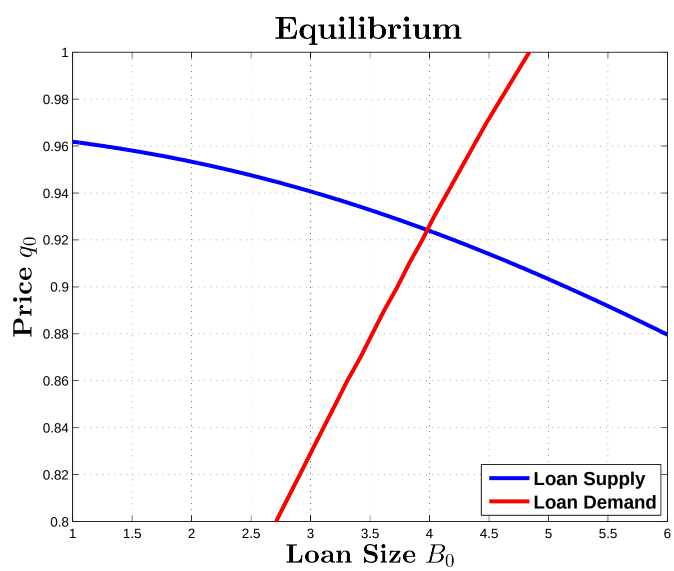

Figure 3: Equilibrium

Proposition 3. (Equilibrium properties) Under assumptions 1 to 3:

a) There exists a unique equilibrium.

b) Equilibrium credit rates $\frac{1}{q_0} = 1 + r$ *increase with m, i.e.,* $\frac{dq_0}{dm} < 0$ *or, equivalently,* $\frac{dr}{dm} > 0$.

c) The marginal change in equilibrium credit B_0 *with respect to m is ambiguous, i.e.,* $\frac{dB_0}{dm} \gtreqless 0$.

The results in proposition 3 follow naturally from the results in propositions 1 and 2. First, because supply and demand have different slopes, showing existence and uniqueness of the equilibrium is straightforward. Second, when m is larger, lenders are only willing to lend at a higher rate and borrowers increase their demand: both effects imply that equilibrium rates must be higher — this is the result in proposition 3b. Because lenders face a constant riskless rate, all changes in equilibrium rates in this model correspond to credit spreads. Third, when m is larger, borrowers increase their demand for a credit at any given rate at the same time that lenders restrict credit at a given rate: hence the value of B_0 can increase or decrease in equilibrium — this is the result in proposition 3c.

3 Optimal bankruptcy exemption m^*

Having characterized the equilibrium given an exemption m, now I turn to the problem of determining the welfare maximizing exemption m^*. Social welfare in this model is given by the indirect utility of the borrowers, since lenders make zero profits in equilibrium. So far, there is no mechanism within the model to pin down m.

Social welfare, denoted by $W(m)$, is a function of m, the single policy parameter. It is given by:

$$W(m) \equiv U(y_0 + q_0 B_0) + \beta \left[\int_{\underline{y_1}}^{m} U(y_1)\, dF(y_1) + \int_{m}^{m+B_0} U(m)\, dF(y_1) + \int_{m+B_0}^{\overline{y_1}} U(y_1 - B_0)\, dF(y_1) \right] \tag{6}$$

The optimal bankruptcy exemption m^* can be found as:

$$m^* = \arg\max_m W(m),$$

subject to the equilibrium conditions (4) and (5), which determine B_0 and q_0, rewritten here for completeness:

$$q_0 B_0 = \frac{\int_m^{m+B_0} (y_1 - m)\, dF(y_1) + B_0 \int_{m+B_0}^{\overline{y_1}} dF(y_1)}{1 + r^*} \tag{3}$$

$$q_0 U'(y_0 + q_0 B_0) = \beta \int_{m+B_0}^{\overline{y_1}} U'(y_1 - B_0)\, dF(y_1), \tag{4}$$

The welfare assessment of marginal changes in the level of exemptions and the characterization of the optimal exemption m^* are the main results of this paper.

Marginal intervention Proposition 4 characterizes the welfare effects of a marginal change in the bankruptcy exemption.

Proposition 4. (Marginal effect on welfare)

a) The marginal welfare change induced by a marginal change in the bankruptcy exemption m is given by:

$$\frac{dW}{dm} = U'(C_0) \frac{dq_0}{dm} B_0 + \int_m^{m+B_0} \beta U'\left(C_1^D\right) dF(y_1) \tag{7}$$

b) The marginal welfare change induced by a marginal change in the bankruptcy exemption m, expressed as a fraction of $t = 0$ consumption units, can be written as:

$$\frac{\frac{dW}{dm}}{U'(C_0)\, C_0} = -\Lambda \varepsilon_{\tilde{r},m} + \frac{1}{m} \frac{\Pi_m \left\{C_1^D\right\}}{C_0}, \tag{8}$$

where $\Lambda \equiv \frac{q_0 B_0}{y_0 + q_0 B_0}$, $\varepsilon_{\tilde{r},m} \equiv \frac{d \log(1+r)}{dm}$ *and* $\frac{\Pi_m\{C_1^D\}}{C_0} \equiv \int_m^{m+B_0} \frac{C_1^D}{C_0} \frac{\beta U'(C_1^D)}{U'(C_0)} dF(y_1)$.[15]

The derivation of (7) relies on equations (2) and (4) as envelope conditions for the default decision and the choice of B_0, respectively.

On the one hand, a marginal increase in the exemption m makes borrowing more expensive through a reduction in the equilibrium price change $\frac{dq_0}{dm}$, which affects the total amount of credit B_0. This change is valued by borrowers at their $t = 0$ marginal utility $U'(C_0)$ — this reduction in borrowing is the marginal cost of a more lenient bankruptcy mechanism. On the other hand, a marginal increase in m increases the resources that borrowers can keep when they

[15] The subscript m in $\Pi_m\{\cdot\}$ denotes the fact that the integration is made only over those states in which borrowers income exceeds the exemption level (equivalently, when lenders receive a positive repayment).

default and have enough resources to repay. Averaging over the pertinent realizations of y_1 and weighting this gain by the marginal utility $\beta U'\left(C_1^D\right)$ at each state, the marginal welfare gain is determined by $\beta \int_m^{m+B_0} U'\left(C_1^D\right) dF\left(y_1\right)$.

Equation (8) provides further intuition by expressing the welfare change as a money-metric — dividing by $U'\left(C_0\right)$ — and normalizing by initial consumption C_0. The term Λ represents borrowers' leverage ratio.[16] The term $\varepsilon_{\tilde{r},m}$ denotes the derivative of the equilibrium log interest rate with respect to the bankruptcy exemption. Given proposition 3, $\varepsilon_{\tilde{r},m}$ is strictly positive. Note that $\varepsilon_{\tilde{r},m}$ captures the complete equilibrium response of interest rates to an exemption change.

$\Pi\left\{C_1^D\right\}$ is the price from the borrower's perspective at $t=0$ of a claim that pays borrowers' consumption only in default states with positive repayments — those in which $y_1 > m$. $\frac{\Pi_m\left\{C_1^D\right\}}{C_0}$ express this price in relative terms to current consumption C_0: this is a measure of the marginal benefit for borrowers of increased leniency.

I refer to the ratio $\frac{\Pi_m\left\{C_1^D\right\}}{C_0}$ as the "price-consumption" ratio. It is determined by the product of two terms. First, it can be high when the ratio of marginal utilities $\frac{U'\left(C_1^D\right)}{U'\left(C_0\right)}$ is high or, second, it can be high when consumption growth in those states $\frac{C_1^D}{C_0}$ is also high. Both reasons are not disjoint and tightly linked in the CRRA or Epstein-Zin cases, as discussed below.

Optimal bankruptcy exemption Although the planner's problem is not necessarily convex in m, at an interior optimum, the optimal bankruptcy exemption can be characterized by $\frac{dW}{dm} = 0$. Non-convexities may arise due to changes in the default decision.[17] See the appendix for a discussion of the conditions required for convexity and uniqueness of the optimal exemption.

Proposition 5. **(Optimal bankruptcy exemption)** *The optimal exemption m^* — given in units of the consumption good, i.e., dollars — is characterized by:*

$$m^* = \frac{\frac{\Pi_m\left\{C_1^D\right\}}{C_0}}{\Lambda \varepsilon_{\tilde{r},m}}, \tag{9}$$

where $\Lambda \equiv \frac{q_0 B_0}{y_0 + q_0 B_0}$, $\varepsilon_{\tilde{r},m} \equiv \frac{d\log(1+r)}{dm}$ *and* $\frac{\Pi_m\left\{C_1^D\right\}}{C_0} \equiv \int_m^{m+B_0} \frac{C_1^D}{C_0} \frac{\beta U'\left(C_1^D\right)}{U'\left(C_0\right)} dF\left(y_1\right)$.

The expression for m^* optimally trades off the marginal benefit of increasing consumption in default states (numerator) against the marginal cost of restricting access to credit (denominator). When m^* is high, borrowers face higher interest rates, which makes borrowing less profitable, at the cost of improved insurance when declaring bankruptcy. A low m^* makes ex-ante borrowing less costly while making bankruptcy more painful.

A high value for m^* is optimal when Λ and $\varepsilon_{\tilde{r},m}$ are large. Intuitively, if equilibrium interest rates are very sensitive to increasing the bankruptcy exemption, making default more attractive by increasing m^* is very costly in terms of curtailed access to credit; this effect is modulated by the amount borrowed Λ. A low value for m^* is optimal when $\frac{\Pi_m\left\{C_1^D\right\}}{C_0}$, the normalized welfare gain of a marginally higher exemption is large. Note that, although equations (8) and (9) must hold at the optimum, they do not provide a characterization as a function of primitives, because all right hand side variables are endogenous.

[16]A more standard definition of leverage is a debt-to-equity ratio, that is, $L \equiv \frac{q_0 B_0}{y_0}$. The variable Λ is simply a monotonic transformation of L: $\Lambda = \frac{1}{\frac{1}{L}+1}$.

[17]In general, the convexity properties of problems with extensive margin choices (in this case, defaulting vs. repaying) depend on the shape of probability distributions; in this case, on the shape of $F\left(\cdot\right)$.

CRRA/logarithmic utility

To build further intuition, I now assume that borrowers have constant relative risk aversion utility (CRRA) utility, defined by $U(C) = \frac{C^{1-\gamma}}{1-\gamma}$, where $\gamma \equiv -C\frac{U''(C)}{U'(C)}$. Assuming a particular utility specification only affects directly the price-consumption ratio $\frac{\Pi_m\{C_1^D\}}{C_0}$ — the marginal cost of increased leniency $\Lambda\varepsilon_{\tilde{r},m}$ does not depend directly on the utility function.

We can thus write:

$$\frac{\Pi_m\left\{C_1^D\right\}}{C_0} = \beta\int_m^{m+B_0^b}\left(\frac{C_1^D}{C_0}\right)^{1-\gamma} dF(y_1)$$

Therefore, an increase in leniency m increases C_1^D and has both discount rate and cash flow effects. When the CRRA coefficient γ is greater than one, the discount rate effect $\left(\frac{C_1^D}{C_0}\right)^{-\gamma}$ dominates — this is the standard parametrization, see Campbell (2003) — but, if γ is less than one the consumption growth term $\frac{C_1^D}{C_0}$ dominates. With logarithmic utility ($\gamma = 1$) both effects exactly cancel out. When $C_1^D = m < C_0$, the natural case,[18] the marginal loss derived from harsher bankruptcy policies is increasing in the risk aversion parameter γ.

The forces that determine the price-consumption ratio are the same that determine the price-dividend ratio in standard consumption based asset pricing models — see Campbell (2003) for a review. In the classic Lucas (1978) model, price dividend ratios are constant and equal to β for borrowers with logarithmic utility. That result directly applies here, but instead of β we have $\beta\pi_m$, since we are interested in the price dividend ratio of an "asset" which only pays in default states in which investors consume the bankruptcy exemption.

Because of its importance as a benchmark, I present in the following corollary the optimal exemption for borrowers with logarithmic utility.

Corollary. (Optimal bankruptcy exemption w/ logarithmic utility) *When borrowers have logarithmic utility, the optimal bankruptcy exemption m^* can be written as:*

$$m^* = \frac{\beta\pi_m}{\Lambda\varepsilon_{\tilde{r},m}},$$

where $\Lambda \equiv \frac{q_0B_0}{y_0+q_0B_0}$, $\varepsilon_{\tilde{r},m} \equiv \frac{d\log(1+r)}{dm}$, β is the borrowers' discount factor and π_m is the probability that a borrower defaults at the same time that his income exceeds the bankruptcy exemption.

In the case of logarithmic utility, the price-consumption ratio can be written as:

$$\frac{\Pi_m\left\{C_1^D\right\}}{C_0} = \beta\pi_m,$$

where $\pi_m \equiv \int_m^{m+B_0^b} dF(y_1)$ is the unconditional probability that a borrower consumes the bankruptcy exemption. It can alternatively be written as: $\pi_m = \pi_D \cdot \pi_{m|D}$, the unconditional probability of default (π_D) times the conditional probability that a borrower's income exceeds the bankruptcy exemption $\pi_{m|D}$. Hence, when borrowers have log utility and lenders always recover some fraction of the amount owed, instead of calculating the price-consumption ratio, the planner would only need to know the probability of default to assess the marginal gain of increasing the bankruptcy exemption.

[18] Combining the loan demand equation (4) with the fact that $U'\left(C_1^{ND}\right) < U'\left(C_1^D\right)$, it is the case that $q_0U'(C_0) < \beta U'\left(C_1^D\right)$, for any value of C_1^D. Using the CRRA specification, this expression can be written as $\frac{C_1^D}{C_0} < \left(\frac{\beta}{q_0}\right)^{\frac{1}{\gamma}}$. Hence, a sufficient condition for $\frac{C_1^D}{C_0} < 1$ is that borrowers are sufficiently impatient, that is, $\beta(1+r) < 1$. This is one of the natural parametrizations that guarantee that assumption 1 holds.

Logarithmic utility provides a benchmark calibration for many models, since it implies plausible values for both relative risk aversion and the elasticity of intertemporal substitution. Note that when borrowers' risk aversion is larger than unity and $m < C_0$, the logarithmic utility formula provides a lower bound for the optimal exemption, because the welfare loss of defaulting with harsh penalties will be larger.

I conclude this section with two remarks.

Remark. (Sufficient statistics) Three observable variables: leverage, the equilibrium interest rate sensitivity and the price-consumption ratio, suffice to determine the optimal exemption, independently of the rest of the structure of the model. For log utility borrowers, the price-consumption simplifies to the probability of default (when lenders recovery is everywhere positive). For instance, the distribution of income shocks or the level of interest rates only affect m^* through these sufficient statistics.

The logic behind these sufficient statistics is similar to the one behind the CAPM, in which the beta of an asset becomes sufficient to determine expected returns. It is also similar to the logic behind consumption based asset pricing models, in which the consumption process, independently of how it is generated, is sufficient to determine asset prices and expected returns.

Remark. (Security design interpretation) The key tradeoff in this paper can be interpreted as a security design problem. Assume that borrowers can choose between two securities, both priced in fair terms. They can issue either a) a noncontingent bond, or b) a bundle formed by the same noncontingent bond as well as a put option with a given strike (the optimally chosen default decision). Given their income process, one of these two securities will generally be preferred; the choice of m adjusts parametrically between both contracts and m^* selects the optimal traded security.

4 Extensions

I now extend the baseline model in multiple dimensions. The goal of this section is to understand which departures from the environment already studied do modify the expressions for the optimal bankruptcy exemption and how. For that, I focus on the most natural extensions — I discuss some others in section 4.9. To ease the exposition, I analyze every extension separately, omit equilibrium definitions and keep the discussion of regularity conditions to a minimum.

4.1 Endogenous income: elastic labor supply and effort choice

In the baseline model, borrowers' income is determined exogenously. However, it is sometimes argued that harsh bankruptcy systems further reduce borrowers' welfare by distorting labor supply decisions. It is true that seizing a fraction of borrowers' labor income has a negative effect on labor supply, however, this is a margin which does not modify the expression for the optimal exemption. Similarly, bankruptcy procedures may distort ex-ante effort choices by borrowers'. Distortions in effort or labor supply choices due to changes in leniency do not change the expression for the optimal exemption, since borrowers make those decisions optimally.

To capture these concerns, I modify in two dimensions the baseline model. First, I allow borrowers to have a meaningful labor supply choice in both periods. Second, I assume that the distribution of income $F(Y_1;a)$ is a differentiable function of an effort choice a, made at $t=0$ by borrowers. Whether the effort choice is observable or not (a form of moral hazard) is irrelevant for the results.

Borrowers can work at exogenously given wages w_0 and w_1. Borrowers' flow utility is now given by $U(C,N;a)$, where N denotes hours worked and a is the quantity of effort exerted. Utility is decreasing and convex in N and

a and there is no Inada condition for hours worked — this allows borrowers to choose $N=0$. For now, I make no assumptions about separability between consumption, leisure or effort. For simplicity, I assume that, in case of bankruptcy, all labor income is transferred from borrowers to lenders.

The problem solved by borrowers is now:

$$\max_{C_0,\{C_1\}_{y_1},B_0^b,\{\xi\}_{y_1},N_0,\{N_1\}_{y_1},a} \quad U\left(C_0,N_0;a\right)+\beta\mathbb{E}_a\left[V\left(C_1,N_1\right)\right]$$

$$\text{s.t.} \qquad C_0=y_0+w_0N_0+q_0^bB_0^b; \quad C_1^{ND}=y_1+w_1N_1^{ND}-B_0^b; \quad C_1^D=\min\left\{y_1,m\right\},$$

where $V\left(C_1,N_1\right)=\max_{\xi\in\{0,1\}}\left\{\xi\max_{C_1^D,N_1^D}U\left(C_1^D,N_1^D\right)+(1-\xi)\max_{C_1^{ND},N_1^{ND}}U\left(C_1^{ND},N_1^{ND}\right)\right\}$. I define three regions depending on the realization of y_1. First, no default, denoted by ND. Second, default with positive recovery by lenders, in which borrowers consume $C_1^D=m$, denoted by D_m. Third, default with no recovery by lenders, in which borrowers consume $C_1^D=y_1$, denoted by D_y. In the bankruptcy regions, investors have no incentive to work, so they always choose $N_1=0$. This is always optimal in the region D_m and it is also optimal here even when $y_1<m$ because I have assumed that all labor income is garnished.[19]

The characterization of the default region is similar to the baseline model. First, let's define the $t=1$ indirect utility of borrowers after choosing optimally consumption and working hour as $\tilde{V}\left(y_1;B_0^b,w_1\right)$, that is:

$$\tilde{V}\left(y_1;B_0^b,w_1\right)\equiv\max U\left(C_1^{ND},N_1^{ND}\right) \text{ s.t. } \quad C_1^{ND}=y_1+w_1N_1^{ND}-B_0^b \tag{10}$$

Given y_1, a static consumption-leisure choice characterizes borrowers' labor supply, that is: $w_1\frac{\partial U}{\partial C}\left(C_1^{ND},N_1^{ND}\right)=-\frac{\partial U}{\partial N}\left(C_1^{ND},N_1^{ND}\right)$.

Second, note that the default region is characterized by an indifference condition in $\tilde{y_1}$ such that:

$$U\left(m\right)=\tilde{V}\left(\tilde{y_1};B_0^b,w_1\right)$$

When $y_1\geq\tilde{y_1}$, it is optimal for a borrower to repay, but when $y_1<\tilde{y_1}$, it is optimal to default — see figure (A.2) in the appendix for a graphical representation.

Given the optimal default decision, borrowers' behavior is characterized by three additional optimality conditions. First, an Euler equation for borrowing:[20]

$$q_0^b\frac{\partial U}{\partial C}\left(C_0,N_0;a\right)=\beta\int_{ND}\frac{\partial U}{\partial C}\left(C_1^{ND},N_1^{ND}\right)dF\left(y_1;a\right)$$

Second, an optimal effort choice:

$$\frac{\partial U}{\partial a}\left(C_0,N_0;a\right)+\beta\int V\left(C_1,N_1\right)f_a\left(y_1;a\right)dy_1=0$$

Third, an optimal consumption-leisure choice at $t=0$:

$$w_0\frac{\partial U}{\partial C}\left(C_0,N_0;a\right)=-\frac{\partial U}{\partial N}\left(C_0,N_0;a\right)$$

From the optimality conditions, it is easy to show that changes in bankruptcy m modify both borrower's consumption, labor supply and effort choices. In particular, when m increases, borrowers' labor supply is lower because the default region, in which no labor is supplied, grows. The pricing equation for lenders is identical to the one in the baseline model.

[19] This assumption can be easily relaxed by allowing borrowers to keep a fraction or all of their labor income. The normative results would not be affected.

[20] Where I have used the envelope condition in (10), implying that $\frac{d\tilde{V}}{dB_0^b}=-\frac{\partial U}{\partial C}\left(C_1^{ND},N_1^{ND}\right)$.

Proposition 6. (Endogenous income)

a) *The marginal welfare change from varying the optimal exemption m when labor supply is endogenous and borrowers have an effort choice is given by:*

$$\frac{dW}{dm} = \frac{\partial U}{\partial C_0}(C_0, N_0; a)\frac{dq_0}{dm}B_0 + \beta \int_{D_m} \frac{\partial U}{\partial C_1}\left(C_1^D, 0\right) dF(y_1; a) \tag{11}$$

b) The optimal exemption m^* when labor supply is endogenous and borrowers have an effort choice is given by:

$$m^* = \frac{\frac{\Pi_m\left\{C_1^D\right\}}{C_0}}{\Lambda \varepsilon_{\tilde{r},m}} \tag{12}$$

where $\Lambda \equiv \frac{q_0 B_0}{y_0 + q_0 B_0}$, $\varepsilon_{\tilde{r},m} \equiv \frac{d\log(1+r)}{dm}$ *and* $\frac{\Pi_m\left\{C_1^D\right\}}{C_0} \equiv \beta \int_{D_m} \frac{m}{C_0} \frac{\frac{\partial U}{\partial C_1}\left(C_1^D, 0\right)}{\frac{\partial U}{\partial C_0}(C_0, N_0; a)} dF(y_1; a)$.

Equations (11) and (12) are almost identical to their counterparts with exogenous income (7) and (9). Unless utility is separable between consumption, leisure and effort, in which case both expressions are identical, now the marginal benefit of the bankruptcy exemption also depends on the values of N and a through the value of marginal utility. There are rich literatures studying the separability properties of the utility function, e.g., see Attanasio and Weber (1989) or Aguiar and Hurst (2007), but separable utility of consumption is often seen as a reasonable approximation.

The intuition behind proposition (12) is simple: because borrowers choose optimally their labor supply and effort, changes in these variables due to changes in m are "enveloped away" and become second-order — the same argument applies to any other static endogenous variable. This is an important takeaway of this paper: we only need to measure consumption to determine the welfare consequences of bankruptcy policies. The labor supply response to changes in leniency is irrelevant once the planner accounts for the consumption response.[21]

4.2 Bankruptcy costs/non-pecuniary losses

In the baseline model, bankruptcy acts as costless transfer of resources between borrowers and lenders. However, bankruptcy may entail pecuniary costs, as legal fees, or may impose non-pecuniary costs on borrowers. For instance, borrowers may feel "bad" about not repaying, perhaps because of stigma, or social pressure. Alternatively, lenders can hound bankrupt borrowers. I now incorporate both features, which make bankruptcy costlier, into the model.

First, I assume that a fraction $\varepsilon \in [0,1]$ of resources transferred from borrowers to lenders in bankruptcy is lost. This captures the possibility bankruptcy costs. Second, I assume that borrowers have state dependent utility. In particular, I assume that the utility of a bankrupt borrower who consumes C units of the consumption good in bankruptcy is given by $U(\phi C)$, where $\phi \in [0,1]$. Alternative forms of state dependent utility deliver similar insights. For both mechanisms to have any bite, it is important to assume that renegotiation is impossible, perhaps because it is too costly. Otherwise, borrowers and lenders would renegotiate to avoid entering in bankruptcy, avoiding the pecuniary or non-pecuniary losses.

Hence, now borrowers maximize:

$$\max_{C_0, \{C_1\}_{y_1}, B_0^b, \{\xi\}_{y_1}} U(C_0) + \beta \mathbb{E}[V(C_1)],$$

[21] This insight is analogous to one made regarding asset prices in the consumption based asset pricing literature: (marginal utility of) consumption is all we need to price any asset. See Campbell (2003) or Cochrane (2005) for forceful defenses of this approach.

where $V(C_1) = \max_{\xi\in\{0,1\}}\left\{\xi U\left(\phi C_1^D\right) + (1-\xi)U\left(C_1^{ND}\right)\right\}$. The default region is characterized as in the baseline model. The optimal default decision is given by:

$$\begin{cases} \text{if } y_1 < \phi m + B_0^b & \text{Default} \quad \xi = 1 \\ \text{if } y_1 \geq \phi m + B_0^b & \text{No Default} \quad \xi = 0 \end{cases}$$

With the exception of the change in the default region, the expression for loan demand is analogous to the one in the baseline model, that is:

$$q_0^b U'\left(y_0 + q_0^b B_0^b\right) = \beta \int_{\phi m + B_0^b}^{\overline{y_1}} U'\left(y_1 - B_0^b\right) dF(y_1)$$

Loan supply now incorporates the loss associated with bankruptcy, therefore:

$$q_0^l B_0^l = \frac{\varepsilon \int_m^{\phi m + B_0^l} (y_1 - m)\, dF(y_1) + B_0^l \int_{\phi m + B_0^l}^{\overline{y_1}} dF(y_1)}{1 + r^*}$$

Proposition 7. (Bankruptcy costs/non-pecuniary losses)

a) *The marginal welfare change from varying the optimal exemption m when bankruptcy is costly and borrowers face non-pecuniary losses is given by:*

$$\frac{dW}{dm} = U'(C_0)\frac{dq_0}{dm}B_0 + \beta \int_m^{\phi m + B_0} \phi U'\left(\phi C_1^D\right) dF(y_1) \tag{13}$$

b) The optimal exemption m^* when bankruptcy is costly and borrowers face non-pecuniary losses is given by:

$$m^* = \frac{\frac{\Pi_m\left\{C_1^D\right\}}{C_0}}{\Lambda \varepsilon_{\tilde{r},m}},$$

where $\Lambda \equiv \frac{q_0 B_0}{y_0 + q_0 B_0}$, $\varepsilon_{\tilde{r},m} \equiv \frac{d\log(1+r)}{dm}$ *and* $\frac{\Pi_m\left\{C_1^D\right\}}{C_0} \equiv \beta \int_m^{\phi m + B_0} \frac{\phi C_1^D}{C_0} \frac{U'\left(\phi C_1^D\right)}{U'(C_0)} dF(y_1)$.

The presence of non-pecuniary costs of default modifies the default region and the expression for the price-consumption ratio, but the formula for the optimal bankruptcy exemption remains unchanged. This conclusion extends to any form of state dependent utility. Likewise, the existence of bankruptcy costs only affects m^* through changes in the sensitivity of credit spreads. These results imply that the presence of additional costs in bankruptcy states only affects m through the sufficient statics described in the baseline model.

4.3 Internalized price response

In the baseline model, borrowers take prices as given, failing to internalize the effect of their own borrowing on the interest rate that they face. Hence, borrowers perceive that they can borrow as much as they wish at a given rate. The price-taking assumption may be reasonably within some range, e.g., credit card rates are constant within some borrowing levels, but it may be not reasonable for all forms of borrowing.

I now assume that borrowers internalize how their demand for credit determines the interest rate they face. Instead of taking prices as given, borrowers know that $q_0^b = q_0^l\left(B_0^b\right)$, where q_0^l is the loan supply function. To avoid a degenerate solution in which borrowers always declare bankruptcy and consume $\min\{y_1, m^*\}$, I keep the assumption that bankruptcy entails either pecuniary or non-pecuniary costs or both. Intuitively, when bankruptcy is costless, a planner who chooses m^* optimally can replicate the perfect insurance outcome between risk neutral lenders and risk averse borrowers. Any form of bankruptcy costs breaks down this extreme result. See the appendix for a more detailed discussion.

Under the new assumptions, loan demand is characterized by borrowers' first order condition, given by:

$$U'(C_0)\left[q_0^b+\frac{\partial q_0^l}{\partial B_0^b}B_0^b\right]=\beta\int_{\phi m+B_0^b}^{\overline{y_1}}U'\left(C_1^{ND}\right)dF(y_1), \tag{14}$$

After substituting $q_0^l\left(B_0^b\right)$, equation (14) directly determines the equilibrium level of B_0. Because $\frac{\partial q_0^l}{\partial B_0^b}<0$, borrowers perceive a lower marginal benefit of borrowing, since they acknowledge that their interest rate increases with every marginal unit of B_0^b. The price elasticity of loan supply now matters at the margin.[22] The marginal cost of borrowing is unchanged. The additional term in (14) acts as a wedge in the borrowing decision. Its presence implies that the envelope theorem cannot be used to eliminate the derivative $\frac{dB_0}{dm}$ from the welfare calculations.

Proposition 8. (Internalized price response)

a) *The marginal welfare change from varying the optimal exemption m when borrowers internalize price responses is given by:*

$$\frac{dW}{dm}=U'(C_0)\left[\frac{dq_0}{dm}B_0+\frac{dB_0}{dm}\frac{\partial q_0^l}{\partial B_0^b}B_0\right]+\beta\int_m^{\phi m+B_0}\phi U'\left(\phi C_1^D\right)dF(y_1)$$

b) The optimal exemption m^* when borrowers internalize price responses is given by:

$$m^*=\frac{\frac{\Pi_m\left\{C_1^D\right\}}{C_0}}{\Lambda\left(\varepsilon_{\tilde{r},m}+\varepsilon_{B_0,m}\hat{\varepsilon}_{q_0^l,B_0}\right)},$$

where $\Lambda\equiv\frac{q_0B_0}{y_0+q_0B_0}$, $\varepsilon_{\tilde{r},m}\equiv\frac{d\log(1+r)}{dm}$, $\varepsilon_{B_0,m}\equiv\frac{\frac{dB_0}{B0}}{dm}$, $\hat{\varepsilon}_{q_0^l,B_0}\equiv\frac{\frac{\partial q_0^l}{q_0}}{\frac{\partial B_0^b}{B_0}}$ *and* $\frac{\Pi_m\left\{C_1^D\right\}}{C_0}\equiv\beta\int_m^{\phi m+B_0}\frac{\phi C_1^D}{C_0}\frac{U'\left(\phi C_1^D\right)}{U'(C_0)}dF(y_1)$.

Because borrowers' are not price takers, a new term appears in the expression for optimal policies. This term, given by the product of the semi-elasticity of credit with an exemption change $\frac{\frac{dB_0}{B0}}{dm}$, which can be positive or negative, with the loan supply elasticity $\frac{\frac{\partial q_0^l}{q_0}}{\frac{\partial B_0^b}{B_0}}$, which is strictly negative — see proposition 2—, arises due to the wedge in the borrowers' Euler equation (14). The planner must now know both the exemption semi-elasticity $\varepsilon_{B_0,m}$ and the supply elasticity $\hat{\varepsilon}_{q_0^l,B_0}$ to implement the optimal bankruptcy exemption.

Hence, it is not clear theoretically that the optimal bankruptcy is more or less lenient when borrowers are not price-takers. For instance, when $\frac{dB_0}{dm}>0$, the planner internalizes that an increase in m endogenously increases equilibrium borrowing B_0, which at the same reduces q_0, curtailing access to credit. In that case, a lower m^* compared to the price-taking case is optimal. The opposite would occur when $\frac{dB_0}{dm}<0$.

4.4 Epstein-Zin utility

The results of the baseline model extend directly to non-expected utility specifications. I analyze here the Epstein-Zin case to disentangle the effects of risk aversion versus intertemporal substitution.[23] While risk aversion plays an important role in pricing the marginal benefit of a bankruptcy exemption increase, intertemporal substitution plays an important role shaping the sensitivity of credit demand to interest rates.

[22] Note that equation (14) can be rewritten as $U'(C_0)q_0^b\left[1+\hat{\varepsilon}_{q_0^l,B_0}\right]=\beta\int_{m+B_0^b}^{\overline{y_1}}U'(y_1-B_0)dF(y_1)$, where $\hat{\varepsilon}_{q_0^l,B_0}\equiv\frac{\frac{\partial q_0^l}{q_0^b}}{\frac{\partial B_0^b}{B_0^b}}$.

[23] There is scope to extend the results in this paper to more general Kreps-Porteus preferences or other types of non-expected utility preferences.

Borrowers' utility is now given by:

$$V_0 = \left[\left(1-\hat{\beta}\right) C_0^{1-\frac{1}{\psi}} + \hat{\beta} \left(\mathbb{E}\left[C_1^{1-\gamma} \right] \right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}} \right]^{\frac{1}{1-\frac{1}{\psi}}},$$

where the parameter γ is the coefficient of relative risk aversion and ψ represents the elasticity of intertemporal substitution for a given non-stochastic consumption path. After imposing the optimal default condition, and substituting budget constraints, the problem solved by a borrower is:

$$\max_{B_0^b} \left[\begin{array}{c} \left(1-\hat{\beta}\right) \left(y_0 + q_0^b B_0^b\right)^{1-\frac{1}{\psi}} + \\ \hat{\beta} \left(\int_{\underline{y_1}}^{m} (y_1)^{1-\gamma} dF(y_1) + \int_{m}^{m+B_0^b} (m)^{1-\gamma} dF(y_1) + \int_{m+B_0^b}^{\overline{y_1}} \left(y_1 - B_0^b\right)^{1-\gamma} dF(y_1) \right)^{\frac{1-\frac{1}{\psi}}{1-\gamma}} \end{array} \right]^{\frac{1}{1-\frac{1}{\psi}}}$$

Borrowers' behavior, under analogous regularity conditions to the baseline model, is fully characterized by the first order condition in B_0^b:

$$q_0^b (C_0)^{-\frac{1}{\psi}} = \beta Q^{\gamma-\frac{1}{\psi}} \int_{m+B_0^b}^{\overline{y_1}} \left(y_1 - B_0^b\right)^{-\gamma} dF(y_1), \tag{15}$$

where I normalize, without loss of generality, $\beta \equiv \frac{\hat{\beta}}{1-\hat{\beta}}$ and denote the certainty equivalent at $t=1$ of consumption by Q — see appendix A for the exact expression. I also show in the appendix that the sensitivity of credit demand to interest rates is controlled by ψ.

The loan supply curve remains invariant. The formulas for the marginal welfare change and the optimal bankruptcy exemption are identical to those in the baseline model.

Proposition 9. (Epstein-Zin utility)

a) *The marginal welfare change from varying the optimal exemption m when borrowers have Epstein-Zin preferences is given by:*

$$\frac{dW}{dm} = V_0^{\frac{1}{\psi}} \left[\left(1-\hat{\beta}\right) (C_0)^{-\frac{1}{\psi}} \frac{dq_0}{dm} B_0 + \hat{\beta} Q^{\frac{\gamma-\frac{1}{\psi}}{1-\gamma}} \int_{m}^{m+B_0^b} (m)^{-\gamma} dF(y_1) \right]$$

b) The optimal exemption m^* when borrowers have Epstein-Zin preferences is given by:

$$m^* = \frac{\frac{\Pi_m\left\{C_1^D\right\}}{C_0}}{\Lambda \varepsilon_{\tilde{r},m}}$$

where $\Lambda \equiv \frac{q_0 B_0}{y_0 + q_0 B_0}$, $\varepsilon_{\tilde{r},m} \equiv \frac{d\log(1+r)}{dm}$ *and* $\frac{\Pi_m\left\{C_1^D\right\}}{C_0} \equiv \left(\frac{Q}{C_0}\right)^{\gamma-\frac{1}{\psi}} \beta \int_{m}^{m+B_0^b} \left(\frac{C_1^D}{C_0}\right)^{1-\gamma} dF(y_1)$.

All the intuition from the baseline model extends directly. Varying preferences only changes the expression for the price-consumption ratio $\frac{\Pi_m\left\{C_1^D\right\}}{C_0}$, which is given by the product of two terms. The first term $\beta \int_{m}^{m+B_0^b} \left(\frac{m}{C_0}\right)^{1-\gamma} dF(y_1)$ is the same as in the CRRA case studied above. The second term $\left(\frac{Q}{C_0}\right)^{\gamma-\frac{1}{\psi}}$ can be interpreted as a correction to the discount factor. Intuitively, when $Q > C_0$, the natural case for borrowers, higher values of $\gamma - \frac{1}{\psi}$ increase the benefit of higher exemptions.[24] As expected, when $\gamma = \frac{1}{\psi}$, the second term of the price-consumption ratio cancels out, recovering the CRRA formulas.

[24] In this static model, the difference $\gamma - \frac{1}{\psi}$ measures the relative preference for smooth consumption within states versus within periods. In dynamic models, a term analogous to $\left(\frac{Q}{C_0}\right)^{\gamma-\frac{1}{\psi}}$ would capture the variation in future continuation utility, which depends on borrowers' preference for early versus late resolution of uncertainty — see Campbell (2003), Bansal and Yaron (2004) or Backus, Routledge and Zin (2005).

An intuitive way to interpret the results with Epstein-Zin preferences, despite the complexity of the problem, is to say that elasticity of intertemporal substitution is important to determine the demand for credit — because ψ controls directly the sensitivity of the loan demand to interest rates — but risk aversion, through the classic CRRA term, and the preference for early versus later resolution of uncertainty, play an important role to determine the marginal changes in welfare induced by varying exemptions. These results relate to the findings by Tallarini (2000), who shows that the EIS determines quantity dynamics but risk aversion is the relevant parameter for asset valuations.

4.5 Multiple arbitrary contracts

I now relax the baseline model in two dimensions. First, borrowers can now invest in a finite – for simplicity — number of contracts, indexed by $j = 1, \dots, J$. Second, every contract j has an arbitrary payoff scheme $z_j(y_1)$, which can take positive or negative value depending on the realization of y_1.[25]

I make three assumptions about lender behavior. First, lenders can fully commit to repay at $t = 1$. Second, lenders can fully observe borrowers portfolio allocations at $t = 0$ to price all assets in a risk neutral way. Third, in case of default, all claimants split the the amount transferred from borrowers to lenders proportionally to their claims. This implies that the price of a given contract j will depend on the amount purchased of all J contracts. If $J = 1$ and $z_1(y_1) = 1$, this extension collapses to the baseline model. If there are as many Arrow-Debreu assets as realizations of y_1, this formulation naturally nests the complete markets benchmark.[26]

Borrowers' budget constraints now read as:

$$C_0 = y_0 + \sum_{j=1}^{J} q_{0j}^b B_{0j}^b; \qquad C_1^{ND} = y_1 - \sum_{j=1}^{J} z_j(y_1) B_{0j}^b; \qquad C_1^D = \min\{y_1, m\}$$

The default decision given a choice of B_{0j}^b parallels the one in the baseline model. Borrowers default in those realizations y_1 in which $C_1^D > C_1^{ND}$. A set of equalities given by $\min\{y_1, m\} = y_1 - \sum_{j=1}^{J} z_j(y_1) B_{0j}^b$ defines three regions depending on the realization of y_1: no default (ND), default with positive recovery by lenders (D_m) — borrowers consume $C_1^D = m$ —, and default with no recovery by lenders (D_y) — borrowers consume $C_1^D = y_1$. See the appendix for an explicit characterization.

I illustrate a possible scenario graphically in figure (4), which is the counterpart of figure (1) for the baseline model. This generalization shows that the results of the baseline do not really on borrowers defaulting for low realizations of y_1. Both forced and strategic default also occur in this case too.

Lenders jointly price all assets in the following way:

$$q_{0j}^l B_{0j}^l = \frac{\eta_j \int_{D_m} (y_1 - m)\, dF(y_1) + \int_{ND} z(y_1) B_{0j}^l dF(y_1)}{1 + r^*}, \quad \forall j,$$

where $\eta_j \equiv \frac{z_j(y_1) B_{0j}^l}{\sum_{j=1}^{J} z_j(y_1) B_{0j}^l}$ is the recovery ratio in bankruptcy. The assumption that lenders recover repayments proportionally in case of default determines the shape of η_j — alternative assumptions would not change the sufficient statistic characterization.

Borrowers' behavior, under analogous regularity conditions to the baseline model, is fully characterized by the first order condition for every asset j:

$$q_{0j}^b U'(C_0) = \beta \int_{ND} z_j(y_1) U'\left(C_1^{ND}\right) dF(y_1), \quad \forall j$$

[25] In the exposition, I assume that both B_{0j} and q_j are strictly positive. This can be easily relaxed.

[26] Starting from the complete markets benchmark and imposing mild regularity conditions, it is easy to show that $\left.\frac{dW}{dm}\right|_{m=0} < 0$. That is, if the economy is already at the full insurance benchmark, allowing for bankruptcy is welfare reducing. Intuitively, the contingent contracts for states in which borrowers default cease to be traded, so $\frac{dq_{0j}}{dm} \approx -\infty$.

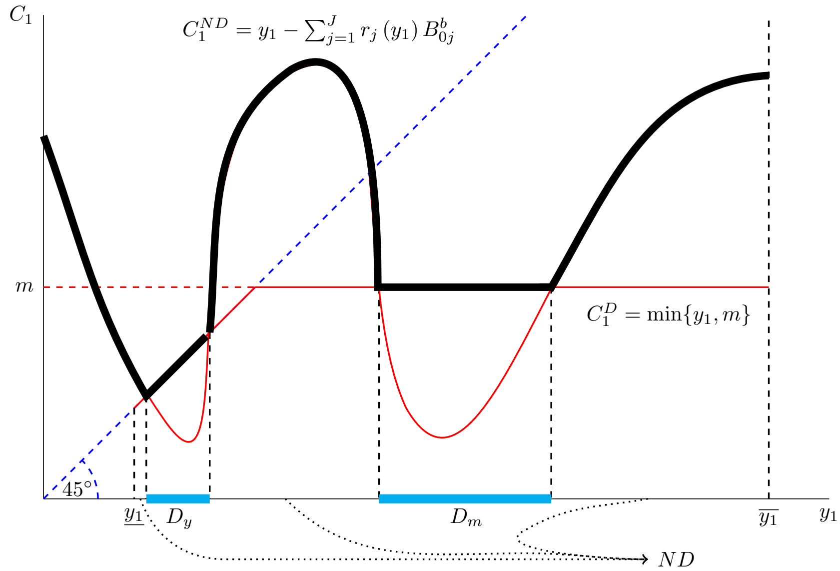

Figure 4: Optimal default decision with multiple arbitrary contracts

Proposition 10. (Multiple arbitrary contracts)

a) *The marginal welfare change from varying the optimal exemption m when borrowers can trade J assets with arbitrary payoffs is given by:*

$$\frac{dW}{dm} = U'(C_0) \sum_{j=1}^{J} \frac{dq_{0j}}{dm} B_{0j} + \beta \int_{D_m} U'\left(C_1^D\right) dF(y_1) \tag{16}$$

b) The optimal exemption m^* when borrowers can trade J assets with arbitrary payoffs is given by:

$$m^* = \frac{\frac{\Pi_m\left\{C_1^D\right\}}{C_0}}{\sum_{j=1}^{J} \Lambda_j \varepsilon_{\tilde{r}_j,m}} \tag{17}$$

where $\Lambda_j \equiv \frac{q_{0j}B_{0j}}{y_0 + \sum_{j=1}^{J} q_{0j}B_{0j}}$, $\varepsilon_{\tilde{r}_j,m} \equiv \frac{d\log(1+r_j)}{dm}$ *and* $\frac{\Pi_m\left\{C_1^D\right\}}{C_0} \equiv \beta \int_D \frac{C_1^D}{C_0} \frac{U'(C_1^D)}{U'(C_0)} dF(y_1)$.

The main difference with respect to the baseline case is that the welfare gain of having harsher penalties is now given by a weighted average of the price sensitivities with respect to the bankruptcy exemption. The weight given to an asset j is determined by the fraction of the amount purchased as a function of total consumption. Intuitively, the welfare associated with higher rates is larger for those assets which account for a larger fraction of borrowers portfolios. The presence of multiple assets with arbitrary payoffs only affects the welfare cost of harsher penalties through changes in the default region.

4.6 Heterogeneous borrowers

The baseline model assumes that all borrowers are ex-ante symmetric, although they may have a different income realizations ex-post. Here, I assume that borrowers are heterogeneous in multiple dimensions and that they are distributed according to G. I index borrowers by i.

I make two assumptions. First, lenders are able to price each (group of) borrower(s) i independently.[27] Second, I restrict the planner to use a single constant bankruptcy exemption. If the planner can set individual exemptions

[27] There is scope to extend the results to cases in which adverse selection, perhaps in the form of credit rationing, as in Stiglitz and Weiss (1981), is present. If rationing happened to be an efficient outcome, the results of this section would remain unchanged.

conditioning in individual characteristics, he would replicate the outcome of the baseline model for each type of borrower. Nonlinear exemptions would also be welfare improving, but I rule them out for tractability reasons.

The planner maximizes a weighted sum of individual utilities, so social welfare W is now given by:

$$W = \int \lambda(i) W(i) \, dG(i),$$

where welfare weights are given by $\lambda(i)$ are welfare weights and $dG(i)$ represents the distribution of the characteristic i in the population.

Proposition 11. (Heterogeneous borrowers)

a) *The marginal welfare change from varying the optimal exemption m when when borrowers are ex-ante heterogeneous is given by:*

$$\frac{dW}{dm} = \int \lambda(i) \left[U_i'(C_{0i}) \frac{dq_{0i}}{dm} B_{0i} + \beta_i \int_m^{m+B_{0i}} U_i'\left(C_{1i}^D\right) dF_i(y_{1i}) \right] dG(i)$$

b) The optimal exemption m^* when borrowers are ex-ante heterogeneous is given by:

$$m^* = \frac{\int h_i \frac{\Pi_{m,i}\left\{C_{1i}^D\right\}}{C_{0i}} dG(i)}{\int h_i \varepsilon_{\tilde{r}_i,m} \Lambda_i dG(i)}$$

where $h_i \equiv \lambda(i) U_i'(C_{0i})$, $\Lambda_i \equiv \frac{q_{0i} B_{0i}}{y_{0i} + q_{0i} B_{0i}}$, $\varepsilon_{\tilde{r},m} \equiv \frac{d \log(1+r)}{dm}$ *and* $\frac{\Pi_m\left\{C_1^D\right\}}{C_0} \equiv \int_m^{m+B_{0i}} \frac{C_1^D}{C_{0i}} \frac{\beta_i U_i'\left(C_{1i}^D\right)}{U_i'(C_{0i})} dF(y_1)$.

As expected, the optimal exemption now contains a weighted average of marginal costs and benefits in the cross-section of borrowers. The weights h_i are a combination of the social welfare weight and the marginal utility of consumption. When $h_i = 1$ for all borrowers, which can be implemented when the planner is utilitarian and he has access to ex-ante lump sum redistribution,[28] the optimal m^* can written as:

$$m^* = \frac{\mathbb{E}_G\left[\frac{\Pi_{m,i}\left\{C_{1i}^D\right\}}{C_{0i}}\right]}{\mathbb{E}_G\left[\varepsilon_{\tilde{r}_i,m}\right] \mathbb{E}_G\left[\Lambda_i\right] + \mathbb{C}ov_G\left[\varepsilon_{\tilde{r}_i,m}, \Lambda_i\right]},$$

where $\mathbb{E}_G[\cdot]$ denotes cross-sectional averages. This expression shows that if the optimal bankruptcy exemption is to be implemented with cross sectional averages, it is important to correct in the denominator for $\mathbb{C}ov_G\left[\varepsilon_{\tilde{r}_i,m}, \Lambda_i\right]$ or at least be aware of the possible bias.

4.7 Bankruptcy exemptions contingent on aggregate risk

Should bankruptcy exemptions be more lenient in recessions? If, for any reason, agents use contracts which do not condition on aggregate shocks, it will be optimal to set bankruptcy exemptions which vary with the realization of the aggregate state.

The income realization for borrowers is now given by $y_1 = \omega + y_{1\omega}$, where ω is the realization of the aggregate shock, which can take a finite of values $\omega \in \Omega$ with probability $p(\omega)$, and $\tilde{y}_{1\omega}$ denotes the idiosyncratic endowment shock. The distribution of idiosyncratic shocks $F_\omega(\cdot)$ can vary with ω, freeing the correlation patterns between aggregate and idiosyncratic shocks. Lenders remain risk neutral — we can think of well diversified investors. The planner now sets exemptions m_ω contingent on value of the aggregate shock; in practice, the key difference between

[28] A similar assumption is used in Davila (2014).

aggregate and idiosyncratic shocks in this model is that the planner can condition bankruptcy policies on the aggregate shocks.

After imposing the optimal default condition for every state ω, and substituting budget constraints, the problem solved by a borrower is:

$$\max_{B_0} U\left(y_0+q_0B_0\right)+\beta\sum_{\omega}p\left(\omega\right)\left[\begin{array}{c}\int_{\underline{y_{1\omega}}}^{m_\omega}\left(y_1\right)^{1-\gamma}dF_\omega\left(y_{1\omega}\right)\\+\int_{m_\omega}^{m_\omega+B_0^b}\left(m_\omega\right)^{1-\gamma}dF_\omega\left(y_{1\omega}\right)\\+\int_{m_\omega+B_0^b}^{\overline{y_{1\omega}}}U\left(y_1-B_0^b\right)dF_\omega\left(y_{1\omega}\right)\end{array}\right]$$

The first order condition, which determines loan demand, is given by:

$$q_0U'\left(y_0+q_0B_0\right)=\beta\sum_{\omega}p\left(\omega\right)\left[\int_{m_\omega+B_0^b}^{\overline{y_{1\omega}}}U'\left(y_1-B_0^b\right)dF_\omega\left(y_1\right)\right]$$

Equation (4) is a direct generalization of the baseline case

Proposition 12. (Bankruptcy exemption contingent on aggregate risk)

a) *The marginal welfare change from varying the optimal state contingent exemption m_ω is given by:*

$$\frac{\partial W}{\partial m_\omega}=U'\left(C_0\right)\frac{dq_0}{dm_\omega}B_0+\beta p\left(\omega\right)\int_{m_\omega}^{m_\omega+B_0}U'\left(C_1^D\right)dF_\omega\left(y_{1\omega}\right),\qquad\forall\omega$$

b) The optimal state contingent exemptions m_ω^* are given by:

$$m_\omega^*=\frac{p\left(\omega\right)\frac{\Pi_{m_\omega}\left\{C_1^D\right\}}{C_0}}{\Lambda\varepsilon_{\tilde{r},m_\omega}},\quad\forall\omega\tag{18}$$

where $\Lambda\equiv\frac{q_0B_0}{y_0+q_0B_0}$, $\varepsilon_{\tilde{r},m_\omega}\equiv\frac{d\log(1+r)}{dm_\omega}$ *and* $\frac{\Pi_{m_\omega}\left\{C_1^D\right\}}{C_0}\equiv\int_{m_\omega}^{m_\omega+B_0}\frac{C_1^D}{C_0}\frac{\beta U'\left(C_1^D\right)}{U'\left(C_0\right)}dF_\omega\left(y_{1\omega}\right)$.

The set of optimal state contingent exemptions is jointly determined by (18). The marginal benefit of increasing the bankruptcy exemption in state ω is now proportional to the probability $p\left(\omega\right)$ of that state occurring. The expression for the marginal cost embeds this change in probabilities in the term $\varepsilon_{\tilde{r},m_\omega}$.

Equation (18) sheds light on whether optimal exemptions should be countercyclical or not by looking at how price-consumption ratios and interest rate sensitivities vary across states ω. If bankruptcies increase in recessions, as occurs naturally in this model, $\Pi_{m_\omega}\left\{C_1^D\right\}$ will be high — under natural parametrizations, for instance, log utility —, which calls for countercyclical penalties as long as interest rate sensitivities $\varepsilon_{\tilde{r},m_\omega}$ do not vary significantly across aggregate states.[29]

4.8 Dynamics

Finally, I extend the results to a dynamic environment. Time is discrete and there is a finite horizon: $t=0,\ldots,T$.[30] Risk averse borrowers trade a single noncontingent one period contract, i.e., short-term debt.[31] Lenders are risk neutral and demand an interest rate $1+r^*$. The income process for y_t has a Markov structure.

[29] This is a natural assumption for risk neutral lenders.

[30] As long as the distribution of wealth across borrowers remains bounded, the results of this section extend naturally to an infinite horizon economy.

[31] There is scope to extend the results of this section to assets with longer maturity. This should not change the relevant tradeoffs that determine the optimal m^*.

In case of default, borrowers debt is fully discharged but they can't borrow in the defaulting period and recover access to credit markets with a stochastic probability α. These are natural assumptions in this environment, as shown for instance in Chatterjee et al. (2007), for US chapter 7 bankruptcy. I restrict the planner to optimally choose a constant exemption level.

In addition to setting m optimally, the planner could improve welfare by optimizing in the amount of debt discharged or the degree of enforcement regarding the exclusion of borrowers from financial markets. Note that I also rule out renegotiation between the parties.

Hence, borrowers maximize:

$$\max \mathbb{E}\left[\sum_{t=0}^{T} \beta^{t} U\left(C_{t}\right)\right],$$

where their consumption when they don't default is given by $C_t^{ND} = y_t + q_t B_t - B_{t-1}$. In the default period, borrowers consume $C_t^D = \min\{y_1, m\}$ and when excluded from credit markets, they simply consume their income y_1.

From a $t = 0$ perspective, we can write borrowers' problem recursively as:

$$V_{ND,0}\left(0, y_0; m\right) = \max_{B_0} U\left(y_0 + q_0 B_0\right) + \beta \mathbb{E}\left[\max\left\{V_{ND,1}\left(B_0, y_1; m\right), V_{D,1}\left(y_1; m\right)\right\}\right],$$

where $V_{ND,1}\left(B_0, y_1; m\right)$ and $V_{D,1}\left(B_0, y_1; m\right)$, which determine indirect utility at $t = 1$ for borrowers' who repay and default, respectively, are explicitly characterized in the appendix. Note that $V_{D,1}$ is independent of B_0 because of the assumption that debts are fully discharged.

I characterize default at $t = 1$ and borrowing at $t = 0$. The results for other periods follow directly. First, note that borrowers default decision is determined by an indifference condition, given by:

$$V_{ND,1}\left(B_0, \hat{y}_1; m\right) = V_{D,1}\left(\hat{y}_1; m\right) \tag{19}$$

For my purposes, the shape of the default region, which is formed by a union of intervals, is irrelevant. As in previous extensions, I denote the repayment region by ND, the default region in which lenders are partially repaid by D_m and the default region in which borrowers consume all their income by D_y. Equation (19) fully captures the forward looking nature of the default decision.

Consequently, the borrowing decision B_0 is characterized by borrowers' first condition:

$$q_0 U'\left(y_0 + q_0 B_0\right) = \beta \int_{ND} U'\left(y_1 + q_1 B_1 - B_0\right) dF\left(y_1\right)$$

Default and borrowing decisions for all others periods are characterized in the same way.

Proposition 13. (Dynamics)

a) *The marginal welfare change from varying the optimal exemption m in the dynamic model is given by:*

$$\frac{dW}{dm} = \sum_{t=0}^{T-1} \mathbb{E}_{ND}\left[\beta^t U'\left(C_t^{ND}\right) \frac{dq_t}{dm} B_t\right] + \sum_{t=1}^{T} \mathbb{E}_{D_m}\left[\beta^t U'\left(C_t^{D}\right)\right]$$

b) The optimal exemption m^* in the dynamic model is given by:

$$m^* = \frac{\sum_{t=1}^{T} \frac{\Pi_m\left\{C_t^D\right\}}{C_0}}{\sum_{t=0}^{T-1} \Pi_{ND}\left\{g_t \Lambda_t \varepsilon_{\tilde{r}_t, m}\right\}},$$

where $\mathbb{E}_{ND}[\cdot]$ denotes the t= 0 expectation of being in a no default state in which borrowers have access to credit, $\mathbb{E}_{D_m}[\cdot]$ denotes the $t = 0$ expectation of defaulting in a given state, $\Lambda_t \equiv \frac{q_t B_t}{y_t + q_t B_t - B_{t-1}}$, $g_t \equiv \frac{C_t}{C_0}$, $\varepsilon_{\tilde{r}_t, m} \equiv \frac{-d\log(1+r_t)}{dm}$ and $\frac{\Pi_m\left\{C_t^D\right\}}{C_0} \equiv \mathbb{E}_{D_m}\left[\frac{C_t^D}{C_0} \frac{\beta^t U'\left(C_t^D\right)}{U'\left(C_0\right)}\right]$ and $\Pi_{ND}\{x\} \equiv \mathbb{E}_{ND}\left[\frac{\beta^t U'\left(C_t^D\right)}{U'\left(C_0\right)} x\right]$.

As expected, the optimal bankruptcy exemption becomes a weighted average across periods/states of the marginal benefits/losses. The numerator of m^*, which captures the marginal benefit of increased leniency, is the price of a claim to an asset that pays consumption in the default states in which investors consume the bankruptcy exemption as a ratio of initial consumption. The denominator, which captures marginal losses, is the $t = 0$ price of weighted credit spread sensitivities with respect to the bankruptcy exemption — with weights given by the product of consumption growth g_t and leverage ratios Λ_t. Intuitively, the optimal formula in the dynamic model trades off marginal welfare gains and losses using borrowers' stochastic discount factor. Consequently, we can interpret the equations obtained in the baseline static setup as valid in the steady state of a dynamic model.

4.9 Additional remarks

I would like to conclude the theoretical analysis with several remarks. Analytical results supporting the first five are available under request.

First, as long as the shape of the assets/contracts traded does not vary with the level of exemptions, there is no loss of generality in assuming an exogenous contract space for the purpose of characterizing the optimal exemption formula. This seems a reasonable assumption for debt contracts, which are widespread in economies with both high and low levels of exemptions. Intuitively, when deriving $\frac{dW}{dm}$, we do not have to account for the response of the shape of the contract.

Second, the precise condition that determines the decision of declaring bankruptcy does not affect the formula for the optimal exemption. For instance, whether borrowers' go bankrupt after high or low income realizations does not change the formula for optimal exemptions. In dynamic models, there exists an option value to wait before defaulting, which complicates the characterization of the default decision but does not modify the formula for the optimal exemption. Likewise, other punishments associated with not repaying, like reduced access to credit markets or wage garnishments, do not modify the formula for optimal exemptions, independently of whether those punishments are set optimally or not.

Third, allowing borrowers to stop repayments (default) without declaring bankruptcy does not affect the formula for the optimal exemption. That possibility will be priced by lenders and borrowers will exercise it optimally, which is crucial to keep the optimal formula for exemptions unchanged.

Fourth, the approach of this paper can handle the case of risk averse lenders. Although the set of sufficient statistics in that case (leverage ratios, credit spread sensitivities, etc.) remains the same, the optimal exemption formula now solves a more complex co-insurance problem.

Fifth, externalities associated with bankruptcy or mistakes made by borrowers (internalities) introduce welfare wedges that can potentially change the optimal exemption formula. Only externalities which are not be priced by lenders would modify the optimal exemption formula, with larger externalities calling for lower exemptions. A planner that acknowledges that borrowers make mistakes when borrowing — for instance, because they are too optimistic or pessimistic about their income realizations — would also tilt the optimal exemption.

Finally, market power on the lender side and adverse selection may generate wedges that potentially modify the optimal exemption formula. Depending on the environment, these may be relevant, so there is further scope to extend the results to handle those cases.

5 Calibration

To show the applicability of my results in practice, I first calibrate the optimal exemption to US data. Subsequently, I study the magnitude of the welfare gains generated by marginal changes in the bankruptcy exemption. In both cases, I use a yearly calibration.

Optimal exemption

Because it entails minimal informational requirements, I calibrate the optimal exemption for the baseline model assuming that borrowers have CRRA utility, which nests the log utility calibration.[32] Section 3 shows that the optimal exemption for CRRA utility borrowers is given by:

$$m^* = \frac{\beta \pi_m \left(\frac{C_1^D}{C_0}\right)^{1-\gamma}}{\Lambda \varepsilon_{\tilde{r},m}} \tag{20}$$

Therefore, four observable variables and two preference parameters, the discount factor β and the coefficient of relative risk aversion γ, need to be calibrated.

Calibrating Λ, β, and γ is straightforward. To calibrate Λ, I use the same average ratio of unsecured debt to personal disposable income as Livshits, MacGee and Tertilt (2007), which is $\frac{q_0 B_0}{y_0} = 8.4\%$. This value implies that $\Lambda = 0.0775$.[33] I use $\beta = 0.96$ as the annual discount factor: this is a standard choice. I also adopt $\gamma = 10$ for the baseline calibration for risk aversion,[34] but I also discuss the log utility case.

Finding appropriate values for π_m , C_1^D/C_0, and $\varepsilon_{\tilde{r},m}$ requires further work. First, note that π_m can be written as the product of the unconditional probability of default π_D with the probability of keeping the full bankruptcy exemption conditional on defaulting $\pi_{m|D}$. To determine π_D, I use the average probability of filling for chapter 7 bankruptcy, also from Livshits, MacGee and Tertilt (2007), which is 0.008. This choice should raise no concerns. I determine $\pi_{m|D}$ by using the fact that roughly 90% of bankruptcies are filed as "no-asset" bankruptcies — see Lupica (2012). This implies that 10% of bankrupt borrowers fully exhaust their exemption.

Second, to calibrate the sensitivity of the credit spread on unsecured credit with respect to the bankruptcy exemption, I follow the results found in Gropp, Scholz and White (1997), who estimate the effect on interest rates of changes in bankruptcy exemptions over time and across states. Although they find that $\varepsilon_{\tilde{r},m}$ is strictly positive, as predicted by the theoretical model, the effect is only statistically significant from zero for part of the population of borrowers. A value of $\varepsilon_{\tilde{r},m} = 2.5 \cdot 10^{-7}$ is within the reasonable range for the average response of credit spreads in the population. This choice implies that increasing the bankruptcy exemption by a hundred thousand dollars would increase the equilibrium credit spread by 250 basis points.

Finally, I use $C_1^D/C_0 = 0.9$ for the change in consumption by bankrupt borrowers. A 10% reduction in consumption may be considered as a large change, but it is within the range of variation documented in Filer and Fisher (2005) using PSID data on changes in food consumption for bankrupt individuals. I purposefully pick a relatively large number for

[32]The results would be identical in a model with Epstein-Zin preferences in which the certainty equivalent of consumption Q, defined in (23), equals C_0.

[33]In actual economies, a substantial fraction of borrowing is collateralized. Under the assumption that collateralized borrowing is fully secured, which implies that the interest rate charged is not sensitive to m, equation (17) implies that the denominator of the optimal exemption only has to account for the fraction of unsecured credit. If collateralized lending is not fully secured, the optimal exemption found in this section should be adjusted downwards,

[34]The preferred parametrization, standard in the asset pricing literature, guarantees that $\psi = \frac{1}{\gamma} > \Lambda$, as required by assumption 3.

C_1^D/C_0 to magnify the effect of risk aversion; otherwise, the results when $C_1^D/C_0 \approx 1$ would be nearly identical to those in the log utility case for sensible choices of γ.

Table 1 summarizes the choices of parameters and variables used in the baseline calibration.

Parameter/Variable		Value	Parameter/Variable		Value
β	Discount factor	0.96	π_D	Probability of default	0.008
C_1^D/C_0	Consumption change	0.9	$\pi_{m\|D}$	Probability of no-asset bankruptcy	0.1
Λ	Leverage	0.0775	$\varepsilon_{\tilde{r},m}$	Credit spread sensitivity	$2.5 \cdot 10^{-7}$

Table 1: Calibrated variables

Using equation (20), table 2 presents the optimal exemption m^* for different values of the risk aversion coefficient γ:

$\gamma = 1$ (log)	$\gamma = 5$	$\gamma = 10$	$\gamma = 20$	$\gamma = 50$
$39,638$	$60,416$	$\mathbf{102,314}$	$293,435$	$6,922,079$

Table 2: Optimal exemption m^* (measured in dollars)

This analysis concludes that the actual average bankruptcy exemption across US states, of roughly $70,000$ dollars, is very close to the optimal value implied by the model. The preferred calibration suggests that the optimal exemption should approximately be $100,000$ dollars. Varying the level of risk aversion can dramatically change the value of m^*.

On the size of welfare gains

The results regarding m^* show that the optimal exemption should be slightly larger than the average exemption across US states. Given that result, a natural question is how large are the welfare gains from changing m at the current level of exemptions. To answer this question, I focus again in the CRRA case and define $\sigma(m) \equiv \frac{\frac{dW}{dm}}{U'(C_0)C_0}$ as the welfare gain, as a fraction of initial consumption, of increasing the bankruptcy exemption starting from a level m. The value of $\sigma(m)$, as shown and discussed at length in section 3, is given by:

$$\sigma(m) = -\Lambda\varepsilon_{\tilde{r},m} + \frac{1}{m}\beta\pi_m\left(\frac{C_1^D}{C_0}\right)^{1-\gamma}$$

For the purposes of this exercise, I freely use $\sigma(m)$ to assess changes of any magnitude, given the difficulty of calibrating $\frac{dW}{dm}$ for all values of m and then integrating over those to actually compute welfare changes.[35] Hence, the results will be more precise for small interventions and should be interpreted otherwise with caution.

Using the same set of parameters described in table 2, and starting from the average exemption of $m = 70,000$ dollars, I now show in table 3 the welfare gains associated to a $10,000$ dollars increase in m.

$\gamma = 1$ (log)	$\gamma = 5$	$\gamma = 10$	$\gamma = 20$	$\gamma = 50$
-0.0084%	-0.0027%	$\mathbf{0.0089}\%$	0.062%	1.897%

Table 3: Welfare gain/loss ($10,000$ dollars change) measured as a fraction of initial consumption

[35] Note that equation (7) is valid for all values of the bankruptcy exemption m. If we could obtain empirical counterparts of the different sufficient statistics for all values of m, we could calculate the exact welfare gain or loss induced by a non-marginal change in m by integrating $\frac{dW}{dm}$ over the relevant range of exemptions. The calibration becomes an approximation only because I impose *constant* values for the sufficient statistics.

Therefore, table 3 provides the welfare gains, measured as a percent of initial consumption C_0, from increasing the bankruptcy exemption m by $10,000$ dollars when $m = 70,000$, for the same parametrization used to derive m^*. As expected, the magnitude of the welfare gain grows with the value of m^*, when $m^* > 70,000$ and viceversa. For instance, for the benchmark calibration, a $10,000$ dollars increase in m increases welfare by 0.0089%. Increasing m by $30,000$ dollars, which would move an economy from the average $70,000$ until the optimum of $100,000$, yields a welfare gain of roughly 0.03%; this measures the total welfare loss incurred by not having the optimal exemption.

6 Conclusion

This paper has characterized the optimal bankruptcy exemption for risk averse borrowers in a model of unsecured credit as a function of sufficient statistics. This paper shows that a simple formula that depends on borrowers' leverage, the sensitivity of the equilibrium credit spread with respect to the level of bankruptcy exemption, the probability of default, and the change in the consumption of bankrupt borrowers is sufficient to characterize the optimal bankruptcy exemption under a wide variety of circumstances. Other features of the environment need not be specified once those variables are known.

Using the theoretical results directly, for the preferred calibration to US data, this paper concludes that the optimal bankruptcy exemption should be approximately $100,000$ dollars. This value is slightly larger than the average exemption level in the US, but it is in same order of magnitude. For the same calibration, moving from the average exemption to the optimal one achieves welfare gains on the order of 0.03% of ex-ante consumption.

The conclusions of this paper have relevant implications for further research for both structural models and reduced form work on bankruptcy. For structural macroeconomic modeling, it would be interesting to understand and report the values of the endogenous variables that this paper has identified as sufficient statistics to determine optimal exemptions. By comparing those sufficient statistics across models, we could understand through which channels different assumptions on primitives affect optimal bankruptcy exemptions. For reduced form empirical work, it seems of vital importance to find consensus values for the sufficient statistics identified in this paper. In particular, it would be valuable to a) find credibly identified estimates of the sensitivity of credit spreads with respect to bankruptcy exemptions, b) measure precisely which fraction of bankrupt borrowers received the whole bankruptcy exemption and c) build precise measures of changes in consumption for bankrupt borrowers. Finally, I have also suggested multiple possibilities for further theoretical research within the framework developed in this paper.

Appendix: Additional figures, proofs and derivations

Equilibrium figure parametrization

Figure (3) plots the market equilibrium, given a level of m, under assumptions 1 to 3. Table 4 contains the parameters chosen to draw figure (3). The distribution $F(\cdot)$ is log-normal.

Preferences	$\gamma = 1$	$\psi = 1$	$\beta = 0.92$	$r^* = 3\%$
Income	$y_0 = 8$	$\mu = 2.7$	$\sigma = 0.5$	
Bankruptcy	$m = 4$	$\varepsilon = 0.9$	$\phi = 1$	

Table 4: Parameters figure (3)

Proofs: Section 2

Assumption 1. (Borrowers borrow)

Under the rest of assumptions made in this section, which guarantee that the borrowers' problem is convex, it is sufficient for B_0^b to be positive that, at the initial endowment, the borrowers' marginal utility increase from a marginal increase in B_0^b is positive. For the rest of the appendix, I define the utility of the borrowers' from a $t = 0$ perspective as function of the amount borrowed B_0^b, taking into account the optimal default decision as $J\left(B_0^b\right)$, that is:

$$J\left(B_0^b\right) \equiv U\left(y_0 + q_0^b B_0^b\right) + \beta \left[\begin{array}{c} \int_{\underline{y_1}}^{m} U(y_1)\, dF(y_1) + \int_{m}^{m+B_0^b} U(m)\, dF(y_1) \\ + \int_{m+B_0^b}^{\overline{y_1}} U\left(y_1 - B_0^b\right) dF(y_1) \end{array} \right] \tag{21}$$

Hence, a sufficient conditions for borrowers' to borrow is that:

$$\left.\frac{dJ}{dB_0^b}\right|_{B_0^b=0} = \frac{U'(y_0)}{1+r^*} - \beta \int_{\underline{y_1}}^{\overline{y_1}} U'(y_1)\, dF(y_1) > 0$$

This can be rewritten as a condition stating that the shadow interest rate of the borrowers' in autarky must be larger than the interest offered by lenders, who do not require a credit spread, since the probability of default at $B_0^b = 0$ equals zero. Formally:

$$\frac{U'(y_0)}{\beta \int_{\underline{y_1}}^{\overline{y_1}} U'(y_1)\, dF(y_1)} > 1 + r^*$$

This condition holds for different combinations of fundamentals. For example, first, when the initial endowment y_0 is sufficiently low. Second, when borrowers' are more impatient than lenders, that is $\beta(1+r^*) < 1$. Third, when expected future income is sufficiently high and not too stochastic (to prevent the precautionary savings effect from dominating).

Assumption 2. (Bounded credit choice)

a) Price-taking borrowers can potentially borrow any given amount at a price q_0^b. However, under this assumption, the optimal solution to (3) for borrowers is to demand $B_0^b = \infty$. This unreasonable behavior can be seen directly from (3), since when $B_0^b \to \infty$, the last term $\int_{m+B_0^b}^{\overline{y_1}} U\left(y_1 - B_0^b\right) dF(y_1)$, which contains the cost of debt, tends to 0 (this is the case as long as $B_0^b \geq \overline{y_1} - m$). Intuitively, borrowers are running a "two-period Ponzi scheme", in which they borrow as much as they can knowing that they won't have to repay in the future. It is easy to observe that lenders will not offer that price q_0^b in equilibrium. Since this happens for any q_0^b, this argument shows that an equilibrium *does not exist* under price taking behavior when the choice set of B_0^b is unrestricted. I rule out this nonexistence problem by assuming that borrowers choice set is the closed interval $\left[\underline{B_0^b}, \overline{B_0^b}\right]$, where $\underline{B_0^b} = 0$ without loss of generality, given assumption 1. Intuitively, we can think that limits on credit card borrowing play the same role of $\overline{B_0^b}$ in this model.

b) The choice of $\overline{B_0^b}$ depends in principle in all other features of the environment. The upper limit $\overline{B_0^b}$ must be chosen so that the interior optimum is the global optimum of the problem. I show in figure (A.1) how $J\left(B_0^b\right)$ looks like for a standard parametrization. In this case, to respect the global convexity from assumption 3, any $\overline{B_0^b}$ in the range 5 to 15 would be valid. Under this assumption, borrowers' use their first order condition to determine loan demand. Note that many form of unsecured credit, e.g. credit cards, come with an associated borrowing limit.

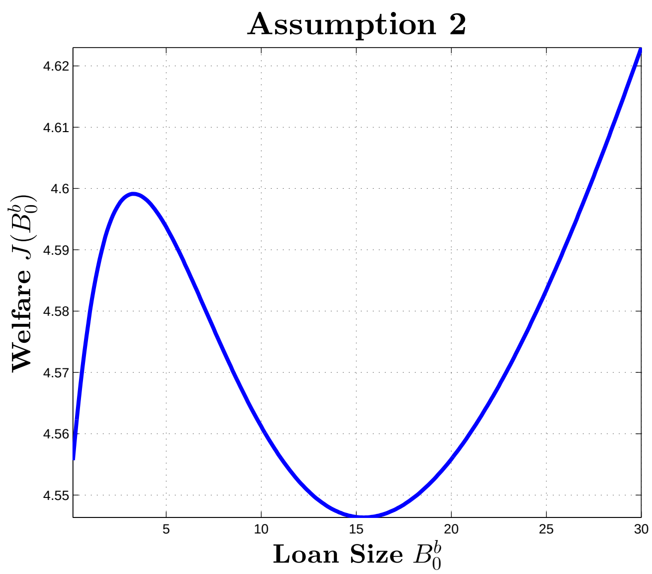

Figure A.1: Optimal default decision given B_0^b

Assumption 3. (Regularity conditions on $F(\cdot)$ and ψ)

a) At $t=0$, borrowers solve $\max_{B_0^b} J\left(B_0^b\right)$, where $J\left(B_0^b\right)$ is defined in equation (21) above. The first order condition to this problem is:

$$J'\left(B_0^b\right) = q_0^b U'\left(y_0 + q_0^b B_0^b\right) - \beta \int_{m+B_0^b}^{\overline{y_1}} U'\left(y_1 - B_0^b\right) dF\left(y_1\right) = 0$$

The second order condition to this problem is:

$$J''\left(B_0^b\right) = \underbrace{\left(q_0^b\right)^2 U''\left(y_0 + q_0^b B_0^b\right)}_{\leq 0} + \underbrace{\beta \int_{m+B_0^b}^{\overline{y_1}} U''\left(y_1 - B_0^b\right) dF\left(y_1\right)}_{\leq 0} + \underbrace{\beta U'(m) f\left(m + B_0^b\right)}_{\geq 0} \tag{22}$$

The first two terms in equation (22) are standard and would be sufficient in a model without default to guarantee that the borrowers' problem is convex. However, the third term, which is strictly positive, may create non-convexities. Hence, when $f(\cdot) \approx 0$ is sufficiently low for all values, we can guarantees that $\beta U'(m) f\left(m+B_0^b\right) < \left(q_0^b\right)^2 U''\left(y_0 + q_0^b B_0^b\right) + \beta \int_{m+B_0^b}^{\overline{y_1}} U''\left(y_1 - B_0^b\right) dF\left(y_1\right)$, making the borrowers' problem convex. Intuitively, a marginal increase in borrowing cannot change the probability of default drastically, so the fact that debt must be repaid in the future is dominant.

b) See the proof of proposition 1 below.

Proposition 1. (Loan demand properties)

a) Assumptions 2 and 3 guarantee that the problem solved by borrowers is convex. Given the convexity of the problem, assumption 1 guarantees that borrowers' loan demand is strictly positive, i.e., $B_0^b > 0$.

b) Taking derivatives in the first order condition given by equation (4), we can find:

$$\frac{\partial q_0^b}{\partial B_0^b} = \frac{-\beta\left[\left(q_0\right)^2 U''\left(C_0\right) + \int_{m+B_0^b}^{\overline{y_1}} U''\left(y_1 - B_0^b\right) dF\left(y_1\right) + U'\left(m\right) f\left(m + B_0^b\right)\right]}{U'\left(y_0 + q_0^b B_0^b\right)\left[1 - \frac{\Lambda}{\psi}\right]},$$

where $\Lambda \equiv \frac{q_0^b B_0^b}{y_0 + q_0^b B_0^b}$ measures leverage and $\frac{1}{\psi} \equiv -C_0 \frac{U''(C_0)}{U'(C_0)}$ defines the elasticity of intertemporal substitution — I show in the Epstein-Zin extension that it is the EIS the one which matters. Under assumption 2a, the numerator of this expression is strictly positive. Hence, the key condition to have a positive denominator is that:

$$1 > \frac{\Lambda}{\psi} \Rightarrow \psi > \Lambda$$

Hence, because $\Lambda < 1$, under assumption 3b, it is always the case that $\psi \geq 1 > \Lambda$, so the denominator is also positive, allowing us to conclude that $\frac{\partial q_0^b}{\partial B_0^b} > 0$. Note that even when $\psi < 1$ loan demand may be upward sloping as long as $\psi > \Lambda$. I use the slightly more restrictive assumption 3b to have a condition which does not involve endogenous quantities.

c) Taking derivatives in equation (4), we can find:

$$\frac{\partial q_0^b}{\partial m} = \frac{-\beta U'\left(m\right) f\left(m + B_0^b\right)}{U'\left(y_0 + q_0^b B_0^b\right)\left[1 - \frac{\Lambda}{\psi}\right]}$$

The numerator of this expression is strictly negative, since marginal utility is strictly positive. As above, under assumption 3b, the denominator is positive, allowing us to conclude that $\frac{\partial q_0^b}{\partial m} < 0$.

Proposition 2. (Loan supply properties)

a) We can rewrite equation (5) as:

$$\frac{1+r^*}{1+r} = \int_m^{m+B_0^l} \frac{\left(y_1 - m\right)}{B_0^l} dF\left(y_1\right) + \int_{m+B_0^l}^{\overline{y_1}} dF\left(y_1\right)$$

But it is easy to show that $\int_m^{m+B_0^l} \frac{(y_1-m)}{B_0^l} dF\left(y_1\right) + \int_{m+B_0^l}^{\overline{y_1}} dF\left(y_1\right) < 1$, which implies directly that $r^* < r$.

b) Taking derivatives in equation (5), we can find:

$$\frac{\partial q_0^l}{\partial B_0^l} = -\frac{1}{1+r^*} \frac{\int_m^{m+B_0^l}\left(y_1 - m\right) dF\left(y_1\right)}{\left(B_0^l\right)^2} < 0$$

This expression is strictly negative.

c) Taking derivatives in equation (5), we can find:

$$\frac{\partial q_0^l}{\partial m} = -\frac{1}{1+r^*} \frac{\int_m^{m+B_0^l} dF\left(y_1\right)}{B_0^l} < 0$$

This expression is strictly negative.

Proposition 3. (Equilibrium)

a) Propositions 1b and 2b guarantee that the slopes of loan demand and loan supply have different sign. This fact, combined with assumption 1, which guarantees that, at $B_0 = 0$, $q_0^b > q_0^l$, and a direct application of the intermediate value theorem guarantees existence and uniqueness of the equilibrium.

b) I first define

$$SOC \equiv (q_0)^2 U''(y_0 + q_0 B_0) + \beta \int_{m+B_0}^{\overline{y_1}} U''(y_1 - B_0)\, dF(y_1) + \beta U'(m) f\left(m + B_0^b\right),$$

which is strictly negative under assumption 3a. Differentiating the equilibrium value of q_0 with respect to m yields:

$$\frac{dq_0}{dm} = \frac{\overbrace{-\int_m^{m+B_0} dF(y_1) + \frac{\beta U'(m) f(m+B_0)}{SOC} \frac{\int_m^{m+B_0} (y_1 - m)\, dF(y_1)}{B_0}}^{<0}}{\underbrace{B_0(1+r^*) - \frac{U'(C_0)\left[1 - \frac{\Lambda}{\psi}\right]}{SOC} \frac{\int_m^{m+B_0} (y_1 - m)\, dF(y_1)}{B_0}}_{>0}} < 0$$

Alternatively, the results of propositions 1b and 2b necessarily imply that $\frac{dq_0}{dm} < 0$.

b) An explicit characterization of $\frac{dB_0}{dm}$ yields:

$$\frac{dB_0}{dm} = -\underbrace{\frac{U'(C_0)\left[1 - \frac{\Lambda}{\psi}\right]}{SOC}}_{>0} \underbrace{\frac{dq_0}{dm}}_{<0} - \underbrace{\frac{\beta U'(m) f(m+B_0)}{SOC}}_{>0},$$

which is in general ambiguous. Alternatively, the results of propositions 1c and 2c imply that the sign of $\frac{dB_0}{dm}$ can be positive or negative.

Proofs: Section 3

Proposition 4. (Marginal effect on welfare)

a) Starting from equation (6), we can write, applying repeatedly Leibniz rule and using the envelope theorem for B_0 and the default decision, we can write:

$$\begin{aligned}
\frac{dW}{dm} &= \left[U'(C_0) \frac{dq_0}{dm} B_0 + \beta \int_m^{m+B_0} U'(m)\, dF(y_1)\right] \\
&+ \underbrace{\left[U'(C_0) q_0 - \beta \int_{m+B_0}^{\overline{y_1}} U'(y_1 - B_0)\, dF(y_1)\right]}_{=0} \frac{dB_0}{dm} \\
&+ \underbrace{\beta \left[U(m) f(m+B_0)\left(1 + \frac{dB_0}{dm}\right) - U(m) f(m+B_0)\left(1 + \frac{dB_0}{dm}\right)\right]}_{=0},
\end{aligned}$$

which corresponds to equation (7) in the paper, noting that $C_1^D = \min\{y_1, m\}$.

b) By rewriting the previous expression, we can find that:

$$\frac{\frac{dW}{dm}}{U'(C_0) C_0} = \underbrace{\frac{\frac{dq_0}{q_0}}{dm}}_{\equiv -\varepsilon_{\tilde{r},m}} \underbrace{\frac{q_0 B_0}{C_0}}_{\equiv \Lambda} + \frac{1}{m} \underbrace{\beta \int_m^{m+B_0} \frac{C_1^D}{C_0} \frac{U'\left(C_1^D\right)}{U'(C_0)} dF(y_1)}_{\equiv \frac{\Pi_m\left\{C_1^D\right\}}{C_0}},$$

where I use the fact that $\frac{\frac{dq_0}{q_0}}{dm} = \frac{d \log q_0}{dm} = \frac{-d \log(1+r)}{dm} = -\varepsilon_{\tilde{r},m}$.[36]

[36]Note that this definition uses the fact that $C_1^D = m$ in the relevant range. If that is not the case – for instance, because households have access to a secret source of income y' – the term $\frac{\Pi_m\{C_1^D\}}{C_0}$ would have to be multiplied by a term $\theta \equiv \frac{m}{m+y'}$, where $\theta \leq 1$. Intuitively, since borrowers' consumption is higher in those states, the optimal exemption is lower.

Proposition 5. (Optimal bankruptcy exemption)

Under assumptions 1 to 3, it is straightforward to show that the planner's problem is differentiable in m. Hence, a sufficient condition for the optimal exemption is found when setting $\frac{dW}{dm}$ in equation (8) to zero. Solving for m^* immediately yields (9). To be added: restriction on $F(\cdot)$ to guarantee the convexity of the planner's problem (see figure (A.3) below, for a parametrization that yields a convex problem for the planner in the case of borrowers who internalize price effects).

Corollary. (Optimal bankruptcy exemption w/ logarithmic utility)

In the logarithmic utility case:

$$\frac{\Pi_m\left\{C_1^D\right\}}{C_0} = \beta \int_m^{m+B_0} \frac{C_1^D}{C_0}\frac{U'\left(C_1^D\right)}{U'\left(C_0\right)} dF\left(y_1\right) = \beta \underbrace{\int_m^{m+B_0} dF\left(y_1\right)}_{\equiv \pi_m},$$

since $U'(C) = \frac{1}{C}$.

Proofs: Section 4

Proposition 6. (Endogenous income)

a) Applying repeatedly Leibniz' rule in the planner's problem and using the envelope theorem for the loan demand B_0, the default decision, the labor supply choices N_1^{ND} and N_0 and the optimal effort choice a, we can write:

$$\frac{dW}{dm} = \frac{\partial U}{\partial C_0}\left(C_0, N_0; a\right)\frac{dq_0}{dm}B_0 + \beta \int_{D_m} \frac{\partial U}{\partial C_1}\left(C_1^D, 0\right) dF\left(y_1; a\right),$$

where $\frac{\partial U}{\partial C_0}\left(C_0, N_0; a\right)$ denotes the partial derivative or flow utility with respect to consumption — equivalently with $\frac{\partial U}{\partial C_1}\left(C_1^D, 0\right)$.

Figure (A.2) illustrates graphically how to characterize the default region when borrowers' labor supply decision is non-trivial. Although it is impossible to solve explicitly for the default region, its characterization is conceptually identical to the baseline model.

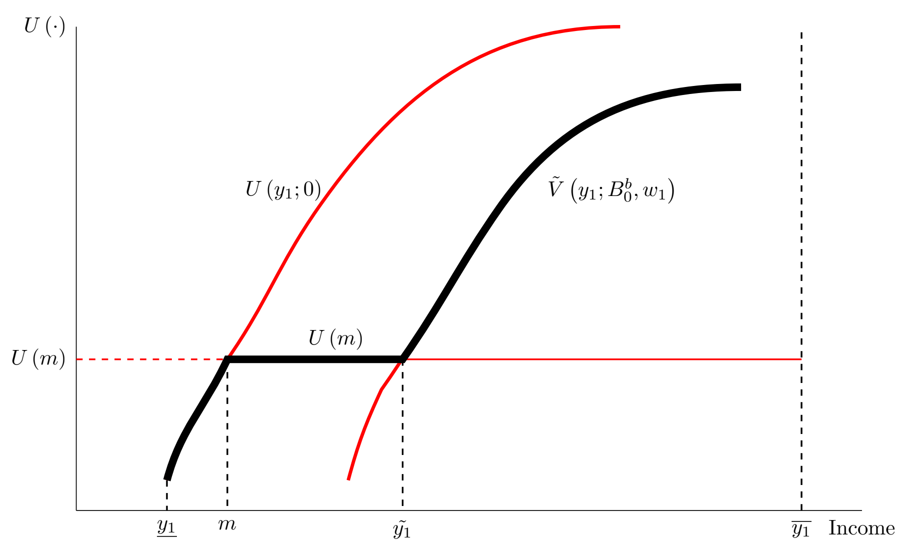

Figure A.2: Optimal default decision given B_0^b

b) Setting $\frac{dW}{dm} = 0$ and solving for m yields directly m^*.

Proposition 7. (Non-pecuniary penalties/state contingent utility)

a) The bankruptcy region is characterized by the indifference condition $\phi m = y_1 - B_0^b$. The derivation of equation (13) follows the same steps as the baseline model. Note that ε only enters through the price derivative term $\frac{dq_0}{dm}$.

b) Setting $\frac{dW}{dm} = 0$ and solving for m yields directly m^*.

Proposition 8. (Internalized price response)

As in the previous case with bankruptcy costs, the indifference condition that determines the default region is $y_1 = \phi m + B_0$. Given the default decision, we can now write borrowers' problem as $\max_{B_0} J(B_0; m)$, where:

$$J(B_0;m) \equiv U\left(y_0 + \frac{1}{1+r^*}\left(\varepsilon \int_m^{\phi m + B_0} (y_1 - m)\, dF(y_1) + B_0 \int_{\phi m + B_0}^{\overline{y_1}} dF(y_1)\right)\right)$$
$$+\beta\left[\int_{\underline{y_1}}^{m} U(\phi y_1)\, dF(y_1) + \int_m^{\phi m + B_0} U(\phi m)\, dF(y_1) + \int_{\phi m + B_0}^{\overline{y_1}} U(y_1 - B_0)\, dF(y_1)\right]$$

This is a function exclusively of B_0 and m. Because the optimal contract turns out to require a flat repayment (leaving aside the region in which $y_1 < m$), by optimizing jointly (or sequentially) in B_0 and m, when ε and ϕ are not equal to one, borrowers can guarantee the first best full insurance outcome. The first and second order conditions in B_0 are given by:

$$\frac{dJ}{dB_0} = U'(C_0)\frac{1}{1+r^*}\left[(\varepsilon m(\phi - 1) + B_0(\varepsilon - 1)) f(\phi m + B_0) + \int_{\phi m + B_0}^{\overline{y_1}} dF(y_1)\right]$$
$$-\beta \int_{\phi m + B_0}^{\overline{y_1}} U'(y_1 - B_0)\, dF(y_1)$$

Note that when there are costs associated to bankruptcy, the term

$$(\varepsilon m(\phi - 1) + B_0(\varepsilon - 1)) f(\phi m + B_0)$$

is strictly negative. When $\varepsilon = \phi = 1$, this term goes away.

$$\frac{d^2 J}{dB_0^2} = \underbrace{U''(C_0)\left(\frac{1}{1+r^*}\left[(\varepsilon m(\phi - 1) + B_0(\varepsilon - 1)) + \int_{\phi m + B_0}^{\overline{y_1}} dF(y_1)\right]\right)^2}_{<0}$$
$$\underbrace{+U'(C_0)\frac{1}{1+r^*}\left[(\varepsilon m(\phi - 1) + B_0(\varepsilon - 1)) f'(\phi m + B_0) - f(\phi m + B_0)\right]}_{\gtreqless 0}$$
$$\underbrace{+\beta \int_{\phi m + B_0}^{\overline{y_1}} U''(y_1 - B_0)\, dF(y_1)}_{<0} + \underbrace{\beta U'(\phi m) f(\phi m + B_0)}_{>0}$$

As in the baseline model, convexity is only guaranteed after restricting the distribution $F(\cdot)$. To provide intuition for why $B_0 = \overline{y_1} - \phi m$ is optimal when there are no bankruptcy costs, note that $\left.\frac{dJ}{dB_0}\right|_{\phi m + B_0 = \overline{y_1}}$ can be written as:

$$\left.\frac{dJ}{dB_0}\right|_{\phi m + B_0 = \overline{y_1}} = U'(C_0)\frac{(\varepsilon m(\phi - 1) + B_0(\varepsilon - 1))}{1+r^*} f(\phi m + B_0) \leq 0$$

When $\varepsilon = \phi = 1$, this expression equals 0 which implies that the level of B_0 has to be at least a local optimum. When m is chosen optimally, this optimum has to be the global optimum, since it replicates the first best outcome. Note that when $\varepsilon < 1$ or $\phi < 1$, this expression has to be negative, which implies that always defaulting cannot be an optimum.

Because it is not easy to provide sufficient conditions for convexity, I illustrate the results with a numerical example. In the left plot of figure (A.3), I show $J(B_0;\cdot)$ for different values of m. In the right plot, I show the how J as a function of m once B_0 is optimally chosen, that is $W(m)$. This is the function optimized by the planner. For this particular parametrization, $m^* = 7.5$.

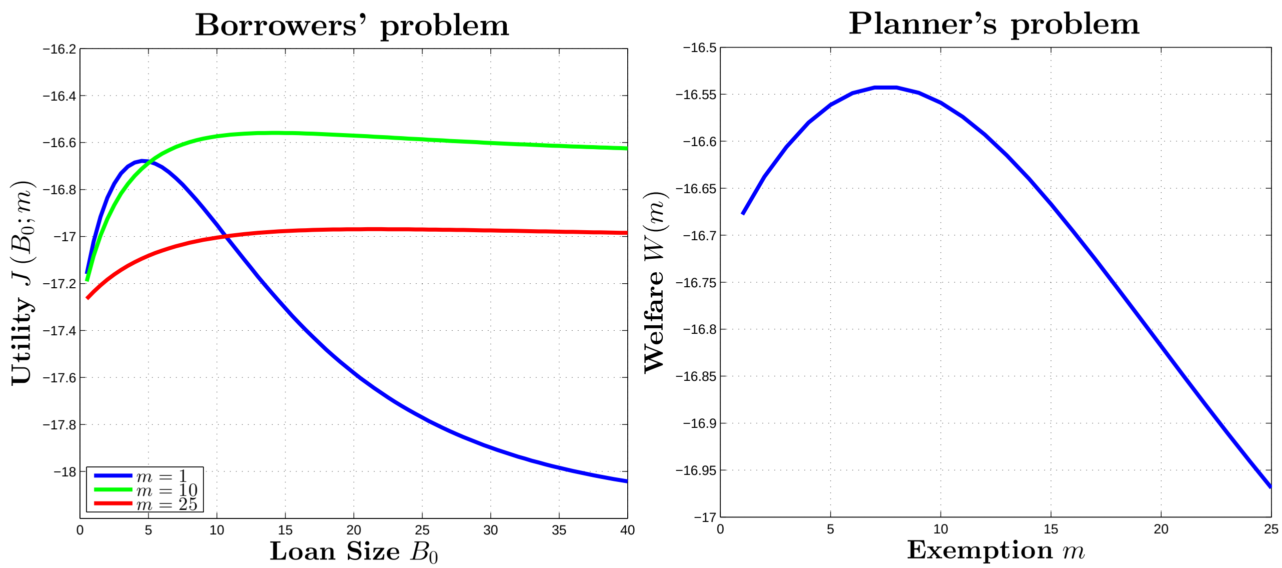

Figure A.3: Borrowers' problem and planner's problem

The parametrization used to plot figure (A.3) is in table 5.

Preferences	$\gamma = 1$	$\psi = 1$	$\beta = 0.96$	$r^* = 4\%$
Income	$y_0 = 1$	$\mu = 2.7$	$\sigma = 0.65$	
Bankruptcy	$\varepsilon = 0.8$	$\phi = 0.8$		

Table 5: Parameters figure (A.3)

Proposition 9. (Epstein-Zin utility)

a) Taking derivatives in equation (15) in the paper yields:

$$\frac{\partial q_0^b}{\partial B_0^b} = \frac{-\beta \left[\begin{array}{c} (q_0)^2 \frac{1}{\psi} \left(y_0 + q_0^b B_0^b\right)^{-\frac{1}{\psi}-1} \\ + \int_{m+B_0^b}^{\overline{y_1}} \gamma \left(y_1 - B_0^b\right)^{-\gamma-1} dF(y_1) + \left(y_1 - B_0^b\right)^{-\gamma} f\left(m + B_0^b\right) \end{array} \right]}{\left(y_0 + q_0^b B_0^b\right)^{-\frac{1}{\psi}} \left[1 - \frac{\Lambda}{\psi}\right]}$$

This shows that it is EIS the relevant parameter for the slope of the loan demand.

The derivation of $\frac{dW}{dm}$ follows the same steps as the baseline model. In this case, I define:

$$Q \equiv \left(\int_{\underline{y_1}}^{m} (y_1)^{1-\gamma} dF(y_1) + \int_{m}^{m+B_0^b} (m)^{1-\gamma} dF(y_1) + \int_{m+B_0^b}^{\overline{y_1}} \left(y_1 - B_0^b\right)^{1-\gamma} dF(y_1) \right)^{\frac{1}{1-\gamma}}, \tag{23}$$

which denotes the certainty equivalent of consumption at $t = 1$.

b) Setting $\frac{dW}{dm} = 0$ and solving for m yields directly m^*.

Proposition 10. (Multiple arbitrary contracts)

a) As stated in the paper, there are three regions depending on the realization of y_1. The indifference condition between the regions D_m and ND in the range $y_1 > m$ is given by $m = y_1 - \sum_{j=1}^{J} z_j(y_1) B_{0j}^b$. The indifference condition between the regions D_y and ND in the range $y_1 \leq m$ is given by $0 = \sum_{j=1}^{J} z_j(y_1) B_{0j}^b$. When $\sum_{j=1}^{J} z_j(m) B_{0j}^b > 0$, at $y_1 = m$ there is a new boundary separating the D_m and D_y regions — this doesn't occur for instance in figure (4). This characterization decomposes the possible set of realizations in $\left[\underline{y_1}, \overline{y_1}\right]$ into multiple non-overlapping intervals. The planner now maximizes:

$$W(m) \equiv U\left(y_0 + \sum_j q_{0j} B_{0j}\right) + \beta \left[\begin{array}{c} \int_{D_y} U(y_1)\, dF(y_1) + \int_{D_m} U(m)\, dF(y_1) \\ + \int_{ND} U\left(y_1 - \sum_j z_j(y_1) B_j(y_1)\right) dF(y_1) \end{array}\right]$$

The derivation of equation (16) follows the same steps as the baseline model. It requires the use of J envelope conditions for the J different assets traded.

b) Setting $\frac{dW}{dm} = 0$ and solving for m yields directly m^*.

Proposition 11. (Heterogeneous borrowers)

a) The derivation of $\frac{dW}{dm}$ follows the same steps as the baseline model. It requires the use of as many envelope conditions regarding the choice of B_0 and default as the number of borrowers.

b) Setting $\frac{dW}{dm} = 0$ and solving for m yields directly m^*.

Proposition 12. (Bankruptcy exemption contingent on aggregate risk)

a) The problem solved by the planner is now:$\max_{\{m_\omega\}} W(\{m_\omega\})$. The derivation of $\frac{\partial W}{\partial m_\omega}, \forall \omega$ follows the same steps as the baseline model. For every ω it requires the use of as envelope conditions regarding the choice of B_0 and default.

b) Setting $\frac{dW}{dm_\omega} = 0$ and solving for m_ω yields directly m_ω^* for every ω.

Proposition 13. (Dynamics)

a) The problem solved by borrowers at $t = 0$ can be written as:

$$V_{ND,0}(0, y_0; m) = \max_{B_0} U(y_0 + q_0 B_0) + \beta \mathbb{E}\left[\max\left\{V_{ND,1}(B_0, y_1; m), V_{D,1}(y_1; m)\right\}\right],$$

where $V_{ND,1}(B_0, y_1; m)$ denotes indirect utility at $t = 1$ when a borrower repays and $V_{D,1}(y_1; m)$ denotes the indirect utility at $t = 1$ when a borrower just defaulted, given by:

$$V_{ND,1}(B_0, y_1; m) = \max_{B_1} U(y_1 + q_1 B_1 - B_0) + \beta \mathbb{E}\left[\max\left\{V_{ND,2}(B_1, y_2; m), V_{D,2}(B_1, y_2; m)\right\}\right]$$
$$V_{D,1}(y_1; m) = U(\min\{y_1, m\}) + \beta\left[\alpha \mathbb{E}\left[V_{D_\alpha,2}(y_2; m) | y_1\right] + (1-\alpha)\mathbb{E}\left[V_{ND,2}(0, y_2; m) | y_1\right]\right],$$

where $V_{D_\alpha,2}(y_2; m)$ denotes the indirect utility at $t = 2$ for a borrower who has no access to credit markets.

$$V_{D_\alpha,2}(y_2; m) = U(y_2) + \beta\left[\alpha \mathbb{E}\left[V_{D_\alpha,3}(y_3; m) | y_2\right] + (1-\alpha)\mathbb{E}\left[V_{ND,3}(0, y_3; m) | y_2\right]\right]$$

The definition of these value functions for future periods is straightforward.

Hence, the derivative $\frac{dW}{dm}$ can be written, after using extensively envelope conditions

$$\frac{dW}{dm} = U'(C_0) \frac{dq_0}{dm} B_0 + \beta \left[\int_{ND} \frac{dV_{ND,1}(B_0, y_1; m)}{dm} dF(y_1) + \int_D \frac{dV_{D,1}(y_1; m)}{dm} dF(y_1)\right], \tag{24}$$

where

$$\frac{dV_{ND,1}(B_0, y_1; m)}{dm} = U'\left(C_1^D\right) \frac{dq_0}{dm} B_1 + \beta \left[\begin{array}{c} \int_{ND} \frac{dV_{ND,2}(B_1, y_2; m)}{dm} dF(y_2 | y_1) \\ + \int_D \frac{dV_{D,2}(y_2; m)}{dm} dF(y_2 | y_1) \end{array}\right]$$

$$\frac{dV_{D,1}(y_1; m)}{dm} = U'(m)\mathbb{I}(y_1 > m) + \beta \left[\begin{array}{c} \alpha \mathbb{E}\left[\left.\frac{dV_{D_\alpha,2}(y_2; m)}{dm}\right| y_1\right] \\ + (1-\alpha)\mathbb{E}\left[\left.\frac{dV_{ND,2}(0, y_2; m)}{dm}\right| y_1\right] \end{array}\right]$$

$$\frac{dV_{D_\alpha,2}(y_2;m)}{dm} = \beta\left[\alpha\mathbb{E}\left[\frac{dV_{D_\alpha,3}(y_3;m)}{dm}\middle| y_2\right] + (1-\alpha)\mathbb{E}\left[\frac{dV_{ND,3}(0,y_3;m)}{dm}\middle| y_2\right]\right]$$

Substituting the last three expressions into equation (24) and iterating forward, we can write:

$$\frac{dW}{dm} = \sum_{t=0}^{T-1}\mathbb{E}_{ND}\left[\beta^t U'\left(C_t^{ND}\right)\frac{dq_t}{dm}B_t\right] + \sum_{t=1}^{T}\mathbb{E}_{D_m}\left[\beta^t U'\left(C_t^D\right)\right],$$

where $\mathbb{E}_{ND}[\cdot]$ denotes the $t=0$ expectation of being in a no default situation at a given state/period and $\mathbb{E}_{D_m}[\cdot]$ denotes the $t=0$ expectation of defaulting while consuming the bankruptcy exemption in a given state.

The previous equation can be rewritten as:

$$\frac{\frac{dW}{dm}}{U'(C_0)C_0} = -\sum_{t=0}^{T-1}\Pi_{ND}\left\{g_t\Lambda_t\varepsilon_{\tilde{r}_t,m}\right\} + \frac{1}{m}\sum_{t=1}^{T}\frac{\Pi_m\left\{C_t^D\right\}}{C_0},$$

where all new variables are defined in proposition 13.

b) Setting $\frac{dW}{dm}=0$ and solving for m yields directly m^*.

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