# *Job Market Paper*

# Capital Asset Pricing under Ambiguity

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**Abstract**

This paper generalizes the standard *mean–variance* paradigm to a *mean–variance–ambiguity* paradigm by relaxing the assumption that probabilities are known and instead assuming that probabilities are themselves random. It extends the CAPM from risk to uncertainty by incorporating ambiguity. This model makes the distinction between *systematic ambiguity* and *idiosyncratic ambiguity* and proves that the *ambiguity premium* is proportional to systematic ambiguity. It introduces a new *measure of uncertainty* that combines risk and ambiguity. Use of this model can be extended to other applications including portfolio selection and performance measurement.

**Keywords and Phrases:** Ambiguity Measure, Uncertainty Measure, Ambiguity premium, Mean-variance, Mean-uncertainty, Capital Market Line (CML), Capital Asset Pricing Model (CAPM).

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# Capital Asset Pricing under Ambiguity

## I Introduction

The assumption underlying modern portfolio theory, including asset pricing models, is that the probability distributions of returns are known such that there is a unique mean–variance space upon which preferences are imposed. In reality, probabilities are usually unknown and an additional premium is required to induce investors to bear *ambiguity* (*Knightian uncertainty*).[1] What is the nature of this premium? Is it related to the entire—systematic and idiosyncratic—ambiguity? Or is it related only to the systematic ambiguity? Can systematic ambiguity and idiosyncratic ambiguity be differentiated? The current paper is motivated by these questions.

This paper contributes to the existing literature in five ways. First, it generalizes the *mean–variance* paradigm to a *mean–uncertainty* paradigm, i.e., a *mean–variance–ambiguity* paradigm, where *uncertainty* is considered to be the aggregation of risk and ambiguity. Within this paradigm, it reestablishes the efficient frontier and the capital market line (CML) and introduces mean–uncertainty preferences. Second, the paper generalizes the classical CAPM to incorporate ambiguity while making the distinction between systematic and idiosyncratic ambiguity. It proves that investors are rewarded for systematic uncertainty (i.e., systematic risk and systematic ambiguity), but not for idiosyncratic uncertainty (i.e., idiosyncratic risk and idiosyncratic ambiguity). Third, the paper combines risk and ambiguity to introduce an objective, empirically applicable *measure of uncertainty.* Fourth, the paper introduces new performance measures by extending the *Treynor* and *Sharpe* ratios from risk to uncertainty. Fifth, for the first time, the paper proposes an empirically tractable asset pricing model under ambiguity.[2]

This paper focuses on the implications of ambiguity for asset pricing. To this end, it employs a model of decision making under ambiguity, called *expected utility with random*

[1] *Risk* is defined as a condition in which the event to be realized is *a-priori* unknown, but the odds of all possible events are perfectly known. *Ambiguity* refers to conditions in which not only is the event to be realized *a-priori* unknown, but the odds of events are also either not uniquely assigned or are unknown.

[2] Other models allow only for calibration (e.g., Epstein and Schneider (2008), Ju and Miao (2012) and Drechsler (2012)).

*probabilities* (EURP). This model, proposed by Izhakian (2012a), is based on Schmeidler's (1989) *Choquet expected utility* and Tversky and Kahneman's (1992) *cumulative prospect theory*. The central concept of EURP is that not only are the returns on assets random but the probabilities of these returns are themselves also random. Its main advantage is that, as the degree of risk can be measured by the variance of outcomes, so too can the degree of ambiguity be measured by the variance of the probability of loss (or gain).[3] The measure of ambiguity derived by EURP is a centerpiece of the theoretical model introduced in this paper.

The reason that the EURP model is employed in this paper is that, unlike other models of decision making under ambiguity, EURP applies preferences concerning ambiguity directly to probabilities and not to expected utility (e.g, Klibanoff *et al.* (2005)) nor to certainty equivalents (e.g., Ju and Miao (2012)), which are subject to risk and preferences concerning risk. The novelty of EURP is that, by applying ambiguity and preferences concerning it solely to probabilities, it allows for separating risk from ambiguity and preferences from beliefs in a way that an applicable measure of ambiguity and a premium that is entirely attributed to ambiguity can be derived. Other models (e.g., Gilboa and Schmeidler (1989) and Klibanoff *et al.* (2005)) do not allow for such derivations.[4] Models of robustness (e.g., Hansen and Sargent (2001)) require the identification of a reference model, while EURP requires only the identification of a reference point, which is easier to obtain, especially in empirical studies. Moreover, except EURP we are not aware of any other model of decision making under ambiguity that has been empirically tested rather than by calibrating the model to the data or by using proxies for ambiguity (e.g., dispersion of analysts forecasts).

The neoclassical finance literature dealing with capital asset pricing assumes away ambiguity and focuses on the risk–return relationship in the mean–variance paradigm. The ability of asset pricing models, including the widely used CAPM (e.g., Sharpe (1964)), to portray a full realistic image of uncertainty has been challenged over the years. The present paper attempts to deal with this challenge by introducing the idea that the probability distributions

[3]Measuring risk by the variance of outcomes is admissible under some conditions; the same is true for measuring ambiguity by the variance of probabilities (see Izhakian (2012a)).

[4]For example, based on Klibanoff *et al.* (2005), Maccheroni *et al.* (2010) propose an ambiguity premium; this premium, however, is also a function of risk aversion and, it ignores the ambiguity regarding all the moments of the probability distribution which are greater than one (i.e., variance skewness and kurtosis).

of returns are random. Particularly, it assumes that the moments (i.e., mean and variance) of normally distributed returns are unknown and random. That is, investors have a prior over probability distributions. As investors are assumed to be ambiguity averse, they don't compound probability distributions (beliefs) with this prior while assessing expected utility. Expected utility is therefore negatively affected by both risk and ambiguity (uncertainty). To measure uncertainty, this paper introduces a new measure that combines risk, derived from beliefs regarding outcomes, with ambiguity, derived from beliefs regarding probabilities. Based on this consolidated measure, the mean-variance paradigm is then generalized to mean-uncertainty.

Using the newly proposed mean-uncertainty paradigm, the paper derives an asset pricing model, referred to as *capital asset pricing model under ambiguity* (ACAPM). This model makes the distinction between *systematic ambiguity*, dominated by economy-wide shocks, and *idiosyncratic ambiguity*, dominated by firm-specific shocks.[5] Since, in ACAPM, risk and ambiguity might be inversely related (see Izhakain (2012c)) the systematic risk and the systematic ambiguity are optimal and not necessarily minimal for a given expected return. Their aggregation to *systematic uncertainty*, however, is the minimal possible degree of uncertainty for a given level of expected return.

The ACAPM asserts that the expected return on an asset depends upon the correlation of its probability of loss with the probability of loss on the market portfolio (market ambiguity) and not on the ambiguity associated with its own probability fluctuations (asset ambiguity). In this model the entire ambiguity of an asset is not the relevant determinant of its expected return, but only its systematic component. Formally, the ambiguity premium is modeled by a *beta ambiguity*, separately from the conventional *beta risk*. A special case of the ACAPM is when probabilities are known, in which beta ambiguity is zero and the model collapses to the classical CAPM.

Decomposing risk and ambiguity into systematic and idiosyncratic components allows for incorporating ambiguity into the *Sharpe ratio* and the *Treynor ratio*, which are commonly used for evaluating portfolio performance. The Sharpe ratio evaluates the premium per unit of total risk (systematic and idiosyncratic) borne by an asset, while the Treynor ratio

[5] The classical CAPM makes the distinction between *systematic risk*, for which investors are rewarded via a higher rate of return, and *idiosyncratic risk*, which is not accompanied by an additional reward.

evaluates the premium per unit of systematic risk. Our extended performance measures evaluate the premium per unit of total uncertainty borne and the premium per unit of systematic uncertainty borne.

The theoretical model introduced in this paper paves the way for further research, especially empirical, into the risk–ambiguity–return relationship. The ACAPM has been tested empirically by Brenner and Izhakian (2012) who study the cross-sectional impact of ambiguity on capital asset return. They find that systematic ambiguity has a significant positive effect on firm-specific expected return.

The rest of the paper is organized as follows. Section II reviews the related literature. For context, Section III reviews the main principles of EURP. Section IV extends the classical mean–standard-deviation space to mean–standard-deviation–ambiguity and forms preferences in this space. Section V establishes the efficient frontier and the CML in mean–uncertainty settings. Section VI generalizes the CAPM to incorporate ambiguity and discusses the implications of the components of ambiguity for capital asset pricing. Section VII discusses the security market line (SML) and performance measures. Section VIII discusses the empirical implications of ambiguity for capital asset pricing. Section IX concludes.

# II Related literature

The implications of ambiguity for asset pricing have been studied mainly in the context of the equity premium. Cao *et al.* (2005), Nau (2006) and Izhakian and Benninga (2011), for example, focus on decomposing the equity premium into two components: risk premium and ambiguity premium.[6] Unlike these papers, which consider the ambiguity premium of an asset independently of the ambiguity of other assets in the market, the current paper studies the nature of asset ambiguity relative to market ambiguity.

Several extensions of the mean–variance approach to incorporate ambiguity have been

[6] Dow and Werlang (1992), Easley and O'Hara (2009) and Ui (2011) attribute limited market participation to ambiguity aversion and study its impact on the equity premium. Uppal and Wang (2003), Maenhout (2004), Taboga (2005) and Gollier (2011), for example, study issues of optimal portfolio selection under ambiguity. Leippold *et al.* (2008), Gagliardini *et al.* (2009) and Ju and Miao (2012), for example, tie ambiguity to puzzling financial phenomena such as the equity premium puzzle, the risk-free rate puzzle and the excess volatility puzzle. Epstein and Schneider (2008) tie ambiguity to the information quality of fundamentals, and Drechsler (2012) ties it to the volatility risk premium.

suggested in the literature. Pflug and Wozabal (2007) add ambiguity to mean–variance preferences by applying the max-min approach of Gilboa and Schmeidler (1989) to a confidence set of probability distributions. Boyle *et al.* (2011) assume a mean–variance space with known variances and unknown means. The paradigm of the current paper is broader—it assumes that both mean and variance are unknown. An unknown variance plays an important role in ambiguity, especially where an asset portfolio is concerned. The importance of random variance is stressed by Bollerslev *et al.* (1988), who show that conditional covariances are quite variable over time and are a significant determinant of time-varying risk premia.

In a related study, Kogan and Wang (2003) propose a CAPM under ambiguity with an unknown mean and a known variance. Their model is based on assessing ambiguity as reflected in investor's preferences. The ACAPM employs an objective measure of ambiguity which is independent of preferences. This measure allows the ACAPM to separate ambiguity from risk and to isolate systematic ambiguity. There are three advantages of ACAPM over Kogan and Wang (2003). ACAPM accounts for random variance, which is an important component of ambiguity. ACAPM makes the distinction between preferences and beliefs and preserves Tobin's (1958) *separation theorem*. ACAPM can be easily employed for empirical studies, whereas it is more difficult with Kogan and Wang (2003) because a reference model must be identified.

Another related study, Maccheroni *et al.* (2010), defines the ambiguity premium, referred to as *alpha*, by the residual between expected return and the risk premium reward for systematic risk. In fact, the literature attributes this residual also to idiosyncratic risk.[7] In an earlier study, Epstein and Schneider (2008) show that the ambiguity premium depends on the idiosyncratic risk in fundamentals, which is practically equivalent to the alpha proposed by Maccheroni *et al.* (2010).[8] These models do not make the distinction between ambiguity and idiosyncratic risk. Therefore, they may not be suitable for empirical studies of the impact of ambiguity on asset return. The ACAPM provides a closed-form solution for the *beta ambiguity*, which measures the systematic ambiguity associated with an asset relative to market ambiguity. This beta, which has been tested empirically (Brenner and

[7] It can be shown that in Maccheroni *et al.* (2010) and Izhakian and Benninga (2011), which are based on Klibanoff *et al.'s* (2005) *smooth model of ambiguity*, that (counterintuitively) increasing risk aversion reduces the ambiguity premium and sometimes also reduces the equity premium.

[8] Epstein and Schneider's (2008) model is based on Gilboa and Schmeidler's (1989) *max-min* model.

Izhakian (2012)), has a significant positive impact on expected return in the stock markets.[9]

Chen and Epstein (2002) generalize the consumption CAPM by building dynamic recursive multiple priors max-min preferences. Maccheroni, *et al.* (2009) use *variational preferences* to derive a version of the CAPM that under monotone mean–variance preferences can be generalized to incorporate ambiguity. In these models an asset's beta is derived by the covariance between its return and the pricing kernel, which makes no distinction between risk and ambiguity. Unlike these models, the ACAPM achieves a complete separation between risk and ambiguity and attains a well-defined beta ambiguity entirely separated from risk.

It is important to emphasize that ACAPM is different than all previous models mentioned above in several regards: (*i*) It assumes that both mean and variance are ambiguous, not just the mean (e.g., Kogan and Wang (2003) and Maccheroni *et al.* (2010)). (*ii*) It constructs an ambiguity premium that does not depend on risk and risk preferences (e.g., Maccheroni *et al.* (2010) and Izhakian and Benninga (2011)). (*iii*) It generalizes the classical CAPM to show that the ambiguity premium on an asset is proportional to comovement of the asset and market ambiguities; whereas in other models, the ambiguity premium is attributed only to the asset's own ambiguity and does not consider the relationship between asset ambiguity and market ambiguity. (*iv*) It makes the distinction between systematic and idiosyncratic ambiguity and proves that investors are rewarded only for the systematic component. (*v*) It has been proven to be empirically tractable (e.g., Brenner and Izhakian (2012)).

Beta ambiguity is also related to Merton (1973) who introduces a dynamic version of the CAPM and shows that the expected returns on risky assets may differ from the risk-free rate even when these assets do not have systematic risk.[10] He attributes this difference to shifts in the investment opportunity set correlated with a *zero-beta portfolio*. The ACAPM suggests an alternative explanation; it attributes the difference between the expected return on a zero-beta portfolio and the risk-free rate to the presence of ambiguity. In particular, it suggests that this additional expected return is proportional to beta ambiguity. Beta ambiguity is fundamentally different from the additional betas in Merton's (1973) model, which form the

[9] In fact, the current paper fully addresses the challenge that Maccheroni *et al.* (2010) poses: "The natural direction of development of this project is therefore the derivation of a robust CAPM, corresponding to robust Mean-Variance preferences, and its calibration."

[10] Other studies extend the CAPM by incorporating various risk factors (e.g., liquidity risk (Acharya and Pedersen (2005) and Liu (2006)) and long-run risks in aggregate consumption (Ai and Kiku (2012))).

unobservable shifts in investment opportunities. The betas in Merton's model are derived from the correlation between the state variables, dominating the instantaneous investment opportunities, and a non-tangency hedging portfolio. Beta ambiguity in the ACAPM is derived from the correlation between the probability of loss on an asset and the probability of loss on the market portfolio. Therefore, additional hedging portfolios are not needed as implied by Merton's three-fund theorem.

# III The model of ambiguity

The model of decision making under ambiguity, called *expected utility with random probabilities* (EURP), with its ambiguity measure provide the theoretical underpinning of this paper. EURP, proposed by Izhakian (2012a), is based on Schmeidler's (1989) *Choquet expected utility* and adds *reference-dependent beliefs*. Like Tversky and Kahneman's (1992) *cumulative prospect theory*, it assumes that investors have a *reference point* relative to which outcomes are classified as a loss or as a gain.[11] Consequences lower than the reference point are considered a loss, and consequences higher than the reference point are considered a gain.[12] The cumulative probability of loss events plays an important role in measuring the degree of ambiguity.

EURP assumes two tiers of uncertainty, one with respect to outcomes and the other with respect to the probabilities of these outcomes, where each tier of uncertainty is modeled by a separate state space. This structure introduces a complete distinction of risk from ambiguity with regard to both beliefs and preferences. The degree of risk and investors' attitudes toward it are then measured with respect to one space, while ambiguity and investors' attitudes toward it are measured with respect to the second space.

Let $(\Omega, \mathcal{F}, \mathrm{P})$ be a probability space, where $\mathrm{P} \in \mathbb{P}$ is a random probability measure, and the set of probability measures $\mathbb{P}$ is closed and convex. $\mathbb{P}$ is equipped with a Borel probability measure, denoted $\chi$, with a bounded support. Given a random variable, $X : \Omega \to \mathbb{R}$, its *random mean*, $\mathrm{E}_{\mathrm{P}}(X)$, and *random variance*, $\mathrm{Var}_{\mathrm{P}}(X)$, are denoted by the Greek letters $\mu$

[11] Unlike cumulative prospect theory, EURP does not assume different attitudes toward risk for losses and for gains (e.g. loss aversion).

[12] Previous literature focuses on the implication of losses and gains for preferences (see, for example Barberis and Huang (2001) and Hirshleifer (2001)), while ACAPM focuses on beliefs.

and $\sigma^2$, respectively. Similarly, the *random covariance* between two random variables $X$ and $Y$, $\mathrm{E}_{\mathrm{P}}\left(X - \mathrm{E}_{\mathrm{P}}\left(X\right)\right)\left(Y - \mathrm{E}_{\mathrm{P}}\left(Y\right)\right)$, is denoted $\sigma_{X,Y}$.

The *expectation*, $\mathrm{E}\left[X\right]$, and the *variance*, $\mathrm{Var}\left[X\right]$, of $X$ are computed using expected probabilities. That is, a double expectation with respect to the first-order random probability distribution, P, of $X$ and the second-order probabilities $\chi$:

$$\mathrm{E}\left[X\right] = \int_{\mathbb{P}} \left( \int_{\Omega} X\left(\omega\right) d\mathrm{P}\left(\omega\right) \right) d\chi\left(\mathrm{P}\right) \tag{1}$$

and

$$\mathrm{Var}\left[X\right] = \int_{\mathbb{P}} \left( \int_{\Omega} \left(X\left(\omega\right) - \mathrm{E}\left[X\right]\right)^2 d\mathrm{P}\left(\omega\right) \right) d\chi\left(\mathrm{P}\right), \tag{2}$$

where $\omega \in \Omega$. Similarly, the covariance between two random variables $X$ and $Y$ is given by

$$\mathrm{Cov}\left[X, Y\right] = \int_{\mathbb{P}} \left( \int_{\Omega} \left(X\left(\omega\right) - \mathrm{E}\left[X\right]\right)\left(Y\left(\omega\right) - \mathrm{E}\left[Y\right]\right) d\mathrm{P}\left(\omega\right) \right) d\chi\left(\mathrm{P}\right). \tag{3}$$

The main idea of EURP is that the probabilities of outcomes are random; thus, as the degree of risk can be measured by the variance of outcomes, the degree of ambiguity can be measured by the variance of probabilities. Let $\mathrm{P}_L$ and $\mathrm{P}_G$ be the random probabilities of loss and gain, respectively. Their expectations, $\mathrm{E}\left[\mathrm{P}_L\right]$ and $\mathrm{E}\left[\mathrm{P}_G\right]$, taken with respect to the second-order probability distribution $\chi$, are

$$\mathrm{E}\left[\mathrm{P}_L\right] = \int_{\mathbb{P}} \mathrm{P}\left(X < k\right) d\chi\left(\mathrm{P}\right) \qquad \text{and} \qquad \mathrm{E}\left[\mathrm{P}_G\right] = \int_{\mathbb{P}} \mathrm{P}\left(X > k\right) d\chi\left(\mathrm{P}\right), \tag{4}$$

where $k$ is the *reference point* distinguishing losses from gains. The *measure of ambiguity*

$$\mho^2\left[X\right] = 4\mathrm{Var}\left[\mathrm{P}_L\right] = 4\mathrm{Var}\left[\mathrm{P}_G\right] \tag{5}$$

is four times the variance of the probability of loss or four times the variance of the probability of gain, taken with respect to $\chi$.[13] This measure, $\mho^2 \in [0, 1]$, attains its minimal value, 0, when probabilities are known, and its maximal value, 1, only in the extreme case of a binomial distribution with random probabilities that can take the values 0 or 1 with equal likelihood.

To illustrate the concept of ambiguity in EURP, consider the following binomial example of an asset with two possible future returns: $d = -10\%$ and $u = 20\%$. Assume for the

[13] This measure, $\mho^2\left[X\right] = 4 \int_{\mathbb{P}} \left(\mathrm{P}\left(X > k\right) - \mathrm{E}\left[\mathrm{P}_G\right]\right) d\chi\left(\mathrm{P}\right)$, is admissible for symmetric probability distributions that satisfy stochastic dominance with respect to ambiguity (see, Izhakian (2012a)).

moment that the probabilities of $d$ and $u$ are known, say $\mathrm{P}(d) = \mathrm{P}(u) = 0.5$. The expected return is thus 5%, and the standard deviation of return (measuring the degree of risk) is 15%. In this case, since the probabilities are precisely known, ambiguity is not present ($\mho = 0$) and investors face only risk. Assume now that the probabilities of $d$ and $u$ can be either $\mathrm{P}(d) = 0.4$ and $\mathrm{P}(u) = 0.6$ or alternatively $\mathrm{P}(d) = 0.6$ and $\mathrm{P}(u) = 0.4$, where these two alternative distributions are equally likely. Investors now face not only risk but also ambiguity. Assuming that negative returns are considered a loss, the degree of ambiguity, measured by twice the standard deviation of the probability of loss, is $\mho = 0.2$. Notice that the degree of risk, computed using the expected probabilities $\mathrm{E}[\mathrm{P}_d] = \mathrm{E}[\mathrm{P}_u] = 0.5$, has not changed.

Preferences are the primitives of the decision-making model we employ to form expected utility. Aversion to ambiguity is exhibited when an investor prefers the expectation of the random probability of each outcome over the random probability itself. These preferences concerning ambiguity are modeled by a strictly-increasing, continuous and twice-differentiable function $\Gamma : [0,1] \to \mathbb{R}$, called the *outlook function*. Ambiguity aversion takes the form of a concave $\Gamma$, while ambiguity loving takes the form of a convex $\Gamma$, and ambiguity neutrality the form of a linear $\Gamma$. Preferences concerning risk are modeled by a strictly-increasing, continuous and twice-differentiable *utility function* $\mathrm{U} : \mathbb{R} \to \mathbb{R}$, which is normalized to $\mathrm{U}(k) = 0$. As usual, risk aversion takes the form of a concave U, risk loving the form of a convex U, and risk neutrality the form of a linear U.

As a consequence of the nonlinear ways in which individuals may interpret random probabilities, perceived probabilities are nonadditive. Ambiguity aversion results in a subadditive probability measure, while ambiguity loving results in a superadditive measure.[14] Formally, in this framework, the expected utility takes the form

$$\begin{aligned} \mathrm{V}(X) \;=\; & -\int_{-\infty}^{k} \Gamma^{-1}\left(\int_{\mathbb{P}} \Gamma\left(\mathrm{P}\left(\mathrm{U}(X) < z\right)\right) d\chi(\mathrm{P})\right) dz \\ & + \int_{k}^{\infty} \Gamma^{-1}\left(\int_{\mathbb{P}} \Gamma\left(\mathrm{P}\left(\mathrm{U}(X) > z\right)\right) d\chi(\mathrm{P})\right) dz, \end{aligned} \tag{6}$$

where $X$ is the outcome of an investment (future wealth). The function V, proposed by Izhakian (2012a), is based on the functional representation of Wakker (2010) and Kothiyal

[14]Nonadditivity means that probabilities do not necessarily add up to 1.

*et al.* (2011). This function applies a two-sided Choquet integration to gains and to losses, relative to a reference point. Note that when investors are ambiguity neutral, i.e., $\Gamma$ is linear, Equation (6) collapses to the conventional expected utility.

In asset pricing ambiguity and aversion to it imply a premium, called the *ambiguity premium*, in addition to the conventional *risk premium*. While *risk premium* is the premium that investors are willing to pay for exchanging a risky asset for its expected outcome, the *ambiguity premium* is the premium that investors are willing to pay for exchanging an ambiguous asset for a risky but non-ambiguous asset that has an identical expected outcome. The *uncertainty premium* is the total premium that investors are willing to pay for exchanging an ambiguous asset for its expected outcome, i.e., the accumulation of the risk premium and the ambiguity premium.

# IV The mean–variance–ambiguity paradigm

This section generalizes the mean–variance paradigm to mean–varance–ambiguity and analyzes the tradeoff between expected return, risk and ambiguity. Underpinned by EURP, it relaxes the standard assumption of modern portfolio theory (MPT) that probabilities of return are known and assumes instead that these probabilities are random.

MPT, introduced by Markowitz (1952, 1959) and Tobin (1958), asserts that a rational investor in an efficient market assembles an asset portfolio that maximizes expected return for a given risk, measured by the variance of return. To allow preferences to be of the mean–variance type and risk to be measured by the variance of return, MPT usually assumes that returns are normally distributed, so that probability distributions are completely characterized by a known mean and a known variance.

In our model, returns on assets are assumed to be normally distributed, where the parameters, mean and variance, governing the distributions are random.[15] Formally, let $r_j$ be

[15]This assumption of normally distributed returns can be replaced by an assumption that investors' utility functions are either quadratic or of the constant absolute risk aversion type, for which preferences consider only the first two moments of the probability distribution. See, for example, Ljungqvist and Sargent (2004, pages 154-155). It can also be replaced an assumption of an elliptical probability distribution, which is characterized by the first two moments, mean and variance. See, for example, Owen and Rabinovitch (1983) and Zhou (1993).

the return on asset $j$.[16] Its probability distribution, P, is random and defined by the normal probability density function $\phi(r_j; \mu_j, \sigma_j)$, where the mean $\mu_j$ and the standard deviation $\sigma_j$ are both random. The reference point $k$ which distinguishes losses from gains, agreed upon by all investors, is the risk-free rate of return, denoted $r_f$. All assets are evaluated by their returns relative to $r_f$. Any return lower than $r_f$, even if it is positive, is considered a loss and any return higher than $r_f$ is considered a gain. The degree of ambiguity associated with asset $j$ is then measured by

$$\mho^2[r_j] = 4\mathrm{Var}\left[\int_{-\infty}^{r_f} \frac{1}{\sqrt{2\pi\sigma_j^2}} e^{-\frac{(r_j-\mu_j)^2}{2\sigma_j^2}} dr_j\right] = 4\mathrm{Var}\left[\Phi(r_f; \mu_j, \sigma_j)\right], \tag{7}$$

where $\Phi$ stands for the cumulative normal probability distribution.

The mean–standard-deviation–ambiguity space is a subset of $\mathbb{R}^3$ defined by the mean $\mathrm{E}[r]$, the standard deviation $\mathrm{Std}[r]$, and the normalized degree of ambiguity $\widehat{\mho}[r]$, which is formed by

$$\widehat{\mho}[r] \equiv \mathrm{Std}[r]\sqrt{\frac{\mho^2[r]}{1-\mho^2[r]}}. \tag{8}$$

This normalization is applied for two reasons. The first is that ambiguity $\mho$ is measured in units of probabilities, while $\mathrm{E}[r]$ and $\mathrm{Std}[r]$ are in units of return; Equation (8) normalizes $\mho$ to the units of return. Second, $\mho$ ranges from 0 to 1, while $\mathrm{E}[r]$ and $\mathrm{Std}[r]$ range from 0 to $\infty$; Equation (8) maps $\mho \in [0,1]$ to $[0,\infty)$.

Let $\mathbf{h} = (h_1, \ldots, h_n)$ be a portfolio consisting of $n$ assets, where $h_j \in \mathbb{R}$ is the proportion of asset $j$ in the portfolio such that $\sum_j h_j = 1$. In the mean–standard-deviation–ambiguity space, portfolio $\mathbf{h}$ is represented by a triplet $\left(\mathrm{E}[r_{\mathbf{h}}], \mathrm{Std}[r_{\mathbf{h}}], \widehat{\mho}[r_{\mathbf{h}}]\right) \in \mathbb{R}^3$, where $r_{\mathbf{h}} = \sum_{j=1}^{n} h_j r_j$ is its return. Assuming for the moment that a risk-free asset in not available, the set of feasible portfolios can be defined by the set of parametric triplets $\mathbb{F} = \left\{\left(\mathrm{E}[r_{\mathbf{h}}], \mathrm{Std}[r_{\mathbf{h}}], \widehat{\mho}[r_{\mathbf{h}}]\right) \mid \sum_j h_j = 1\right\}$.[17] Each point in $\mathbb{D}$, defined by $\mathrm{E}[r_{\mathbf{h}}]$, $\mathrm{Std}[r_{\mathbf{h}}]$ and $\widehat{\mho}[r_{\mathbf{h}}]$, designates an investment opportunity.

The set of feasible portfolios satisfies $\mathbb{F} \subset \mathbb{R}^3$, i.e., this set is less than the whole $\left\{\mathrm{E}, \mathrm{Std}, \widehat{\mho}\right\}$. To show this, the case of two perfectly correlated returns with different means

[16] To save on notations, when it is clear from the context, $r$ is also referred to as the random variable and we write, for example, $\mathrm{E}[r]$.

[17] In this case, a zero return can be taken as a reference point.

has to be ruled out. This case implies that one could short one asset, long the other asset, and create an infinite expected return with no uncertainty. But, such a case is a violation of the *law of one price*, which must be satisfied since markets are in equilibrium. In other words, the law of one price implies that there is a bounded set of feasible portfolios in the mean–standard-deviation–ambiguity space. Proposition 2 below proves it formally.

In a two-asset economy the set of feasible portfolios, $\mathbb{F}$, draws a *curve* in the mean–standard-deviation–ambiguity space; for three or more assets, conditional on the relationships between their probability moments, it draws a *surface* or a *volume*. For example, if no ambiguity is present and there are at least three assets, $\mathbb{F}$ draws a plane in the mean–standard-deviation–ambiguity space. $\mathbb{F}$ is not necessarily convex over the entire domain defined by the parameter $\mathbf{h}$, i.e., it can possibly be non-convex for a subset of $\mathbf{h}$. $\mathbb{F}$ is bounded by a hyperbola in the mean–standard-deviation section. In the mean–ambiguity section, $\mathbb{F}$ is also bounded but not necessarily by a concave shape. The upper boundary of $\mathbb{F}$, i.e., the collection of portfolios with maximal expected return $\mathrm{E}[r]$ for a given risk $\mathrm{Std}[r]$ and ambiguity $\mho[r]$, is referred to as the *uncertain asset frontier*. This frontier takes the shape of a curve or a surface in $\mathbb{R}^3$. A portfolio that lies on the uncertain asset frontier is denoted $\mathbf{e}$.

Investment decisions are considered in the context of a static model: investments are made at the beginning of a period, and the outcomes occur at end of a period. Investors are assumed to maximize the expected utility of the end-of-period consumption. Since life ends at the end of a period, there is no difference between consumption and wealth: all end-of-period wealth is consumed. Investors can borrow or lend unlimited quantities at the risk-free rate of return, $r_f$, which is exogenous (see, for example, Sharpe (1964)). All available assets for trading are risky and ambiguous except for the risk-free asset, which has a constant rate of return across all states of nature. All assumptions of the CAPM are maintained except for the assumption that probabilities of returns are known.[18]

Investors are assumed to be *risk averse* and *ambiguity averse*, characterized by the utility

[18] The CAPM assumes that markets are efficient in the sense that all information is available to all investors. All of them have equal access to all assets in a market with no taxes and no commissions, and they can short any asset and hold any fraction of any asset. Investors behave competitively and are faced with a perfect capital market in the sense that they can buy and sell as much as they want of any asset without affecting its price.

function U and the outlook function $\Gamma$, respectively. Given a decision to save an amount $w$ and invest it in portfolio $\mathbf{h}$, the future consumption $c$, determined by the one period (random) return $r_{\mathbf{h}}$, is given by $c = w\left(1 + r_{\mathbf{h}}\right)$. The expected utility associated with this choice is then

$$\begin{aligned} \mathrm{V}(c) &= -\int_{-\infty}^{w\left(1+r_f\right)} \Gamma^{-1}\left(\int_{\mathbb{P}} \Gamma\left(\mathrm{P}\left(\mathrm{U}\left(w\left(1+r_{\mathbf{h}}\right)\right) < z\right)\right) d\chi\left(\mathrm{P}\right)\right) dz \\ &\quad + \int_{w\left(1+r_f\right)}^{\infty} \Gamma^{-1}\left(\int_{\mathbb{P}} \Gamma\left(\mathrm{P}\left(\mathrm{U}\left(w\left(1+r_{\mathbf{h}}\right)\right) > z\right)\right) d\chi\left(\mathrm{P}\right)\right) dz, \end{aligned} \tag{9}$$

where $\mathrm{U}\left(w\left(1+r_f\right)\right) = 0$, and the random probabilities are defined by $\mathrm{P}\left(\mathrm{U}\left(w\left(1+r_{\mathbf{h}}\right)\right) < z\right) = \Phi\left(\frac{\mathrm{U}^{-1}(z)}{w} - 1; \mu_{\mathbf{h}}, \sigma_{\mathbf{h}}\right)$ and $\mathrm{P}\left(\mathrm{U}\left(w\left(1+r_{\mathbf{h}}\right)\right) > z\right) = 1 - \Phi\left(\frac{\mathrm{U}^{-1}(z)}{w} - 1; \mu_{\mathbf{h}}, \sigma_{\mathbf{h}}\right)$.

Given two portfolios with identical risk and ambiguity, a *rational* (risk and ambiguity averse) investor prefers the portfolio with the higher expected return; given two portfolios with identical expected return and risk, she prefers the portfolio with the lower ambiguity; given two portfolios with identical expected return and ambiguity, she prefers the portfolio with the lower risk. These preferences are implied by the next theorem.[19]

**Theorem 1.** *Assume that the reference point $0 \leq r_f \leq \mathrm{E}\left[r_{\boldsymbol{h}}\right]$ is relatively close to $\mathrm{E}\left[r_{\boldsymbol{h}}\right]$. For relatively small returns with relatively small probabilities, the expected utility of $c$ is*

$$\mathrm{V}(c) \approx \mathrm{U}\left(w\left(1 + \mathrm{E}\left[r_{\boldsymbol{h}}\right] - \mathcal{K}\right)\right), \tag{10}$$

*where*

$$\mathcal{K} \approx -\frac{1}{2}\frac{\mathrm{U}''\left(\mathrm{E}\left[r_{\boldsymbol{h}}\right]\right)}{\mathrm{U}'\left(\mathrm{E}\left[r_{\boldsymbol{h}}\right]\right)}\mathrm{Var}\left[r_{\boldsymbol{h}}\right] - \frac{1}{8}\left[\frac{\Gamma''\left(\mathrm{E}\left[\mathrm{P}_L\right]\right)}{\Gamma'\left(\mathrm{E}\left[\mathrm{P}_L\right]\right)} + \frac{\Gamma''\left(\mathrm{E}\left[\mathrm{P}_G\right]\right)}{\Gamma'\left(\mathrm{E}\left[\mathrm{P}_G\right]\right)}\right]\mho^2\left[r_{\boldsymbol{h}}\right] \tag{11}$$

*is the uncertainty premium.*[20,21]

This theorem introduces a general form of the approximated uncertainty premium. In this paper, since returns are normally distributed and depend only on the first two moments of the probability distribution, the approximation in Equation (11) is accurate. An exact

[19] This theorem extends Theorem 8 in Izhakian (2012a) to the case of state space with infinite support.

[20] The proof of this theorem applies the same method as used by Arrow (1965) and Pratt (1964). The Arrow-Pratt coefficient of absolute risk aversion is defined by $-\frac{\mathrm{U}''}{\mathrm{U}'}$. Similarly, the coefficient of absolute ambiguity aversion is defined by $-\frac{\Gamma''}{\Gamma'}$.

[21] Maccheroni *et al.* (2010) and Izhakian and Benninga (2011), which employ Klibanoff *et al.'s* (2005) *smooth model of ambiguity*, also have derived an uncertainty premium. In their models, as in Equation (11), the first component is the Arrow-Pratt risk premium. However, while in their models the ambiguity premium is also a function of risk attitude; Whereas in the model of Equation (11), the ambiguity premium is a function of only ambiguity and preferences concerning ambiguity.

expression in Equation (11) is obtained when both the utility function, U, and the outlook function, Γ, are quadratic.[22] Notice that, since returns are normally distributed, the ordinal values of the degree of ambiguity, $\mho^2$, and the uncertainty premium, $\mathcal{K}$, are not sensitive to changes in the reference point.

Theorem 1 states that a higher expected return results in a higher expected utility. It also proposes that higher risk results in a lower expected utility, since investors are risk-averse ($-\frac{\mathrm{U}''}{\mathrm{U}'} < 0$). In particular, expected utility generated by a normally distributed portfolio is a declining function of the standard deviation of return. A second source of uncertainty is ambiguity, which also has a negative impact on expected utility. This happens since (random) probabilities, as perceived by ambiguity-averse investors, are subadditive and are a decreasing function of the degree of ambiguity, $\mho^2$, and aversion to it, $-\frac{\Gamma''}{\Gamma'} < 0$.[23] The conclusion derived from Theorem 1 is, therefore, that a risk and ambiguity averse investors would aim to minimize both risk and ambiguity for a given expected return as they assemble their optimal asset portfolios.

Preferences concerning both risk and ambiguity define a set of portfolios over which the investor is indifferent. Each such set draws a shell in $\mathbb{R}^3$, referred to as the *indifference surface*. A rational investor chooses from among all feasible portfolios the one placing her on the indifference surface that represents the highest level of expected utility. Assuming that there are at least three risky–ambiguous assets, Figure 1 illustrates the set of feasible portfolios and the indifference surface in the mean–standard-deviation–ambiguity space in an economy without a risk-free asset. The lower, horizontal paraboloid volume represents all feasible portfolios. The upper concave shell represents the indifference surface describing the tradeoff between risk, ambiguity, and expected return, as guided by preferences. The higher the indifference surface, the higher the level of expected utility.

Theorem 1 implies that risk and ambiguity have a negative impact on expected utility, while expected return has a positive impact; this means that expected return compensates for bearing risk and ambiguity. Thus, Theorem 1 characterizes mean–standard-deviation–ambiguity preferences, which in turn enable the definition of efficient portfolios. A portfolio $\mathbf{h}$, characterized by the mean $\mathrm{E}[r_{\mathbf{h}}]$, the standard deviation $\mathrm{Std}[r_{\mathbf{h}}]$ and the ambiguity

[22] An exact form of Equation (11) can also be proven by constructing the state prices.

[23] *Subadditive* means that probabilities add up to a number lower than 1.

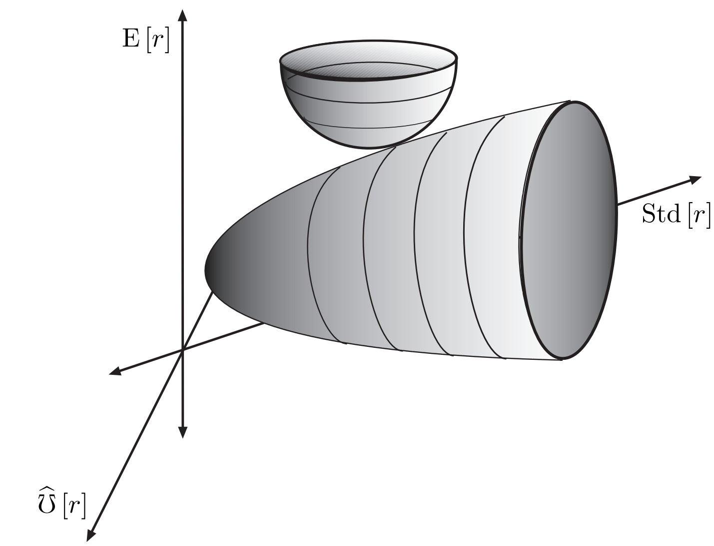


Figure 1: **Feasible portfolios and the indifference surface**
This figure illustrates the set of feasible portfolios in the mean–standard-deviation–ambiguity space ($\mathbb{R}^3$) in an economy without a risk-free asset. The x-axis describes the normalized degree of ambiguity. The y-axis describes the degree of risk, measured by standard deviation. The z-axis describes the expected return. The lower, horizontal paraboloid volume represents the set of feasible portfolios, and the upper concave shell represents the indifference surface.

$\mho[r_{\mathbf{h}}]$, is *efficient* if there is no other portfolio with the same standard deviation and the same ambiguity that has a higher expected return. The set of efficient portfolios establishes the *efficient frontier*, which takes the shape of a surface in $\mathbb{R}^3$. In an economy without a risk-free asset, the efficient frontier is the upper boundary of the set of feasible portfolios—the uncertain asset frontier.

Inclusion of a risk-free asset in the economy draws infinitely many lines emanating from the point $(r_f, 0, 0)$ and tangent to the uncertain asset frontier. Each line is tangent to the uncertain asset frontier at a different point $\left(\mathrm{E}[r_{\mathbf{e}}], \mathrm{Std}[r_{\mathbf{e}}], \widehat{\mho}[r_{\mathbf{e}}]\right)$ and satisfies

$$\left(\mathrm{E}[hr_f + (1-h)\, r_{\mathbf{e}}], \mathrm{Std}[hr_f + (1-h)\, r_{\mathbf{e}}], \widehat{\mho}[hr_f + (1-h)\, r_{\mathbf{e}}]\right). \tag{12}$$

These tangent lines emanating from $r_f$ are linear in $h$, as the following proposition asserts.

**Proposition 1.** *Assuming that the reference point is $r_f$ and that an efficient portfolio $\boldsymbol{e}$ is normally distributed (with random mean and random variance), the line drawn by the parametric triplet*

$$\left(\mathrm{E}[hr_f + (1-h)\, r_{\boldsymbol{e}}], \mathrm{Std}[hr_f + (1-h)\, r_{\boldsymbol{e}}], \widehat{\mho}[hr_f + (1-h)\, r_{\boldsymbol{e}}]\right), \tag{13}$$

*is linear in h.*

This proposition implies that when a risk-free asset exists, any efficient portfolio lies on a straight line emanating from $r_f$ and tangent to the uncertain asset frontier. In such an economy the efficient frontier is drawn by the collection of these lines. As Theorem 1 shows, expected utility is an increasing function of $\mathrm{E}[r_{\mathbf{h}}]$ and a decreasing function of $\mathrm{Std}[r_{\mathbf{h}}]$ and $\mho[r_{\mathbf{h}}]$. Hence, expected utility maximization implies that any optimal portfolio must lie on the efficient frontier. A particular optimal portfolio is selected by each investor individually according to her preferences concerning risk and ambiguity.

Figure 2 illustrates the uncertain asset frontier and the efficient frontier in an economy in which a risk-free asset exists. The straight dashed lines emanating from the risk-free rate point, $r_f$, and tangent to the curved surface describe the set of efficient portfolios, i.e., the efficient frontier. A rational investor selects a portfolio from this set guided by her preferences concerning risk and ambiguity. In other words, her selection is determined by the tangency point of the indifference surface and the efficient frontier.

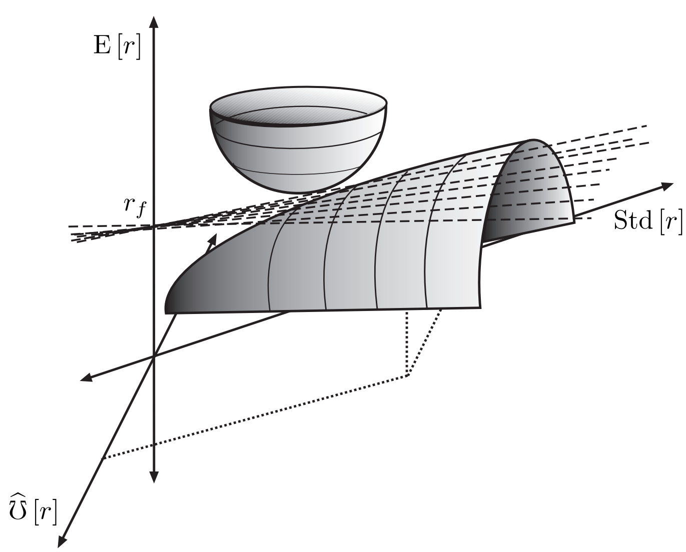


Figure 2: **The uncertain asset frontier and the efficient frontier**
This figure illustrates the uncertain asset frontier and the efficient frontier in the mean–standard-deviation–ambiguity space ($\mathbb{R}^3$) when a risk-free asset exists. The x-axis describes the normalized degree of ambiguity. The y-axis describes the degree of risk, measured by standard deviation. The z-axis describes the expected return. The upper concave shell is the indifference surface, and the lower, horizontal shaded concave surface is the uncertain asset frontier. The straight lines emanating from $r_f$ and tangent to the lower, horizontal concave surface describe the set of efficient portfolios, i.e., the efficient frontier.

# V The mean–uncertainty space and the capital market line

To construct the *capital market line* (CML) the mean–standard-deviation-ambiguity space is projected to the mean–uncertainty space in $\mathbb{R}^2$, where preferences in the latter space are induced by preferences in the former space.

**Assumption 1.** *All investors aggregate risk and ambiguity into uncertainty in the same way, and each investor has her own preferences concerning this uncertainty.*

This assumption asserts that all investors employ the same methodology to consolidate their beliefs regarding the degree of uncertainty based on their common beliefs regarding risk and ambiguity. In other words, given a degree of risk and a degree of ambiguity, every investor translates this pair to the same degree of uncertainty. Hence, they translate the set of feasible portfolios, the uncertain asset frontier and the efficient frontier to three unique sets in the mean–uncertainty space. It is important to note that beliefs, which are common to all investors, are shaped independently of investors' preferences. Then, given these beliefs about the degree of uncertainty, each investor selects a portfolio guided by her personal preferences concerning uncertainty, which are derived from her preferences concerning risk and ambiguity. It is assumed that all investors solve the same optimization problem to maximize expected return conditional on the degree of uncertainty. Since all investors have the same investment opportunities to choose from, the same information and the same decision procedure, every selected portfolio lies on the same efficient frontier.

Now we can simplify the $\mathbb{R}^3$ mean–standard-deviation–ambiguity space to the $\mathbb{R}^2$ mean–uncertainty space by projecting the former space onto the latter. The projection is obtained by the mapping $\left(\mathrm{E}[r], \mathrm{Std}[r], \widehat{\mho}[r]\right) \mapsto \mathbb{R}^2$. This mapping, defined by the Euclidean norm $\left|\left(\mathrm{Std}[r], \widehat{\mho}[r]\right)\right| = \sqrt{\mathrm{Var}[r] + \widehat{\mho}^2(r)}$, can be written (by substituting for $\widehat{\mho}$) as

$$\left|\left(\mathrm{Std}[r], \widehat{\mho}[r]\right)\right| = \sqrt{\frac{\mathrm{Var}[r]}{1-\mho^2[r]}}. \tag{14}$$

This expression aggregates risk and ambiguity to a natural combined *uncertainty measure*.[24]

[24] I thank Xavier Gabaix for his relevant comment.

**Theorem 2.** *The aggregated measure of risk and ambiguity, called the* ***uncertainty measure****,can be defined by*

$$\Upsilon[r] \equiv \sqrt{\frac{\mathrm{Var}[r]}{1-\mho^2[r]}}. \tag{15}$$

This theorem provides a unified measure of uncertainty in units of return. The ordering of assets or asset portfolios by the measure of uncertainty $\Upsilon$ coincides with the ordering provided by an uncertainty-averse (risk and ambiguity avers) investor. When no ambiguity is present, i.e., $\mho^2[r]=0$, the uncertainty measure collapses to the simple standard deviation, which measures risk. In the other extreme case, when the degree of ambiguity is maximal, i.e., $\mho^2[r]=1$, the degree of uncertainty is infinite. An infinite degree of uncertainty is also attained when the standard deviation tends to infinity. It is important to note that $\Upsilon$ is an objective measure, which captures only *beliefs*, so that subjective preferences are not involved in measuring the degree of uncertainty.

Equation (14) maps every point $\left(\mathrm{E}[r], \mathrm{Std}[r], \widehat{\mho}[r]\right) \in \mathbb{R}^3$ in the mean–standard-deviation–ambiguity space to a point $\left(\mathrm{E}[r], \frac{\mathrm{Std}[r]}{\sqrt{1-\mho^2[r]}}\right) \in \mathbb{R}^2$ in the mean–uncertainty space. In particular, the surface defining the uncertain asset frontier results in a unique curve in the mean–uncertainty space. When incorporating a risk-free asset, the surface defining the efficient frontier results in a unique line in the mean–uncertainty space called the *capital market line* (CML). Lastly, each indifference surface results in a single indifference curve in the mean–uncertainty space.

Take, for example, an two-asset economy with a single uncertain asset, denoted $j$, and a risk-free asset. Figure 3 shows the opportunity set available to investors in the mean–uncertainty space. The slope of the line describing the opportunity set generated by these two assets is given by

$$\frac{d\mathrm{E}[r]}{d\Upsilon[r]} = \frac{\mathrm{E}[r_j]-r_f}{\Upsilon[r_j]}, \tag{16}$$

and its intercept is the risk-free rate. That is, the intercept is obtained by a portfolio consisting of only the risk-free asset—the only portfolio bearing no uncertainty. The dashed lines depict the opportunities that are only possible if short sales are allowed.

The set of feasible portfolios in the mean–uncertainty space is a subset of $\mathbb{R}^2$. The uncertain asset frontier in the mean–uncertainty space takes the form of a curve which

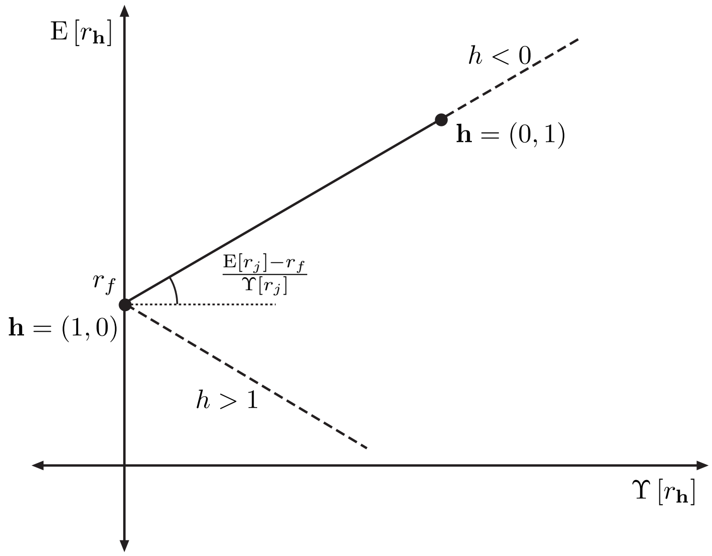


Figure 3: **The opportunity set generated by an uncertain asset and a risk-free asset**
This figure describes the opportunity set in the mean–uncertainty space, when there is only one uncertain asset and a risk-free asset. The x-axis describes the degree of uncertainty, measured by $\Upsilon$, and the y-axis describes the expected return. A portfolio $\mathbf{h} = (h, 1-h)$ consists of a portion $h$ of the risk-free asset and a proportion $1-h$ of the uncertain asset. In particular, $\mathbf{h} = (1, 0)$ and $\mathbf{h} = (0, 1)$ represent a portfolio consisting of only the risk-free asset and a portfolio consisting of only the uncertain asset, respectively. Asset proportions range from zero to one along the solid portion of the opportunity set. Short sales of either asset extend the opportunity set along the dashed line.

defines the minimal degree of uncertainty, $\Upsilon$, for every level of expected return. The degree of uncertainty and the expected return on an optimal portfolio are always nonnegative so that this frontier lies in the first quadrant. The uncertain asset frontier exists since the law of one price is satisfied in equilibrium such that there are no two perfectly correlated assets with different expected returns. The next proposition articulates this.

**Proposition 2.** *Assume that there are no redundant assets. If the expected variance–covariance matrix of returns is nonsingular and the degree of ambiguity associated with any asset portfolio is not equal to* 1*, then the uncertain asset frontier and the efficient frontier exist.*

Writing the uncertain asset frontier as a function of expected return, i.e., expressing the minimal degree of uncertainty as a function of expected return (see Equation (37)), shows that in almost all cases the frontier is concave over the entire domain. Exceptions, in which the curve is non-concave over a relative small subdomain, might occur in the extreme case where the correlation between every two assets is close to 1. Such a case can happen since ambiguity and variance are not independent.[25] The uncertain asset frontier, however, is always bounded, as the next proposition asserts.

**Proposition 3.** *The uncertain asset frontier is bounded by a hyperbola, defined by risk, such that for any given level of expected return the minimal degree of uncertainty is higher than the minimal degree of risk.*

The CML takes the form of a line in $\mathbb{R}^2$ originating from $r_f$ and tangent to the uncertain asset frontier at the point $\mathbf{m}$, which is referred to as the *market portfolio*. In equilibrium the expected return on the market portfolio is at least as high as the risk-free rate. The risk-free rate is lower than the expected return on the portfolio with the minimal possible uncertainty, i.e., the *global minimum uncertainty portfolio*; otherwise, investors with a mean–uncertainty objective would attempt to short the uncertain assets, which cannot represent an equilibrium (see Cochrane (2005)). The market portfolio, $\mathbf{m}$, is unique. To recognize this, note that since the market has already reached an equilibrium, which is governed by supply and demand, the proportion of each asset in the market portfolio is determined by its capital market value divided by the capital value of the whole market. The capital market value of an asset (total worth of its shares) is unique, which implies that the proportion of each asset in the portfolio is unique. Therefore, the market portfolio is unique.[26]

All portfolios lying on the CML are efficient in the sense that they attain the minimal degree of uncertainty for a given level of expected return. In the mean–uncertainty space a *rational* investor minimizes the degree of uncertainty for a given expected return such that any portfolio $\mathbf{h}$ she chooses lies on the CML. Therefore, any optimal portfolio $\mathbf{h}$ satisfies

$$\frac{\mathrm{E}\left[r_{\mathbf{h}}\right]-r_f}{\sqrt{\frac{\mathrm{Var}[r_{\mathbf{h}}]}{1-\mho^2[r_{\mathbf{h}}]}}} = \frac{\mathrm{E}\left[r_{\mathbf{m}}\right]-r_f}{\sqrt{\frac{\mathrm{Var}[r_{\mathbf{m}}]}{1-\mho^2[r_{\mathbf{m}}]}}}, \tag{17}$$

where $\mathrm{E}\left[r_{\mathbf{h}}\right]-r_f$, defines the *uncertainty premium* which rewards for the uncertainty, $\Upsilon\left[r_{\mathbf{h}}\right] = \sqrt{\frac{\mathrm{Var}[r_{\mathbf{h}}]}{1-\mho[r_{\mathbf{h}}]}}$, associated with portfolio $\mathbf{h}$. The CML defines the relationship between the expected return and the degree of uncertainty of any optimal portfolio. By Equation (17), the

[25] See Izhakian (2012c).

[26] The composition of the market portfolio can be solved numerically by equating the slope of the CML to the slope of the uncertain asset frontier, which can be extracted from Equation (37).

CML can be written as

$$\mathrm{E}\left[r_{\mathbf{h}}\right] \;=\; r_f + \left(\frac{\mathrm{E}\left[r_{\mathbf{m}}\right] - r_f}{\Upsilon\left[r_{\mathbf{m}}\right]}\right)\Upsilon\left[r_{\mathbf{h}}\right], \tag{18}$$

which implies a linear relationship between the expected return, $\mathrm{E}\left[r_{\mathbf{h}}\right]$, on portfolio $\mathbf{h}$ and its degree of uncertainty, $\Upsilon\left[r_{\mathbf{h}}\right]$. The slope of the CML, $\frac{\mathrm{E}[r_{\mathbf{m}}]-r_f}{\Upsilon[r_{\mathbf{m}}]}$, defines the compensation per unit of uncertainty borne in the market. This compensation is the same for any investor, no matter how uncertainty averse she is.

The CML is steeper if the economy is less uncertain, i.e., if the return on the market portfolio or its probabilities are less volatile. The reason is that investors demand a relatively high premium for bearing asset uncertainty when the alternative—the market portfolio—is characterized by relatively low uncertainty. A portfolio with $\Upsilon\left[r_{\mathbf{h}}\right] = 0$ corresponds to a portfolio consisting of only the risk-free asset, implying that its expected return is the risk-free rate. On the other hand, $\Upsilon\left[r_{\mathbf{h}}\right] = \Upsilon\left[r_{\mathbf{m}}\right]$ corresponds to a portfolio consisting of only the market portfolio and thus its expected return satisfies $\mathrm{E}\left[r_{\mathbf{h}}\right] = \mathrm{E}\left[r_{\mathbf{m}}\right]$. All other efficient portfolios are obtained along the CML.

Figure 4 describes the uncertain asset frontier and the CML in the mean–uncertainty space. The dotted external concave frontier depicts the uncertain asset frontier when ambiguity is not present, and the solid internal concave frontier depicts the uncertain asset frontier when the economy is imbued with ambiguity.[27] The shaded area is the set of all feasible portfolios in an ambiguous economy without a risk-free asset. One can observe that the minimal uncertainty that accompanies any given expected return is higher when ambiguity is present. The solid straight line is the CML in an ambiguous economy, and the dotted straight line is the CML in a non-ambiguous economy. The slope of the CML is defined by Equation (18). Any point on the CML to the right of $\mathbf{m}$ implies borrowing at the risk-free rate (recall that the risk-free rate is exogenously given).

Tobin's (1958) *separation theorem* asserts that any investor should hold the risk-free asset and a unique optimal portfolio of risky assets—the market portfolio. Equation (18) generalizes Tobin's theorem from risk to uncertainty. It implies that investment decisions can be broken down into two separate phases: the first concerns the choice of a unique optimal uncertain asset portfolio, and the second concerns the allocation of funds to the risk-free

[27]In a non-ambiguous economy the uncertain asset frontier is the *risky asset frontier*.

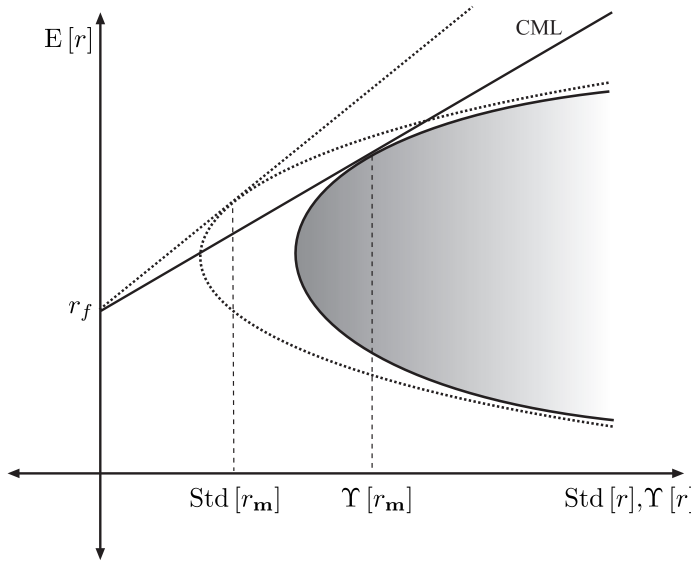


Figure 4: **The uncertain asset frontier and the CML**
This figure describes the uncertain asset frontier and the CML in the mean–uncertainty space. The dotted external frontier depicts the uncertain asset frontier in a non-ambiguous economy, and the solid internal frontier depicts the uncertain asset frontier in an ambiguous economy. The shaded area is the set of feasible portfolios in an ambiguous economy without a risk-free asset. The solid straight line is the CML in an ambiguous economy, and the dotted straight line is the CML in a non-ambiguous economy.

asset and the uncertain portfolio. Investors are different only in their decisions regarding the proportions allocated to the risk-free asset and the uncertain portfolio. Thus, every investor holds uncertain assets in the same proportion, as defined by the market portfolio. The nature of the market portfolio in an ambiguous economy, however, is different from Tobin's market portfolio. Tobin's market portfolio is a portfolio with the minimal risk for a given expected return, whereas in the current model, the market portfolio has the minimal uncertainty for a given expected return, but not necessarily the minimal risk.

Individuals are different in their attitude toward uncertainty, thus the proportions of the risk-free asset and the market portfolio they choose are different. More conservative investors, for example, will choose to allocate a higher portion of their wealth to the risk-free asset. More aggressive investors may decide to borrow capital on the money market, i.e., to make a negative allocation to the risk-free asset, and invest in the market portfolio. In any case, every investment decision is made on the CML.

The CML defines the reward for an efficient portfolio per unit of the entire (systematic and idiosyncratic) uncertainty borne. The next section refines the systematic uncertainty associated with individual assets and defines the premium per unit of systematic uncertainty borne.

# VI Capital asset pricing

This section introduces a new model of capital asset pricing under ambiguity, called the *capital asset pricing model under ambiguity* (ACAPM). This model is based on the mean–uncertainty framework and proposes that expected return on an asset corresponds to its uncertainty relative to the market and not to its own uncertainty. The classical CAPM is a special case of the ACAPM, in which ambiguity is not present.

Risk and ambiguity can be inversely related such that risk reduction incurs higher ambiguity (see Izhakian (2012c)). The implication of this inverse relationship is that full risk diversification is necessarily optimal. Hence, we suggest the following definition.

**Definition 1.** *For a given expected return: systematic risk is the risk component that is optimal not to diversify, and idiosyncratic risk is the component that is optimal to diversify; likewise, systematic ambiguity is the ambiguity component that is optimal not to diversify, and idiosyncratic ambiguity is the component that is optimal to diversify. The aggregation of this systematic risk and systematic ambiguity is systematic uncertainty. Likewise, the aggregation of this idiosyncratic risk and idiosyncratic ambiguity is idiosyncratic uncertainty.*

Systematic uncertainty is the minimal possible uncertainty for a given expected return. Considering the tradeoff between risk and ambiguity in ACAPM, the systematic components of risk and ambiguity are the components that are optimal to not diversify, even though parts of them can potentially be diversified. Note that when ambiguity is not present, risk minimization is optimal, systematic risk is the non-diversifiable risk and idiosyncratic risk is the diversifiable risk.

The next theorem introduces a closed-form pricing model in which only the systematic component of uncertainty is priced. That is, compensation is provided only for the systematic risk and the systematic ambiguity, each independently.

**Theorem 3.** *Let $r_{\boldsymbol{m}}$ be the return on the market portfolio, $r_f$ be the risk-free rate of return, and $r_j$ be the return on asset $j$. Assume risk and ambiguity averse investors whose reference point is $r_f$. If the returns on all assets are normally distributed with random mean, $\mu$, and random variance, $\sigma^2$, then the expected return on asset $j$ is*

$$\mathrm{E}\left[r_j\right] = r_f + \beta_{\mathcal{R},j}\left(\mathrm{E}\left[r_{\boldsymbol{m}}\right] - r_f\right) + \beta_{\mathcal{A},j}\left(\mathrm{E}\left[r_{\boldsymbol{m}}\right] - r_f\right). \tag{19}$$

*The **beta risk** is*

$$\beta_{\mathcal{R},j} = \frac{\mathrm{Cov}\left[r_{\boldsymbol{m}}, r_j\right]}{\mathrm{Var}\left[r_{\boldsymbol{m}}\right]}, \tag{20}$$

*and the **beta ambiguity** is*

$$\beta_{\mathcal{A},j} = 4\frac{\mathrm{Cov}\left[\Phi\left(r_f;\mu_{\boldsymbol{m}},\sigma_{\boldsymbol{m}}\right), \phi\left(r_f;\mu_{\boldsymbol{m}},\sigma_{\boldsymbol{m}}\right)\left(\frac{\sigma_{m,j}}{\sigma_{\boldsymbol{m}}^2}\left(\mu_{\boldsymbol{m}} - r_f\right) - \left(\mu_j - r_f\right)\right)\right]}{1 - \mho^2\left[r_{\boldsymbol{m}}\right]}, \tag{21}$$

*where*

$$\Phi\left(r_f;\mu_{\boldsymbol{m}},\sigma_{\boldsymbol{m}}\right) = \int_{-\infty}^{r_f} \frac{1}{\sqrt{2\pi\sigma_{\boldsymbol{m}}^2}} e^{-\frac{\left(r-\mu_{\boldsymbol{m}}\right)^2}{2\sigma_{\boldsymbol{m}}^2}}\, dr \tag{22}$$

*is the random cumulative probability of loss on the market portfolio, and*

$$\phi\left(r_f;\mu_{\boldsymbol{m}},\sigma_{\boldsymbol{m}}\right) = \frac{1}{\sqrt{2\pi\sigma_{\boldsymbol{m}}^2}} e^{-\frac{\left(r_f-\mu_{\boldsymbol{m}}\right)^2}{2\sigma_{\boldsymbol{m}}^2}} \tag{23}$$

*is its probability density at the reference point.*

This theorem asserts that the expected return on an asset is a function of its systematic uncertainty, formulated by $\beta_{\mathcal{R}}$ and $\beta_{\mathcal{A}}$. Beta risk, $\beta_{\mathcal{R}}$, is a function of the covariance between an asset return and the market return, computed using expected probabilities, i.e., the compounded first-order and second-order probabilities. Beta ambiguity, $\beta_{\mathcal{A}}$, is derived from the correlation between the probability of loss on an asset and the probability of loss on the market portfolio. It is a function of the covariance between various elements of random probabilities of loss, computed using second-order probabilities, i.e., the probability distribution of the random parameters $\mu$ and $\sigma$.

The ACAPM decomposes the price of an asset in terms of expected return into three components: the price of time, the price of risk and the price of ambiguity. The *price of time*, formed by $r_f$, is the pure interest rate on the risk-free asset. The *price of risk*, formed by

$$\mathcal{R} \;=\; \frac{\operatorname{Cov}\left[r_{\mathbf{m}}, r_j\right]}{\operatorname{Var}\left[r_{\mathbf{m}}\right]}\left(\operatorname{E}\left[r_{\mathbf{m}}\right]-r_f\right), \tag{24}$$

is an additional expected return rewarding for the systematic risk borne, referred to as the *risk premium*. The *price of ambiguity*, formed by

$$\mathcal{A}=4\frac{\operatorname{Cov}\left[\Phi\left(r_f;\mu_{\mathbf{m}},\sigma_{\mathbf{m}}\right),\phi\left(r_f;\mu_{\mathbf{m}},\sigma_{\mathbf{m}}\right)\left(\frac{\sigma_{\mathbf{m},j}}{\sigma_{\mathbf{m}}^2}\left(\mu_{\mathbf{m}}-r_f\right)-\left(\mu_j-r_f\right)\right)\right]}{1-\mho^2\left[r_{\mathbf{m}}\right]}\left(\operatorname{E}\left[r_{\mathbf{m}}\right]-r_f\right), \tag{25}$$

is a second additional expected return rewarding for the systematic ambiguity borne, referred to as the *ambiguity premium*. The uncertainty premium on the market portfolio $\mathbf{m}$, formed by $\operatorname{E}\left[r_{\mathbf{m}}\right]-r_f$, is the aggregate excess return rewarding for both risk and ambiguity borne by $\mathbf{m}$. The risk and the ambiguity premiums on asset $j$ are proportional to the uncertainty premium on $\mathbf{m}$, where their proportions are determined by the coefficients $\beta_{\mathcal{R},j}$ and $\beta_{\mathcal{A},j}$, respectively.

In ACAPM the ambiguity premium, defined by Equation (25), is derived from the co-movement of the probability of loss on the asset and the probability of loss on the market portfolio. The premium of an asset is proportional to the ambiguity of the asset relative to the ambiguity of the market portfolio, not to its own ambiguity. That is, investors are rewarded only for the systematic ambiguity and not for the entire, systematic and idiosyncratic, ambiguity measured by $\mho^2$. Investors do not ask for, nor does the market propose, compensation for idiosyncratic ambiguity since it is optimal to diversify it away.

Assume for the moment a non-ambiguous economy, i.e., $\mu$ and $\sigma$ are constants for all assets. In this case, $\mu_{\mathbf{m}}$ is also constant. Therefore, for any asset, $\beta_{\mathcal{A}}=0$ such that Theorem 3 collapses to the classical CAPM, in which only a reward for systematic risk is provided.[28] If $\beta_{\mathcal{R},j}=1$, then the expected return on asset $j$ equals the the expected return on the market portfolio, i.e., $\operatorname{E}\left[r_j\right]=\operatorname{E}\left[r_{\mathbf{m}}\right]$. If $\beta_{\mathcal{R},j}>1$, then $\operatorname{E}\left[r_j\right]>\operatorname{E}\left[r_{\mathbf{m}}\right]$; and if $\beta_{\mathcal{R},j}<1$, then $\operatorname{E}\left[r_j\right]<\operatorname{E}\left[r_{\mathbf{m}}\right]$—exactly as in the classical CAPM. If the return on asset $j$ is negatively correlated with the return on the market portfolio, i.e., $\beta_{\mathcal{R},j}<0$, then $\operatorname{E}\left[r_j\right]<r_f$, implying that investors hold this asset as insurance against a decrease in the market return.

Beta ambiguity, $\beta_{\mathcal{A}}$, is a function of the correlation between the marginal probability of loss and the cumulative probability of loss on the market portfolio. The sign and intensity of

[28] To observe this, recall that the covariance between a constant and a random variable is always 0.

this correlation is influenced by two elements. First, the higher the cumulative probability of loss on the market portfolio, $\Phi(r_f; \mu_{\mathbf{m}}, \sigma_{\mathbf{m}})$, the greater $\beta_{\mathcal{A}}$. The reason is that assets that are positively correlated with the market portfolio are required to provide higher premiums to induce investors to hold them when the probability of loss on the market portfolio is high. Second, assuming a positive expected excess return, i.e., $r_f - \mu_{\mathbf{m}} \leq 0$, a higher probability of loss on the market portfolio implies a higher probability density $\phi(r_f; \mu_{\mathbf{m}}, \sigma_{\mathbf{m}})$ and therefore a higher $\beta_{\mathcal{A}}$. Higher values of the random covariance, $\sigma_{\mathbf{m},j}$, between the return on asset $j$ and the return on the market portfolio $\mathbf{m}$ also implies a higher beta ambiguity, which in turn implies a higher expected return on asset $j$.

An interesting insight emerges in the formulation of beta ambiguity in Equation (21). The component $(\mu_j - r_f) - \frac{\sigma_{\mathbf{m},j}}{\sigma_{\mathbf{m}}^2}(\mu_{\mathbf{m}} - r_f)$ of this equation, suggests that this difference is the *unexpected mean return*—that is, an unexpected shift of the entire distribution of an asset's returns. We can write $\Delta = (\mu_j - r_f) - \frac{\sigma_{\mathbf{m},j}}{\sigma_{\mathbf{m}}^2}(\mu_{\mathbf{m}} - r_f)$, where $\Delta$ is random. A higher $\Delta$ implies a higher absolute value of $\beta_{\mathcal{A},j}$. If $\Delta$ and the probability of loss on the market portfolio, $\Phi(r_f; \mu_{\mathbf{m}}, \sigma_{\mathbf{m}})$, are positively correlated, then $\beta_{\mathcal{A},j}$ is negative, as is its impact on the expected return $\mathrm{E}[r_j]$ of asset $j$. If they are negatively correlated then $\beta_{\mathcal{A},j}$ has a positive impact on $\mathrm{E}[r_j]$. The intuition in this relation is that a positive correlation between the probability of loss on the market portfolio and $\Delta$ compensates for events that induce a high probability of loss. Therefore, the price of an asset with a positive correlation between $\Phi(r_f; \mu_{\mathbf{m}}, \sigma_{\mathbf{m}})$ and $\Delta$ is relatively high and, accordingly, its expected return is relatively low. Recall that $\Delta$ is not a shift of return, but a shift of the entire distribution of returns, which shifts the expected return. That is, it can be understood as a shock to the parameter $\mu$, governing the probability distribution.

The ACAPM allows for an asset $j$ to have $\beta_{\mathcal{R},j} \neq 0$ and $\beta_{\mathcal{A},j} = 0$ at the same time. This can happen, for example, when either the probability of return on the market portfolio or the probability of return on asset $j$ are perfectly known.[29] It is also possible to observe an asset characterized by $\beta_{\mathcal{R},j} = 0$ and $\beta_{\mathcal{A},j} \neq 0$ at the same time. This can happen, for example, when the covariance between $r_j$ and $r_{\mathbf{m}}$ is symmetrically volatile around zero or simply when $r_j$ and $r_{\mathbf{m}}$ are not correlated (their covariance is zero). In this case the expected

[29] Consider, for example, the case where $\mu_j$ and $\sigma_j$ are both constant (known). In this case, the covariance between $r_j$ and $r_{\mathbf{m}}$ is also constant, which implies that $(\mu_j - r_f) = \frac{\sigma_{\mathbf{m},j}}{\sigma_{\mathbf{m}}^2}(\mu_{\mathbf{m}} - r_f)$ and thus $\beta_{\mathcal{A},j} = 0$.

return on asset $j$ is not equal to the risk-free rate (see, for example, Merton (1973)). The following corollary defines the $\beta_{\mathcal{R}}$ and $\beta_{\mathcal{A}}$ of the market portfolio.

**Corollary 1.** *Beta risk and beta ambiguity of the market portfolio* $\boldsymbol{m}$ *satisfy* $\beta_{\mathcal{R},\boldsymbol{m}} = 1$ *and* $\beta_{\mathcal{A},\boldsymbol{m}} = 0$*, respectively.*

Beta risk, $\beta_{\mathcal{R}}$, and beta ambiguity, $\beta_{\mathcal{A}}$, of an asset portfolio are both linear in the betas of the individual assets composing the portfolio. To see this, consider a portfolio $\mathbf{h} = (h_1, \ldots, h_n)$ consisting of $n$ assets. The expected return on portfolio $\mathbf{h}$ can be expressed as

$$\mathrm{E}\left[r_{\mathbf{h}}\right] - r_f = \sum_{j=1}^{n} h_j \left(\mathrm{E}\left[r_j\right] - r_f\right) = \left(\mathrm{E}\left[r_{\mathbf{m}}\right] - r_f\right) \left(\sum_{j=1}^{n} h_j \beta_{\mathcal{R},j} + \sum_{j=1}^{n} h_j \beta_{\mathcal{A},j}\right), \tag{26}$$

implying a linear beta pricing model, even when ambiguity is involved. In other words, beta risk and beta ambiguity of an asset portfolio are equal to the weighted sum of the betas of the individual assets composing the portfolio.

The ACAPM, modeled by Equation (19) in Theorem 3, can be written as

$$\mathrm{E}\left[r_j\right] - r_f \quad = \quad \beta_{\mathcal{K},j} \left(\mathrm{E}\left[r_{\mathbf{m}}\right] - r_f\right), \tag{27}$$

where $\beta_{\mathcal{K},j} = \beta_{\mathcal{R},j} + \beta_{\mathcal{A},j}$ is referred to as *beta uncertainty* of asset $j$. Beta uncertainty makes the distinction between the *systematic uncertainty* and the *idiosyncratic uncertainty* associated with an asset, as the following theorem proposes.

**Theorem 4.** *The uncertainty associated with an asset $j$ can be decomposed by*

$$\Upsilon^2\left[r_j\right] \quad = \quad \beta_{\mathcal{K},j}^2 \frac{1 - \mho^2\left[r_{\boldsymbol{m}}\right]}{1 - \mho^2\left[r_j\right]} \Upsilon^2\left[r_{\boldsymbol{m}}\right] + \Upsilon^2\left[\epsilon\right], \tag{28}$$

*where* $\mathrm{E}\left[\epsilon\right] = 0$*, and* $\epsilon$ *and* $r_{\boldsymbol{m}}$ *are independent for every feasible probability distribution. The term* $\beta_{\mathcal{K},j}^2 \frac{1-\mho^2[r_m]}{1-\mho^2[r_j]} \Upsilon^2\left[r_{\boldsymbol{m}}\right]$ *is referred to as the* ***systematic uncertainty****, and the term* $\Upsilon^2\left[\epsilon\right]$ *is referred to as the* ***idiosyncratic uncertainty****, associated with asset $j$.*

In the special case when ambiguity is not involved, as in the classical CAPM, the risk associated with an asset $j$ can be decomposed by

$$\mathrm{Var}\left[r_j\right] \quad = \quad \beta_{\mathcal{R},j}^2 \mathrm{Var}\left[r_{\mathbf{m}}\right] + \mathrm{Var}\left[\epsilon\right], \tag{29}$$

where $\mathrm{E}\left[\epsilon\right] = 0$ and $\epsilon$ and $r_{\mathbf{m}}$ are independent. The non-diversifiable *systematic risk* associated with asset $j$ takes the form $\beta_{\mathcal{R},j}^2 \mathrm{Var}\left[r_{\mathbf{m}}\right]$, and the diversifiable *idiosyncratic risk* takes the form $\mathrm{Var}\left[\epsilon\right]$. It is important to note that Equation (29) does not hold when stocks are imbued with ambiguity: the residual defined by $\text{Var}[\epsilon] = \text{Var}[r_j] - (\beta_{\mathcal{R},j} + \beta_{\mathcal{A},j})^2 \text{Var}[r_{\mathbf{m}}]$ cannot be interpreted as idiosyncratic risk since $\beta_{\mathcal{A},j}$ is involved.

# VII The security market line and performance measures

In ACAPM the *security market line* (SML) characterizes the linear relation between systematic uncertainty, formed by beta uncertainty, and expected return. Using Theorem 3, the SML of the classical CAPM can be generalized from risk to uncertainty. Formally, when ambiguity is involved the SML is defined by

$$\text{E}[r_j] = r_f + \beta_{\mathcal{K},j}(\text{E}[r_{\mathbf{m}}] - r_f). \tag{30}$$

The intercept, $r_f$, is the price of time; the slope, $\text{E}[r_{\mathbf{m}}] - r_f$, is the uncertainty premium on the market portfolio; and the coefficient $\beta_{\mathcal{K},j}$ measures the level of systematic uncertainty associated with asset $j$.

Figure 5 provides a graphical representation of the SML. The x-axis depicts the magnitude of $\beta_{\mathcal{K}}$, and the y-axis depicts expected return. The solid slope line shows the SML in an ambiguous economy, and the dashed line shows it in a non-ambiguous economy. In an ambiguous economy the slope of the SML is steeper than in the case of a non-ambiguous economy, indicating a higher uncertainty premium (equity premium) on the market portfolio. Recall that the risk-free rate is exogenously given. Also notice that for $\beta_{\mathcal{K}} = 1$ the proportions of assets composing the market portfolio are different in an ambiguous economy, as compared with a non-ambiguous economy. Furthermore, the return on each individual asset is different since systematic risk in an ambiguous economy is the risk component that is optimal, but not necessarily minimal, to bear.

All possible portfolios, efficient and non-efficient, lie on the SML. Effectively, the prices (and the uncertainty premiums) of assets in the market are determined by the investors who minimize uncertainty for a given expected return, since they are willing to pay a relatively high price for any asset (implying a relatively low expected return). The SML can also be viewed as representing the opportunity cost of various investments. Every point on the SML

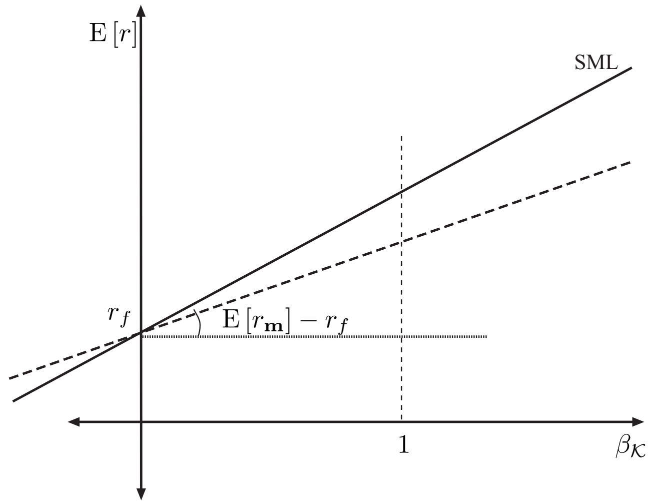


Figure 5: **The security market line**
This figure provides a graphical representation of the SML. The x-axis depicts the magnitude of $\beta_{\mathcal{K}}$, and the y-axis depicts expected return. The solid slope line shows the SML in an ambiguous economy, and the dashed line shows it in a non-ambiguous economy. The SML intercepts the y-axis at the risk-free rate $r_f$, and its slope is equal to the uncertainty premium on the market portfolio $\mathrm{E}[r_{\mathbf{m}}] - r_f$.

represents a combined investment in the market portfolio and the risk-free asset. Assets above the SML are considered undervalued, since for a given degree of uncertainty they yield a relatively high return, implying a relatively low price. Assets below the SML are considered overvalued, since for a given degree of uncertainty they yield a relatively low return, implying a relatively high price.

A point on the SML can be interpreted as the uncertainty premium on a portfolio per unit of systematic uncertainty associated with it. A natural application of this finding is for measuring the performance of asset portfolios. In an economy without ambiguity, the *Treynor ratio* measures the risk premium on an asset relative the systematic risk associated with it. This ratio can now be extended to incorporate ambiguity. Equation (30) can be formed as

$$\frac{\mathrm{E}[r_{\mathbf{h}}] - r_f}{\beta_{\mathcal{K},\mathbf{h}}} = \mathrm{E}[r_{\mathbf{m}}] - r_f, \tag{31}$$

such that its left-hand side—the ratio of the uncertainty premium and the degree of systematic uncertainty—takes the meaning of the uncertainty premium per unit of uncertainty borne. When ambiguity is not present ($\beta_{\mathcal{A}} = 0$), Equation (31) collapses to the original

Treynor ratio.

A second widely used performance measure is the *Sharpe ratio*, which measures the risk premium on a portfolio relative to the entire (systematic and idiosyncratic) risk associated with it. Using the uncertainty measure proposed by Theorem 2, the Sharpe ratio can also be extended from risk to uncertainty. The definition of the CML in Equation (18) implies that the uncertainty premium relative to the entire (systematic and idiosyncratic) uncertainty can be measured by

$$\frac{\mathrm{E}\left[r_{\mathbf{h}}\right]-r_f}{\Upsilon\left[r_{\mathbf{h}}\right]} = \frac{\mathrm{E}\left[r_{\mathbf{h}}\right]-r_f}{\frac{\mathrm{Std}\left[r_{\mathbf{h}}\right]}{\sqrt{1-\mho^2\left[r_{\mathbf{h}}\right]}}}. \tag{32}$$

This extended ratio characterizes the uncertainty premium on a portfolio per unit of its entire uncertainty.

# VIII Empirical implications

The pricing model proposed in Theorem 3 can help us improve our understanding of capital asset prices in financial markets. Brenner and Izhakian (2012), for example, employ this model to study the impact of ambiguity on capital asset pricing in the stock market. They develop an empirical methodology to estimate from the data the entire ambiguity and the systematic ambiguity associated with stocks. Given the estimated ambiguity of each stock, their study employs the Fama and MacBeth (1993) methodology to examine the effect of ambiguity on the cross section of equity returns. To demonstrate the empirical implementations of ACAPM, this section reviews the methodology proposed by Brenner and Izhakian (2012) and their findings.

Using intraday stock data, Brenner and Izhakian (2012) extract the monthly degree of ambiguity associated with each stock and with the market portfolio by utilizing the following four-step methodology for each asset separately. The first step is to sample 20 to 22 groups, each comprising 26 observations of returns (at 15-minute intervals) from the monthly data, where groups can be selected by trading days. The second step is to compute the mean and variance of each group. The third step is to compute the probability of a return lower than the risk-free rate (loss) for each group, using its mean and variance and assuming normally distributed returns. At this point, for each stock and each month there are 20-22 probabilities of loss. The last step is to compute the variance of these probabilities to obtain the monthly degree of ambiguity, $\mho^2$, associated with each stock. Given $\mho^2$, the degree of uncertainty, $\Upsilon^2$, is directly obtained.

One can also compute the monthly systematic ambiguity associated with each stock by applying the following procedure to each stock separately. First, as in the first and the second steps above, compute 20-22 means and variances of a given stock and 20-22 means and variances of the market portfolio for each month. Next, substitute this data into Equation (21) to obtain each asset's beta ambiguity, which measures systematic ambiguity.[30]

Having estimated the degrees of risk and ambiguity and their systematic and idiosyncratic components, Brenner and Izhakian (2012) conduct a series of tests to study the explanatory power of these factors on equity returns. They find that systematic risk and systematic ambiguity each have a significant positive impact on equity returns. They also find that while aggregating risk and ambiguity into uncertainty, investors are rewarded through higher expected returns only for the systematic component of uncertainty and not for the idiosyncratic component. These findings coincide with the results predicted by ACAPM.

# IX Conclusion

This paper introduces a new pricing model that allows for improved pricing of capital assets by accounting for ambiguity—a real word condition in which probabilities of outcomes are not uniquely assigned. It generalizes the classical CAPM form risk to uncertainty by incorporating ambiguity and proves that investors are only rewarded for systematic ambiguity as well as for systematic risk.

This paper relaxes the main assumption of modern portfolio theory that the probabilities of returns are known and instead assumes that probabilities are unknown and are random. It generalizes the *mean–variance* paradigm to *mean–variance–ambiguity* and defines preferences in this space. In the context of this paradigm the paper defines a *mean-uncertainty* space and proposes an aggregation of risk and ambiguity into a new combined measure of

[30]This methodology implicitly assumes that the random means and random variances are uniformly distributed.

uncertainty.

The mean-uncertainty space then provides the theoretical underpinning for a new capital asset pricing model under ambiguity (ACAPM). In this model, a simple formalization of *beta ambiguity* in addition to the conventional *beta risk* is derived such that systematic ambiguity is distinguished from idiosyncratic ambiguity. An asset price in this model is proportional to the systematic components of its risk and ambiguity ensuring that investors are rewarded for neither idiosyncratic risk nor idiosyncratic ambiguity. To the best of our knowledge, the current study is the first to make the distinction between systematic and idiosyncratic ambiguity.

A notable advantage of ACAPM is that it is empirically testable. It can be employed to improve our understanding of capital asset prices in financial markets. Brenner and Izhakian (2012), for example, use this model to study the effect of ambiguity on the cross section of equity returns. A natural outcome of our model is an extension of the Treynor and Sharpe ratios from risk to uncertainty, allowing for the measurement of portfolio performance relative to the uncertainty borne. Furthermore, the model can be used in other applications such as portfolio selection and value at risk.

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# A Appendix

## A.1 Supporting claims

**Theorem 5.** *Let* $p_x = \mathrm{E}\left[\mathrm{P}\left(X \leq x\right)\right]$ *and* $\zeta_x^2 = \mathrm{Var}\left[\mathrm{P}\left(X \leq x\right)\right]$*, where* $\mathrm{P}_x = \mathrm{P}\left(X \leq x\right)$ *is the random probability of a random variable* $X$*. Assume a continuous twice-differentiable outlook function* $\Gamma$*, satisfying* $\frac{1}{2}\left(\frac{\Gamma''(p_x)}{\Gamma'(p_x)}\zeta_x^2 - \frac{\Gamma''(p_z)}{\Gamma'(p_z)}\zeta_z^2\right) \leq p_z$ *for any* $x \leq z \leq k$ *and any* $k \leq z \leq x$*. For a relatively small* $\mathrm{P}_x$*, the perceived probability of a random variable* $X \leq x$ *is then*

$$\mathrm{Q}\left(X \leq x\right) = \Gamma^{-1}\left(\int_{\mathbb{P}} \Gamma\left(\mathrm{P}\left(X \leq x\right)\right) d\chi\left(\mathrm{P}\right)\right) \approx p_x + \frac{1}{2}\frac{\Gamma''\left(p_x\right)}{\Gamma'\left(p_x\right)}\zeta_x^2. \tag{33}$$

## A.2 Proofs

***Proof of Corollary 1.*** The proof that $\beta_{\mathcal{R},\mathbf{m}} = 1$ it is trivial. As for $\beta_{\mathcal{A},\mathbf{m}}$, from Equation (21) it can be observed that substituting $\mu_j = \mu_{\mathbf{m}}$ and $\sigma_{\mathbf{m},j} = \sigma_{\mathbf{m},\mathbf{m}}$ into $\left(\left(\mu_j - r_f\right) - \frac{\sigma_{\mathbf{m},j}}{\sigma_{\mathbf{m}}^2}\left(\mu_{\mathbf{m}} - r_f\right)\right)$ implies that $\beta_{\mathcal{A},\mathbf{m}} = 0$. □

***Proof of Proposition 1.*** To prove that the parametric line defined by

$$\left(\mathrm{E}\left[hr_f + (1-h)\,r_{\mathbf{e}}\right], \mathrm{Std}\left[hr_f + (1-h)\,r_{\mathbf{e}}\right], \widehat{\mho}\left[hr_f + (1-h)\,r_{\mathbf{e}}\right]\right)$$

is linear, it has to be shown that each coordinate of this triplet is linear in $h$. The proof that the first two coordinates, the mean $\mathrm{E}\left[hr_f + (1-h)\,r_{\mathbf{e}}\right]$ and the standard deviation $\mathrm{Std}\left[hr_f + (1-h)\,r_{\mathbf{e}}\right]$, are each linear in $h$ is trivial since $r_f$ is constant. To prove that the third coordinate, $\widehat{\mho}\left[hr_f + (1-h)\,r_{\mathbf{e}}\right]$, is also linear in $h$, the measure of ambiguity $\mho^2$ can be explicitly written as

$$\mho^2\left[hr_f + (1-h)\,r_{\mathbf{e}}\right] \quad = \quad 4\mathrm{Var}\left[\int_{-\infty}^{r_f} \frac{1}{\sqrt{2\pi\left(1-h\right)^2\sigma_{\mathbf{e}}^2}} e^{-\frac{\left(r-\left(hr_f+(1-h)\mu_{\mathbf{e}}\right)\right)^2}{2(1-h)^2\sigma_{\mathbf{e}}^2}} dr\right].$$

Changing the integration variable to $r = (1-h)\,r_{\mathbf{e}}$ provides

$$\mho^2\left[hr_f + (1-h)\,r_{\mathbf{e}}\right] = 4\mathrm{Var}\left[\int_{-\infty}^{r_f} \frac{1}{\sqrt{2\pi\sigma_{\mathbf{e}}^2}} e^{-\frac{(r_{\mathbf{e}}-\mu_{\mathbf{e}})^2}{2\sigma_{\mathbf{e}}^2}} dr_{\mathbf{e}}\right] = \mho^2\left[r_{\mathbf{e}}\right].$$

Since $r_f$ is constant, $\mathrm{Std}\left[hr_f + (1-h)\,r_{\mathbf{e}}\right] = (1-h)\,\mathrm{Std}\left[r_{\mathbf{e}}\right]$. This implies that

$$\widehat{\mho}\left[hr_f + (1-h)\,r_{\mathbf{e}}\right] \quad = \quad (1-h)\,\widehat{\mho}\left[r_{\mathbf{e}}\right].$$

□

***Proof of Proposition 2.*** To prove the existence of the uncertain asset frontier one can write the uncertainty minimization problem for a given expected return $E$ as a function of an asset portfolio $\mathbf{h}$. Using matrix notation this minimization problem can be defined by

$$\min_{\mathbf{h}} \mathbf{h}^T \frac{\mathbf{Cov}}{1 - 4\mathrm{Var}\left[\Phi\left(-\frac{\mathbf{h}^T\boldsymbol{\mu}}{\mathbf{h}^T\boldsymbol{\Sigma}\mathbf{h}}; 0, 1\right)\right]} \mathbf{h} \qquad \text{s.t.} \qquad \mathbf{E}^T\mathbf{h} = E; \quad \mathbf{1}^T\mathbf{h} = 1,$$

where bold font designates vectors and matrices: $\boldsymbol{\mu}$ is a vector of random means, $\boldsymbol{\Sigma}$ is a random variance–covariance matrix, and $\mathbf{Cov}$ is the expected variance–covariance matrix. Letting $\lambda$ and $\kappa$ be the Lagrangian multiplier, the first-order condition of the minimization problem is

$$\frac{2\mathbf{h}^T\mathbf{Cov}\left(1-4\mathrm{Var}\left[\Phi\left(-\frac{\mathbf{h}^T\boldsymbol{\mu}}{\mathbf{h}^T\boldsymbol{\Sigma}\mathbf{h}};0,1\right)\right]\right)+2\mathbf{h}^T\mathbf{Cov}\mathbf{h}\mathrm{Cov}\left[\Phi\left(-\frac{\mathbf{h}^T\boldsymbol{\mu}}{\mathbf{h}^T\boldsymbol{\Sigma}\mathbf{h}};0,1\right),\phi\left(-\frac{\mathbf{h}^T\boldsymbol{\mu}}{\mathbf{h}^T\boldsymbol{\Sigma}\mathbf{h}};0,1\right)\left(\frac{\boldsymbol{\mu}\mathbf{h}^T\boldsymbol{\Sigma}\mathbf{h}-\mathbf{h}^T\boldsymbol{\mu}\mathbf{h}^T\boldsymbol{\Sigma}}{\left(\mathbf{h}^T\boldsymbol{\Sigma}\mathbf{h}\right)^2}\right)\right]}{\left(1-4\mathrm{Var}\left[\Phi\left(-\frac{\mathbf{h}^T\boldsymbol{\mu}}{\mathbf{h}^T\boldsymbol{\Sigma}\mathbf{h}};0,1\right)\right]\right)^2} \tag{34}$$

$$-\lambda\mathbf{E}^T - \kappa\mathbf{1}^T \;=\; 0.$$

One can notice that Equation (34) has a solution if the following two conditions hold. First, as $\mathbf{Cov}$ is not singular, it has an inverse matrix. Second, the degree of ambiguity is not equal to 1. That is, $1 - 4\mathrm{Var}\left[\Phi\left(-\frac{\mathbf{h}^T\boldsymbol{\mu}}{\mathbf{h}^T\boldsymbol{\Sigma}\mathbf{h}}; 0, 1\right)\right] = 1 - \mho^2\left[r_{\mathbf{h}}\right] \neq 0$. These conditions are satisfied by the hypothesis of the theorem and therefore uncertain asset frontier exists. Proposition 3 proves that the uncertain asset frontier is bounded by a hyperbola, which implies that there exists a line originating from $r_f$ and tangent to the uncertain asset frontier. □

***Proof of Proposition 3.*** For simplicity this proof considers a portfolio consisting of two assets. It can then be extended to any number of assets by considering each of the two assets as a mutual fund. Let $\mathbf{h} = (h, 1-h)$ be a portfolio. Its expected return is then

$$\mathrm{E}\left[r_{\mathbf{h}}\right] \;=\; h\mathrm{E}\left[r_1\right] + (1-h)\,\mathrm{E}\left[r_2\right], \tag{35}$$

and its variance is

$$\mathrm{Var}\left[r_{\mathbf{h}}\right] \;=\; h^2\mathrm{Var}\left[r_1\right] + 2h\,(1-h)\,\mathrm{Cov}\left[r_1, r_2\right] + (1-h)^2\,\mathrm{Var}\left[r_2\right],$$

where $r_1$ and $r_2$ stand for the return on asset 1 and 2, respectively. By Theorem 2, the degree of uncertainty associated with portfolio $\mathbf{h}$ is

$$\Upsilon^2\left[r_{\mathbf{h}}\right] \;=\; \frac{h^2\mathrm{Var}\left[r_1\right] + 2h\,(1-h)\,\mathrm{Cov}\left[r_1, r_2\right] + (1-h)^2\,\mathrm{Var}\left[r_2\right]}{1 - 4\mathrm{Var}\left[\Phi\left(\frac{r_f - h\mu_1 - (1-h)\mu_2}{\sqrt{h^2\sigma_1^2 + 2h(1-h)\sigma_{1,2} + (1-h)^2\sigma_2^2}}; 0, 1\right)\right]}. \tag{36}$$

Substituting $h = 0$ into Equation (36) provides the degree of uncertainty associated with asset 1:

$$\Upsilon^2\left[r_1\right] \;=\; \frac{\mathrm{Var}\left[r_1\right]}{1-4\mathrm{Var}\left[\Phi\left(\frac{r_f-\mu_1}{\sigma_1};0,1\right)\right]}.$$

Substituting $h = 1$ into Equation (36) provides the degree of uncertainty associated with asset 2:

$$\Upsilon^2\left[r_2\right] \;=\; \frac{\mathrm{Var}\left[r_2\right]}{1-4\mathrm{Var}\left[\Phi\left(\frac{r_f-\mu_2}{\sigma_2};0,1\right)\right]}.$$

Since $\mho^2\left[r\right] = 4\mathrm{Var}\left[\Phi\left(\frac{r_f-\mu}{\sigma};0,1\right)\right] \in [0,1]$, the degree of uncertainty satisfies $\Upsilon^2\left[r_1\right] \geq \mathrm{Var}\left[r_1\right]$ and $\Upsilon^2\left[r_2\right] \geq \mathrm{Var}\left[r_2\right]$. This proves that for portfolios consisting of a single asset, the degree of uncertainty is bounded by the degree of risk. Next, we prove it for any portfolio.

Since the parameters $\mu_1$ and $\mu_2$ are random, the proportions of the two assets in the portfolio are selected according to their expectations. By Equation (35), the proportion of asset 1 in the portfolio can be writing as $h = \frac{\mathrm{E}[r_{\mathbf{h}}]-\mathrm{E}[r_2]}{\mathrm{E}[r_1]-\mathrm{E}[r_2]}$. Substituting for the proportion $h$ into Equation (36) produces

$$\Upsilon^2\left[r_{\mathbf{h}}\right] \;=\; \frac{\left(\frac{\mathrm{E}[r_{\mathbf{h}}]-\mathrm{E}[r_2]}{\mathrm{E}[r_1]-\mathrm{E}[r_2]}\right)^2\mathrm{Var}[r_1]+2\frac{\mathrm{E}[r_{\mathbf{h}}]-\mathrm{E}[r_2]}{\mathrm{E}[r_1]-\mathrm{E}[r_2]}\left(1-\frac{\mathrm{E}[r_{\mathbf{h}}]-\mathrm{E}[r_2]}{\mathrm{E}[r_1]-\mathrm{E}[r_2]}\right)\mathrm{Cov}[r_1,r_2]+\left(1-\frac{\mathrm{E}[r_{\mathbf{h}}]-\mathrm{E}[r_2]}{\mathrm{E}[r_1]-\mathrm{E}[r_2]}\right)^2\mathrm{Var}[r_2]}{1-\mathrm{Var}\left[\Phi\left(\frac{r_f-\frac{\mathrm{E}[r_{\mathbf{h}}]-\mathrm{E}[r_2]}{\mathrm{E}[r_1]-\mathrm{E}[r_2]}\mu_1-\left(1-\frac{\mathrm{E}[r_{\mathbf{h}}]-\mathrm{E}[r_2]}{\mathrm{E}[r_1]-\mathrm{E}[r_2]}\right)\mu_2}{\sqrt{\left(\frac{\mathrm{E}[r_{\mathbf{h}}]-\mathrm{E}[r_2]}{\mathrm{E}[r_1]-\mathrm{E}[r_2]}\right)^2\sigma_1^2+2\frac{\mathrm{E}[r_{\mathbf{h}}]-\mathrm{E}[r_2]}{\mathrm{E}[r_1]-\mathrm{E}[r_2]}\left(1-\frac{\mathrm{E}[r_{\mathbf{h}}]-\mathrm{E}[r_2]}{\mathrm{E}[r_1]-\mathrm{E}[r_2]}\right)\sigma_{1,2}+\left(1-\frac{\mathrm{E}[r_{\mathbf{h}}]-\mathrm{E}[r_2]}{\mathrm{E}[r_1]-\mathrm{E}[r_2]}\right)^2\sigma_2^2}};0,1\right)\right]},$$

which implies

$$\Upsilon^2\left[r_{\mathbf{h}}\right] = \left(\frac{1}{\mathrm{E}[r_1]-\mathrm{E}[r_2]}\right)^2\frac{(\mathrm{E}[r_{\mathbf{h}}]-\mathrm{E}[r_2])^2\mathrm{Var}[r_1]-2(\mathrm{E}[r_{\mathbf{h}}]-\mathrm{E}[r_2])(\mathrm{E}[r_{\mathbf{h}}]-\mathrm{E}[r_1])\mathrm{Cov}[r_1,r_2]+(\mathrm{E}[r_{\mathbf{h}}]-\mathrm{E}[r_1])^2\mathrm{Var}[r_2]}{1-\mathrm{Var}\left[\Phi\left(\frac{r_f(\mathrm{E}[r_1]-\mathrm{E}[r_2])-(\mathrm{E}[r_{\mathbf{h}}]-\mathrm{E}[r_2])\mu_1+(\mathrm{E}[r_{\mathbf{h}}]-\mathrm{E}[r_1])\mu_2}{\sqrt{(\mathrm{E}[r_{\mathbf{h}}]-\mathrm{E}[r_2])^2\sigma_1^2-2(\mathrm{E}[r_{\mathbf{h}}]-\mathrm{E}[r_2])(\mathrm{E}[r_{\mathbf{h}}]-\mathrm{E}[r_1])\sigma_{1,2}+(\mathrm{E}[r_{\mathbf{h}}]-\mathrm{E}[r_1])^2\sigma_2^2}};0,1\right)\right]}. \quad (37)$$

The numerator of Equation (37) defines the risk, $\mathrm{Var}\left[r_{\mathbf{h}}\right]$, associated with portfolio $\mathbf{h}$. It forms a parabola in the space defined by $\left(\mathrm{E}\left[r\right], \Upsilon^2\left[r\right]\right)$, such that its square root is a hyperbola in the mean–uncertainty space, defined by $\left(\mathrm{E}\left[r\right], \Upsilon\left[r\right]\right)$. The denominator ranges between 0 and 1, implying that $\Upsilon^2\left[r_{\mathbf{h}}\right] \geq \mathrm{Var}\left[r_{\mathbf{h}}\right]$ for any $h$. □

***Proof of Theorem 1.*** By Theorem 5, the expected utility of Equations (6) and (9) can be written

$$\mathrm{V}\left(w\left(1+\mathrm{E}\left[Z\right]-\mathcal{K}\right)\right) \;\approx\; -\int_{-\infty}^{k}\left[p_z+\frac{1}{2}\frac{\Gamma''\left(p_z\right)}{\Gamma'\left(p_z\right)}\zeta_z^2\right]dz+\int_{k}^{\infty}\left[p_z+\frac{1}{2}\frac{\Gamma''\left(p_z\right)}{\Gamma'\left(p_z\right)}\zeta_z^2\right]dz,$$

where $p_z = \mathrm{E}\left[\mathrm{P}_z\right]$, $\mathrm{P}_z = \Phi\left(\frac{\mathrm{U}^{-1}(z)}{w}-1:\mu,\sigma\right)$ for losses, and $\mathrm{P}_z = 1-\Phi\left(\frac{\mathrm{U}^{-1}(z)}{w}-1:\mu,\sigma\right)$

for gains. Changing the integration variable to $z = \mathrm{U}(1+r)$ provides

$$\begin{aligned}\mathrm{V}(w(1+\mathrm{E}[r]-\mathcal{K})) &\approx -\int_{-\infty}^{k}\left[p_r + \frac{1}{2}\frac{\Gamma''(p_r)}{\Gamma'(p_r)}\zeta_r^2\right]\mathrm{U}'(w(1+r))\,dr + \\ &\quad \int_{k}^{\infty}\left[1-p_r + \frac{1}{2}\frac{\Gamma''(1-p_r)}{\Gamma'(1-p_r)}\zeta_r^2\right]\mathrm{U}'(w(1+r))\,dr,\end{aligned}$$

where $p_r = \mathrm{E}[\Phi(r;\mu,\sigma)]$ and $\zeta_r^2 = \mathrm{Var}[\Phi(r;\mu,\sigma)]$.[31] Integrating by parts provides

$$\begin{aligned}\mathrm{V}(w(1+\mathrm{E}[r]-\mathcal{K})) &\approx -p_r\mathrm{U}(w(1+r))|_{-\infty}^{k} + \int_{-\infty}^{k}\mathrm{E}[\phi(r,\mu,\sigma)]\,\mathrm{U}(w(1+r))\,dr \qquad (38)\\ &\quad -\int_{-\infty}^{k}\frac{1}{2}\frac{\Gamma''(p_r)}{\Gamma'(p_r)}\zeta_r^2\mathrm{U}'(w(1+r))\,dr \\ &\quad +(1-p_r)\,\mathrm{U}(w(1+r))|_{k}^{\infty} + \int_{k}^{\infty}\mathrm{E}[\phi(r,\mu,\sigma)]\,\mathrm{U}(w(1+r))\,dr \\ &\quad +\int_{k}^{\infty}\frac{1}{2}\frac{\Gamma''(1-p_r)}{\Gamma'(1-p_r)}\zeta_r^2\mathrm{U}'(w(1+r))\,dr.\end{aligned}$$

Because $\mathrm{U}(w(1+k)) = 0$, the sum of the first element in the first line and the first element in the third line of Equation (38) is zero. Since $k$ is relatively close to $\mathrm{E}[r]$, taking a first-order Taylor approximation (around $\mathrm{E}[r]$) of the second and the fourth lines of Equation (38) provides[32]

$$\begin{aligned}I \;\approx\;& -\int_{-\infty}^{k}\tfrac{1}{2}\tfrac{\Gamma''(p_k)}{\Gamma'(p_k)}\zeta_k^2\mathrm{U}'(w(1+\mathrm{E}[r]))\,dr - \int_{-\infty}^{k}\tfrac{1}{2}\left[\tfrac{\Gamma''(p_k)}{\Gamma'(p_k)}\zeta_k^2\mathrm{U}'(w(1+\mathrm{E}[r]))\right]'(r-\mathrm{E}[r])\,dr \\ &+\int_{k}^{\infty}\tfrac{1}{2}\tfrac{\Gamma''(1-p_k)}{\Gamma'(1-p_k)}\zeta_k^2\mathrm{U}'(w(1+\mathrm{E}[r]))\,dr + \int_{k}^{\infty}\tfrac{1}{2}\left[\tfrac{\Gamma''(1-p_k)}{\Gamma'(1-p_k)}\zeta_k^2\mathrm{U}'(w(1+\mathrm{E}[r]))\right]'(r-\mathrm{E}[r])\,dr.\end{aligned}$$

Thus, $I$ satisfies

$$I \approx -\frac{1}{8}\frac{\Gamma''(p_L)}{\Gamma'(p_L)}\mho^2[r]\,\mathrm{U}'(w(1+\mathrm{E}[r])) - \frac{1}{8}\frac{\Gamma''(p_G)}{\Gamma'(p_G)}\mho^2[r]\,\mathrm{U}'(w(1+\mathrm{E}[r])),$$

where $p_L = \mathrm{E}[\mathrm{P}_L]$ and $p_G = \mathrm{E}[\mathrm{P}_G]$ are the expected probabilities of loss and gain, respectively. The second-order Taylor approximation (around $\mathrm{E}[r]$) of the second component in the first line and the second component in the third line of Equation (38) provides

$$\begin{aligned}II &= \int_{-\infty}^{\infty}\mathrm{E}[\phi(r,\mu,\sigma)]\,\mathrm{U}(w(1+r))\,dr \\ &\approx \int_{-\infty}^{\infty}\mathrm{E}[\phi(r,\mu,\sigma)]\begin{pmatrix}\mathrm{U}(\mathrm{E}[w(1+\mathrm{E}[r])]) + \mathrm{U}'(\mathrm{E}[w(1+\mathrm{E}[r])])(r-\mathrm{E}[r]) + \\ \tfrac{1}{2}\mathrm{U}''(\mathrm{E}[w(1+\mathrm{E}[r])])(r-\mathrm{E}[r])^2\end{pmatrix}dr \\ &= \mathrm{U}(\mathrm{E}[w(1+\mathrm{E}[r])]) + \frac{1}{2}\mathrm{U}''(\mathrm{E}[w(1+\mathrm{E}[r])])\,\mathrm{Var}[r].\end{aligned}$$

[31] To simplify notations, the subscript $\mathbf{h}$ is omitted.

[32] Notice that the first-order Taylor approximation of $\mathrm{U}'$ is equivalent to the second-order approximation of $\mathrm{U}''$.

As in Arrow (1965) and Pratt (1964), the first-order Taylor approximation (around 0) of the LHS of Equation (38) with respect to $\mathcal{K}$ is

$$LHS = \int_{\infty}^{\infty} \mathrm{E}\left[\phi\left(r;\mu,\sigma\right)\right] \mathrm{U}\left(w\left(1+\mathrm{E}\left[r\right]-\mathcal{K}\right)\right) dr \approx \mathrm{U}\left(w\left(1+\mathrm{E}\left[r\right]\right)\right) - \mathcal{K}\mathrm{U}'\left(w\left(1+\mathrm{E}\left[r\right]\right)\right).$$

Combining the LHS, $I$ and $II$, the uncertainty premium is

$$\mathcal{K} \approx -\frac{1}{2}\frac{\mathrm{U}''\left(w\left(1+\mathrm{E}\left[r\right]\right)\right)}{\mathrm{U}'\left(w\left(1+\mathrm{E}\left[r\right]\right)\right)}\mathrm{Var}\left[r\right] - \frac{1}{8}\left[\frac{\Gamma''\left(p_L\right)}{\Gamma'\left(p_L\right)} + \frac{\Gamma''\left(p_G\right)}{\Gamma'\left(p_G\right)}\right]\mho^2\left[r\right]. \qquad \square$$

***Proof of Theorem 2.*** To show that $\Upsilon$ measures uncertainty it has to be shown that for a given expected return, E, an uncertainty-averse investor prefers the asset with the lower $\Upsilon$ over the asset with the higher $\Upsilon$. By Assumption 1, all investors aggregate risk and ambiguity to uncertainty in the same way and each has a preference concerning this uncertainty. Investors are averse to the uncertainty (aggregated from risk and ambiguity) since they are averse to both risk and ambiguity. Assume two assets, denoted $i$ and $j$, such that

$$\Upsilon^2\left[r_j\right] = \frac{\mathrm{Var}\left[r_j\right]}{1-\mho^2\left[r_j\right]} \geq \frac{\mathrm{Var}\left[r_i\right]}{1-\mho^2\left[r_i\right]} = \Upsilon^2\left[r_i\right].$$

Assumption 1 implies that the source of uncertainty (risk or ambiguity) does not play a role for investors. Therefore, the uncertainty of $j$ is equivalent to the variance of an risky and non-ambiguous asset $x$, i.e. $\Upsilon^2\left[r_j\right] = \mathrm{Var}\left[x\right]$. Similarly, $\Upsilon^2\left[r_i\right] = \mathrm{Var}\left[y\right]$ for some risky and non-ambiguous asset $y$. Theorem 1 implies that, for a given E and a given $\mho^2$, a rational risk-averse investor prefers the asset with lower Var over the asset with the higher Var. That is, $\mathrm{V}\left(w\left(1+x\right)\right) \geq \mathrm{V}\left(w\left(1+y\right)\right)$ and by transitivity $\mathrm{V}\left(w\left(1+r_j\right)\right) \geq \mathrm{V}\left(w\left(1+r_i\right)\right)$. $\square$

***Proof of Theorem 3.*** The random probability of loss, $\mathrm{P}\left(L_{\mathbf{m}}\right) = \mathrm{P}\left(r_{\mathbf{m}} \leq r_f\right)$, on the market portfolio, $\mathbf{m}$, is defined by

$$\mathrm{P}\left(L_{\mathbf{m}}\right) = \int_{-\infty}^{r_f} \frac{1}{\sqrt{2\pi\sigma_{\mathbf{m}}^2}} e^{-\frac{\left(r-\mu_{\mathbf{m}}\right)^2}{2\sigma_{\mathbf{m}}^2}} dr. \qquad (39)$$

First, we assemble a portfolio, denoted $\mathbf{a}$, consisting of a proportion $-h$ of the risk-free asset, and the remainder, $1+h$, is allocated to the uncertain market portfolio, where $h > 0$. The expected return on portfolio $\mathbf{a}$ is

$$\mathrm{E}\left[r_{\mathbf{a}}\right] = \left(1+h\right)\mathrm{E}\left[r_{\mathbf{m}}\right] - hr_f,$$

and the variance of its return is

$$\mathrm{Var}\left[r_{\mathbf{a}}\right] = (1+h)^2\,\mathrm{Var}\left[r_{\mathbf{m}}\right].$$

The degree of ambiguity associated with portfolio **a** is

$$\mho^2\left[r_{\mathbf{a}}\right] = 4\mathrm{Var}\left[\int_{-\infty}^{r_f}\frac{1}{\sqrt{2\pi\left(1+h\right)^2\sigma_{\mathbf{m}}^2}}e^{-\frac{\left(r-\left((1+h)\mu_{\mathbf{m}}-hr_f\right)\right)^2}{2(1+h)^2\sigma_{\mathbf{m}}^2}}dr\right]. \tag{40}$$

We also assemble a second portfolio, denoted **b**, consisting of a proportion 1 of **m** and a proportion $0<h$ of some asset $j$, which is financed by a proportion $h$ of the risk-free asset. Asset $j$ is assumed to be non-efficient. That is, in the mean–uncertainty space it lies in the set of feasible portfolios, but not on the efficient frontier. The expected return on portfolio **b** is

$$\mathrm{E}\left[r_{\mathbf{b}}\right] = \mathrm{E}\left[r_{\mathbf{m}}\right] + h\mathrm{E}\left[r_j\right] - hr_f,$$

and the variance of its return is

$$\mathrm{Var}\left[r_{\mathbf{b}}\right] = \mathrm{Var}\left[r_{\mathbf{m}}\right] + h^2\mathrm{Var}\left[r_j\right] + 2h\mathrm{Cov}\left[r_{\mathbf{m}}, r_j\right].$$

The ambiguity associated with portfolio **b** is

$$\begin{aligned}\mho^2\left[r_{\mathbf{b}}\right] &= 4\mathrm{Var}\left[\int_{-\infty}^{r_f}\frac{1}{\sqrt{2\pi\left(\sigma_{\mathbf{m}}^2+h^2\sigma_j^2+2h\sigma_{\mathbf{m},j}\right)}}e^{-\frac{\left(r-\left(\mu_{\mathbf{m}}+h\mu_j-hr_f\right)\right)^2}{2\left(\sigma_{\mathbf{m}}^2+h^2\sigma_j^2+2h\sigma_{\mathbf{m},j}\right)}}dr\right] \\ &= 4\mathrm{Var}\left[\int_{-\infty}^{\frac{(1+h)r_f-\mu_{\mathbf{m}}-h\mu_j}{\sqrt{\sigma_{\mathbf{m}}^2+h^2\sigma_j^2+2h\sigma_{\mathbf{m},j}}}}\frac{1}{\sqrt{2\pi}}e^{-\frac{r^2}{2}}dr\right],\end{aligned} \tag{41}$$

where the second equality is obtained by changing the integration variable.

As $h\to 0$, the curve drawn by portfolio **a** is tangent to the CML at the point $\left(\mathrm{E}\left[r_{\mathbf{m}}\right], \Upsilon\left[r_{\mathbf{m}}\right]\right)$. Thus,

$$\frac{\frac{d\mathrm{E}[r_{\mathbf{a}}]}{dh}}{\frac{d\sqrt{\frac{\mathrm{Var}[r_{\mathbf{a}}]}{1-\mho^2[r_{\mathbf{a}}]}}}{dh}}|_{h=0} = \frac{\mathrm{E}\left[r_{\mathbf{m}}\right]-r_f}{\sqrt{\frac{\mathrm{Var}[r_{\mathbf{m}}]}{1-\mho^2[r_{\mathbf{m}}]}}}. \tag{42}$$

The derivative of the numerator in the LHS of Equation (42) with respect to $h$ satisfies

$$\frac{d\mathrm{E}\left[r_{\mathbf{a}}\right]}{dh} = \mathrm{E}\left[r_{\mathbf{m}}\right] - r_f. \tag{43}$$

Changing the integration variable of Equation (40) implies that

$$\int_{-\infty}^{r_f} \frac{1}{\sqrt{2\pi\left(1+h\right)^2\sigma_{\mathbf{m}}^2}} e^{-\frac{\left(r-\left((1+h)\mu_{\mathbf{m}}-hr_f\right)\right)^2}{2(1+h)^2\sigma_{\mathbf{m}}^2}} dr \quad = \quad \int_{-\infty}^{r_f} \frac{1}{\sqrt{2\pi\sigma_{\mathbf{m}}^2}} e^{-\frac{(r-\mu_{\mathbf{m}})^2}{2\sigma_{\mathbf{m}}^2}} dr.$$

Therefore, the derivative of the denominator in the LHS of Equation (42) with respect to $h$ satisfies

$$\frac{d\sqrt{\frac{\mathrm{Var}[r_{\mathbf{a}}]}{1-\mho^2[r_{\mathbf{a}}]}}}{dh}|_{h=0} \quad = \quad \frac{\frac{\mathrm{Var}[r_{\mathbf{m}}]}{1-\mho^2[r_{\mathbf{m}}]}}{\sqrt{\frac{\mathrm{Var}[r_{\mathbf{m}}]}{1-\mho^2[r_{\mathbf{m}}]}}}. \tag{44}$$

Together, Equations (43) and (44) imply that

$$\frac{\frac{d\mathrm{E}[r_{\mathbf{a}}]}{dh}}{\frac{d\sqrt{\frac{\mathrm{Var}[r_{\mathbf{a}}]}{1+\mho^2[r_{\mathbf{a}}]}}}{dh}}|_{h=0} \quad = \quad \frac{\mathrm{E}\left[r_{\mathbf{m}}\right]-r_f}{\sqrt{\frac{\mathrm{Var}[r_{\mathbf{m}}]}{1-\mho^2[r_{\mathbf{m}}]}}}. \tag{45}$$

As $h \to 0$, the curve drawn by portfolio $\mathbf{b}$ is also tangent to the CML at the point $(\mathrm{E}\left[r_{\mathbf{m}}\right], \Upsilon\left[r_{\mathbf{m}}\right])$. Thus,

$$\frac{\frac{d\mathrm{E}[r_{\mathbf{b}}]}{dh}}{\frac{d\sqrt{\frac{\mathrm{Var}[r_{\mathbf{b}}]}{1+\mho^2[r_{\mathbf{b}}]}}}{dh}}|_{h=0} \quad = \quad \frac{\mathrm{E}\left[r_{\mathbf{m}}\right]-r_f}{\sqrt{\frac{\mathrm{Var}[r_{\mathbf{m}}]}{1-\mho^2[r_{\mathbf{m}}]}}}. \tag{46}$$

The derivative of the numerator in the LHS of Equation (46) with respect to $h$ satisfies

$$\frac{d\mathrm{E}\left[r_{\mathbf{b}}\right]}{dh} = \mathrm{E}\left[r_j\right] - r_f. \tag{47}$$

The derivative of the denominator in the LHS of Equation (46) with respect to $h$ satisfies

$$\frac{d\sqrt{\frac{\mathrm{Var}[r_{\mathbf{b}}]}{1-\mho^2[r_{\mathbf{b}}]}}}{dh}|_{h=0} = \left(\begin{array}{c} \frac{\mathrm{Cov}[r_{\mathbf{m}},r_j]}{1-\mho^2[r_{\mathbf{m}}]} + 4\frac{\mathrm{Var}[r_{\mathbf{m}}]}{(1-\mho^2[r_{\mathbf{m}}])^2}\times \\ \int_{\mathbb{P}}\left(\mathrm{P}\left(L_{\mathbf{b}}\right)-\mathrm{E}\left[\mathrm{P}\left(L_{\mathbf{b}}\right)\right]\right)\left(\mathrm{P}'\left(L_{\mathbf{b}}\right)-\mathrm{E}\left[\mathrm{P}'\left(L_{\mathbf{b}}\right)\right]\right)d\chi\left(\mathrm{P}\right) \end{array}\right)\frac{1}{\sqrt{\frac{\mathrm{Var}[r_{\mathbf{m}}]}{1-\mho^2[r_{\mathbf{m}}]}}}, \tag{48}$$

where $\chi$ is the second-order probability distribution,

$$\mathrm{P}\left(L_{\mathbf{b}}\right)|_{h=0} = \int_{-\infty}^{\frac{r_f-\mu_{\mathbf{m}}}{\sigma_{\mathbf{m}}}} \frac{1}{\sqrt{2\pi}} e^{-\frac{r^2}{2}} dr = \mathrm{P}\left(L_{\mathbf{m}}\right)$$

and

$$\begin{aligned} \mathrm{P}'\left(L_{\mathbf{b}}\right)|_{h=0} \quad &= \quad \frac{1}{\sqrt{2\pi}} e^{-\frac{\left(r_f-\mu_{\mathbf{m}}\right)^2}{2\sigma_{\mathbf{m}}^2}} \left(\frac{r_f-\mu_j}{\sigma_{\mathbf{m}}} + \frac{\sigma_{\mathbf{m},j}}{\sigma_{\mathbf{m}}^3}\left(\mu_{\mathbf{m}}-r_f\right)\right) \\ &= \quad \phi\left(r_f;\mu_{\mathbf{m}},\sigma_{\mathbf{m}}\right)\left(\frac{\sigma_{\mathbf{m},j}}{\sigma_{\mathbf{m}}^2}\left(\mu_{\mathbf{m}}-r_f\right)-\left(\mu_j-r_f\right)\right). \end{aligned}$$

Equation (48) can then be written as

$$\frac{d\sqrt{\frac{\mathrm{Var}[r_{\mathbf{b}}]}{1-\mho^2[r_{\mathbf{b}}]}}}{dh}|_{h=0} = \left(\begin{array}{l}\frac{\mathrm{Cov}[r_{\mathbf{m}},r_j]}{1-\mho^2[r_{\mathbf{m}}]}+4\frac{\mathrm{Var}[r_{\mathbf{m}}]}{(1-\mho^2[r_{\mathbf{m}}])^2}\times \\ \mathrm{Cov}\left[\Phi\left(r_f;\mu_{\mathbf{m}},\sigma_{\mathbf{m}}\right),\phi\left(r_f;\mu_{\mathbf{m}},\sigma_{\mathbf{m}}\right)\left(\frac{\sigma_{\mathbf{m},j}}{\sigma^2_{\mathbf{m}}}\left(\mu_{\mathbf{m}}-r_f\right)-\left(\mu_j-r_f\right)\right)\right]\end{array}\right)\frac{1}{\sqrt{\frac{\mathrm{Var}[r_{\mathbf{m}}]}{1-\mho^2[r_{\mathbf{m}}]}}},$$

which implies

$$\frac{\frac{d\mathrm{E}[r_{\mathbf{b}}]}{dh}}{\frac{d\sqrt{\frac{\mathrm{Var}[r_{\mathbf{b}}]}{1+\mho^2[r_{\mathbf{b}}]}}}{dh}}|_{h=0} = \frac{\mathrm{E}[r_j]-r_f}{\frac{\frac{\mathrm{Cov}[r_{\mathbf{m}},r_j]}{1-\mho^2[r_{\mathbf{m}}]}+4\frac{\mathrm{Var}[r_{\mathbf{m}}]}{(1-\mho^2[r_{\mathbf{m}}])^2}\mathrm{Cov}\left[\Phi\left(r_f;\mu_{\mathbf{m}},\sigma_{\mathbf{m}}\right),\phi\left(r_f;\mu_{\mathbf{m}},\sigma_{\mathbf{m}}\right)\left(\frac{\sigma_{\mathbf{m},j}}{\sigma^2_{\mathbf{m}}}\left(\mu_{\mathbf{m}}-r_f\right)-\left(\mu_j-r_f\right)\right)\right]}{\sqrt{\frac{\mathrm{Var}[r_{\mathbf{m}}]}{1-\mho^2[r_{\mathbf{m}}]}}}}. \tag{49}$$

Equating Equations (45) and (49) yields

$$\frac{\mathrm{E}[r_j]-r_f}{\frac{\mathrm{Cov}[r_{\mathbf{m}},r_j]}{1-\mho^2[r_{\mathbf{m}}]}+4\frac{\mathrm{Var}[r_{\mathbf{m}}]}{(1-\mho^2[r_{\mathbf{m}}])^2}\mathrm{Cov}\left[\Phi\left(r_f;\mu_{\mathbf{m}},\sigma_{\mathbf{m}}\right),\phi\left(r_f;\mu_{\mathbf{m}},\sigma_{\mathbf{m}}\right)\left(\frac{\sigma_{\mathbf{m},j}}{\sigma^2_{\mathbf{m}}}\left(\mu_{\mathbf{m}}-r_f\right)-\left(\mu_j-r_f\right)\right)\right]} = \frac{\mathrm{E}[r_{\mathbf{m}}]-r_f}{\frac{\mathrm{Var}[r_{\mathbf{m}}]}{1-\mho^2[r_{\mathbf{m}}]}}.$$

Arranging terms provides

$$\mathrm{E}\left[r_j\right]-r_f = \begin{array}{l}\frac{\mathrm{Cov}[r_{\mathbf{m}},r_j]}{\mathrm{Var}[r_{\mathbf{m}}]}\left(\mathrm{E}\left[r_{\mathbf{m}}\right]-r_f\right)+ \\ 4\frac{\mathrm{Cov}\left[\Phi\left(r_f;\mu_{\mathbf{m}},\sigma_{\mathbf{m}}\right),\phi\left(r_f;\mu_{\mathbf{m}},\sigma_{\mathbf{m}}\right)\left(\frac{\sigma_{\mathbf{m},j}}{\sigma^2_{\mathbf{m}}}\left(\mu_{\mathbf{m}}-r_f\right)-\left(\mu_j-r_f\right)\right)\right]}{1-\mho^2[r_{\mathbf{m}}]}\left(\mathrm{E}\left[r_{\mathbf{m}}\right]-r_f\right).\end{array} \qquad \square$$

***Proof of Theorem 4.*** By Theorem 3, the return on asset $j$ can be written as

$$r_j - r_f = \beta_{\mathcal{K},j}\left(r_{\mathbf{m}}-r_f\right)+\hat{\epsilon}, \tag{50}$$

where $\mathrm{E}\left[\hat{\epsilon}\right] = \mathrm{E}\left[r_{\mathbf{m}}\hat{\epsilon}\right] = 0$, which implies that $\mathrm{E}\left[r_j-r_f\right] = \beta_{\mathcal{K},j}\mathrm{E}\left[r_{\mathbf{m}}-r_f\right]$. Taking the variance of both sides of Equation (50) using expected probabilities yields

$$\mathrm{Var}\left[r_j\right] = \beta^2_{\mathcal{K},j}\mathrm{Var}\left[r_{\mathbf{m}}\right]+\mathrm{Var}\left[\hat{\epsilon}\right].$$

Normalizing by $1-\mho^2\left[r_j\right]$, provides

$$\frac{\mathrm{Var}\left[r_j\right]}{1-\mho^2\left[r_j\right]} = \beta^2_{\mathcal{K},j}\left(\frac{1-\mho^2\left[r_{\mathbf{m}}\right]}{1-\mho^2\left[r_j\right]}\right)\left(\frac{\mathrm{Var}\left[r_{\mathbf{m}}\right]}{1-\mho^2\left[r_{\mathbf{m}}\right]}\right)+\frac{\mathrm{Var}\left[\hat{\epsilon}\right]}{1-\mho^2\left[r_j\right]}.$$

Normalizing $\hat{\epsilon}$ by $\epsilon = \hat{\epsilon}\sqrt{\frac{1-\mho^2[\epsilon]}{1-\mho^2[r_j]}}$, yields

$$\Upsilon^2\left[r_j\right] = \beta^2_{\mathcal{K},j}\frac{1-\mho^2\left[r_{\mathbf{m}}\right]}{1-\mho^2\left[r_j\right]}\Upsilon^2\left[r_{\mathbf{m}}\right]+\Upsilon^2\left[\epsilon\right]. \qquad \square$$

***Proof of Theorem 5.*** The LHS of Equation (33) can be written as

$$\mathrm{Q}\left(X\leq x\right) = \Gamma^{-1}\left(\Gamma\left(p_x-\varphi\right)\right) = \Gamma^{-1}\left(\int_{\mathbb{P}}\Gamma\left(\mathrm{P}_x\right)d\chi\left(\mathrm{P}\right)\right), \tag{51}$$

for some $\varphi$. Ignoring $\Gamma^{-1}$ and taking the first-order Taylor approximation of $\Gamma\left(p_x-\varphi\right)$

around 0 with respect to $\varphi$ yields

$$\Gamma\left(p_x-\varphi\right)\approx\Gamma\left(p_x\right)+\Gamma'\left(p_x\right)\left(p_x-\varphi-p_x\right)=\Gamma\left(p_x\right)-\varphi\Gamma'\left(p_x\right). \tag{52}$$

Ignoring $\Gamma^{-1}$ and the integration in the RHS of Equation (51) for the moment, the second-order Taylor approximation of $\Gamma\left(\mathrm{P}_x\right)$ around $p_x$ is

$$\Gamma\left(\mathrm{P}_x\right)\approx\Gamma\left(p_x\right)+\Gamma'\left(p_x\right)\left(\mathrm{P}_x-p_x\right)+\frac{1}{2}\Gamma''\left(p_x\right)\left(\mathrm{P}_x-p_x\right)^2.$$

Since $\Gamma\left(p_x\right)$, $\Gamma'\left(p_x\right)$ and $\Gamma''\left(p_x\right)$ are constants, applying the integration provides

$$\int_{\mathbb{P}}\Gamma\left(\mathrm{P}_x\right)d\chi\left(\mathrm{P}\right)\approx\Gamma\left(p_x\right)+\frac{1}{2}\Gamma''\left(p_x\right)\zeta_x^2. \tag{53}$$

Equating (52) to (53) and organizing terms yields $\varphi\approx-\frac{1}{2}\frac{\Gamma''(p_x)}{\Gamma'(p_x)}\zeta_x^2$. Substituting $\varphi$ into Equation (51) proves the theorem. $\square$