Large Investors: Implications for Equilibrium Asset Returns, Shock Absorption, and Liquidity

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February 27, 2009

Abstract

We model illiquid asset markets where institutional investors account for their price-impact when trading. The model explains why liquidity beta's and market prices of liquidity-risk are time varying, and elevated during periods of market turbulence and heightened trading. We extend the distressed investor literature to a setting where all investors are optimizing, rational, and aware of distressed sales. Distressed sales cause pricing relationships to breakdown and front-running and predatory trading sometimes emerge, although in small amounts. We also introduce the concept "market-structure of risk-bearing capacity", and show it affects liquidity, the persistence of shocks to pricing relationships, and optimal liquidations.

Keywords: Strategic Investors, Contagion, Cournot Competition
JEL Classification Numbers: F36, G14, G15, D82, and D84

*Board of Governors of the Federal Reserve System. The views expressed in this paper are those of the author but not necessarily those of the Board of Governors of the Federal Reserve System, or other members of its staff. The author thanks conference participants and discussants at the Bank for International Settlements, the Washington Area Finance Association, and the Western Finance Association; as well as seminar participants at the University of Houston, the University of Texas at Austin, The Federal Reserve Board, the University of Colorado at Boulder, McGill University, and Brigham Young University. Address correspondence to Matt Pritsker, The Federal Reserve Board, Mail Stop 91, Washington DC 20551. Matt may be reached by telephone at (202) 452-3534, or Fax: (202) 452-3819, or by email at mpritsker@frb.gov.

1 Introduction

An increasingly large share of financial assets are owned or managed by large institutional investors whose desired orderflow can be large enough to move prices – and who account for their price impact when trading [Chan and Lakonishok (1995), Keim and Madhaven (1996)]. The price impacts are one of the most important determinants trading costs for institutional investors in markets around the world [Chiyachantana et al. (2004)]. Because large investors trades move prices, they cannot buy or sell all of the assets that they desire at prevailing prices, and markets are not completely liquid from their perspective.

In this paper I model how the liquidity problems faced by large investors affect financial market equilibrium, and then use the model to study an asset pricing puzzle and market dynamics when there is a distressed investor. The puzzle involves the pricing of systematic liquidity risk, which is the risk that many assets will become less liquid at the same time.[1] Assets' risk exposures and the price of this risk are both low about 90% of the time, and are otherwise elevated for short periods that are associated with heightened trading volume and market turbulence [Watanabe and Watanabe (2008), Longstaff et. al. (2000)].[2] At the present time there are very few theories that can explain the time variation in assets' factor loadings and risk prices, and its relationship to trading volume and market turbulence.

The pricing puzzle also appears to be associated with distressed liquidations on the part of large investors. The most prominent distressed liquidation was by the hedge fund Long Term Capital Management (LTCM) in the Fall of 1998. During the period of LTCM's troubles, asset pricing models experienced temporary breakdown that took the form of the emergence of additional temporary factors were need to price asset returns in some markets [Longstaff et. al. (2000)]. The emergence of new factors is related to the pricing puzzle because it is observationally equivalent to a change in existing factor loadings from zero. In addition to the pricing-model breakdown, there were widespread rumors as well as suggestive empirical evidence that LTCM's trades were front-run [Cai (2003)]. [3]

[1] Pastor and Stambaugh (2003) is one of the earliest papers to study whether "systematic liquidity risk" is priced, but they do not provide a theory.

[2] Watanabe and Watanabe (2006) find that 90% of the time risk exposures are low and risk prices are near 0, the other 10% of the time has high risk exposures, risk prices, return volatility, and volume. In the interest rate swaptions market, Longstaff, Santa Clara, and Schwartz (2000) found that an additional "event-related" pricing factor was required to explain the behavior of swaption prices during periods of severe market stress such as in the Fall of 1998.

[3] Empirical support that LTCM's trades may have been front-run is provided by Cai (2003). Part of LTCM's positions reportedly involved a short position in U.S. treasuries, and a long position in Danish mortgage backed securities and other high-yield fixed income instruments (Edwards, 1999). When LTCM was in financial distress, it presumably (their trades are private information) liquidated its positions by purchasing U.S. treasuries and selling high-yield debt. Cai's data can only be used to make inference on LTCM's trades in the Treasury bond futures markets. Her findings suggest that LTCM's purchases of U.S.

To analyze how the liquidity problems that face large investors affect equilibrium asset pricing and market dynamics when there is a distressed investor, I build a multi-asset, multi-participant, dynamic model of asset markets. The model relaxes the standard competitive assumption that all investors are price-takers, and instead assumes that markets are imperfectly competitive in the sense that the market contains both small price-taking investors and large investors who account for their price-impact when trading. All participants in the model are risk averse, and fully rational.

The results on asset pricing provide an economic interpretation of the asset pricing puzzle. The results show that under normal circumstances when large investors asset holdings are nearly consistent with optimal risk sharing, and large shocks are rare, then most of the time asset returns will satisfy a factor model (a one-factor CAPM in this model) and liquidity considerations will have little effect on expected asset returns even though markets are not perfectly liquid. However, when there are occasional significant shocks to investors asset demands (such as margin calls, mutual fund redemptions, etc...) that create a need to trade, liquidity will matter. Following such shocks, trading volume will increase but illiquidity will slow the adjustment of asset holdings towards those that are consistent with optimal risk-sharing. During the transition, additional “liquidity factors” will temporarily emerge that proxy for the deviations from imperfect risk-sharing. The factors take the form of factor-mimicking portfolios, and assets covariances (risk exposures) with the returns on these portfolios will temporarily be priced. Because illiquidity affects asset returns following large shocks, this mechanism is distinct from other channels through which illiquidity affects asset prices.[4] To the best of my knowledge, Watanabe and Watanabe (2008) were the first to provide an explanation for the pricing puzzle. Their explanation builds on Gallmeyer, Hollifield, and Seppi (2005) and is based on how preference uncertainty, learning through trade, and exogenous shocks to assets' liquidity, interact to generate time variation in assets' price sensitivities to changes in liquidity. Our theory is a conceptually distinct explanation that relies principally on large investors accounting for their own price impact, and trading toward optimal risk-sharing. Our explanation relies on neither learning, uncertainty, nor on exogenous shocks to asset liquidity.

The results on distressed investors are roughly consistent with the stylized facts during the period of LTCM's troubles. The anticipation of distressed sales cause asset prices to become depressed and to overshoot their long-run fundamental values; asset pricing models also experience temporary breakdown that take the form of the emergence of additional pricing factors; and in some circumstances there are de minimis amounts of front-running, although

Treasury futures were front-run.

[4]See Amihud, Mendelsohn and Pedersen (2005) for a review of these channels.

the amount of front-running is small relative to the amount of distressed sales. These results contribute to a recent theoretical literature on distressed investors [Attari, Mello, and Ruckes (AMR) 2005; Brunnermeier and Pedersen (BP) 2006; Carlin, Lobo, and Viswanathan (CLV) 2007]. The main findings within the literature are that strategic investors can take advantage of distressed investors that face financing constraints (AMR), and they can profit by following front-running or predatory trading strategies that involve selling at high prices ahead of or at the same time as the distressed seller, and then purchasing back at low prices afterwards [Brunnermeier and Pedersen (BP and CLV). One critique of the three papers in this literature is that the models rely on the presence of long-term investors with exogenously given downward sloping demand curves. These investors buy shares from the strategic investors at high prices, and later sell them back at low prices. Presumably, they might choose to behave differently if they are fully optimizing and/or aware of the distressed sales. This raises the question of how the nature of how the distressed trading equilibrium might differ when all investors are optimizing and aware of the distressed sales. This paper shows the results in the earlier literature are qualitatively robust: front-running and predatory trading sometimes emerge when all investors optimize and are aware of the distressed sales. However, the amounts of front-running and predatory trading that emerge are quantitatively small. Additionally the bulk of the distressed sellers losses are not a result of predatory trading and front-running, but are instead the result of how imperfect competition in asset markets affects market liquidity and lowers the prices received by the distressed seller.

The liquidity costs paid by the distressed seller depend on liquidity providers risk aversion. In many competitive models, the only way that risk aversion affects liquidity costs is through the market's total risk bearing capacity, as measured by the sum total of liquidity providers risk tolerances.[5] One contribution of this paper is that it shows that when liquidity is provided by large investors that behave strategically, then liquidity depends on total risk bearing capacity, and on how this capacity is distributed among large institutional investors. This distribution of risk bearing capacity across investors is the "market structure" of risk bearing. When this capacity is concentrated, then investors with high risk-bearing capacity have market power that they can use to reduce liquidity provision and raise the reward for providing liquidity. When risk-bearing capacity is more diffuse, this market power is lessened. The results in the paper show that the "market structure" of risk bearing is an important determinant of equilibrium liquidity provision. We also show that "market structure" matters for the length of time that transitory risk-factors persist following a large shock, and it affects the optimal trading strategy to following when optimally liquidating a position.

[5] See for example, Grossman and Miller (1988).

This paper is related to the voluminous literature on asset market liquidity. Illiquidity is usually modeled as deriving from some combination of the following three sources: exogenous transaction costs, asymmetric information, or imperfect competition in asset markets. Recent research also emphasizes how constraints on traders funding can lead to the erosion of market liquidity.[6] This paper is most closely related to the literature on imperfect competition in asset markets when information is symmetric.[7] Lindenberg (1979) models the behavior of many large investors trading many assets in a single period mean-variance setting. Basak(1997) and Kihlstrom(2001) expand the analysis to allow for multiple time periods, but they only consider a setting in which there is a single large investor. The model presented here can be understood as an extension of Lindenberg, Basak, and Kihlstrom, which allows for multiple time periods, multiple risky assets, and multiple large investors who vary in their risk aversion. Although I did not do so, the model could also have been derived by extending the dynamic models of Urosevic (2002a) or Vayanos (2001) to allow for multiple assets and multiple large investors who vary in their risk aversion while removing the moral hazard elements of Urosevic's model, or by removing the noise traders and information asymmetry that are present in Vayanos' model.[8]

The remainder of the paper consists of seven sections. To help fix ideas, the next section presents some of the main ideas in a basic model. Section 3 presents the general model. Section 4 studies the implications of the model for asset pricing and the asset pricing puzzle.

[6] The literature is far too large to cite all the relevant papers. For the relationship between funding and market liquidity, see Brunnermeier and Pedersen (2008). For recent reviews of the liquidity literature see Amihud, Mendelsohn, and Pedersen (2005), and Cochrane (2005). Illiquidity resulting from exogenous transaction costs is studied by Constantinides (1986), Heaton and Lucas (1996), Vayanos (1998), Vayanos and Vila (1999), Huang (1999), and Milne and Neave (2003). One type of transaction costs is search costs; this is studied by Duffie et. al. (2001). Illiquidity resulting from asymmetric information about asset payoffs is studied in many papers including Glosten and Milgrom (1985), Kyle (1985), Kyle (1989), Eisfeldt (2001). Asymmetric information about market participants asset holdings is studied by Cao and Lyons (1999), Vayanos (1999), and Vayanos (2001). Finally, illiquidity resulting from imperfect competition in asset markets is studied by Lindenberg(1979), Kyle (1985), Kyle (1989), Basak (1997), Cao and Lyons (1999), Vayanos (1999), DeMarzo and Urosevic (2006), Urosevic(2002a), Urosevic(2002b), Vayanos (2001), and Kihlstrom (2001). Although less common, illiquidity has also been modeled as resulting from Knightian uncertainty [Cherubini and Della Lunga (2001, forthcoming), Routledge and Zin (2001)]; or as the outcome of optimal security design [Boudoukh and Whitelaw (1993), DeMarzo and Duffie (1999)].

[7] One strand of this literature which is not pursued here follows Grossman and Miller (1988) and models liquidity as a function of the number of non-strategic market makers, which is itself endogenous. Fernando (2003) and Fernando and Herring (2003) make the number of market makers an endogenous function of the relative importance of idiosyncratic and systematic valuation shocks.

[8] Urosevic's (2002a) extends DeMarzo and Urosevic's (2006) dynamic model of a single large investor to a setting in which several large shareholders choose to trade the assets of one or more firms while simultaneosly choosing how carefully to monitor the behavior of the firms' management. Urosevic (2002a) does not examine a setting with many large investors and many risky assets. He instead examines a setting with one large investor and many risky assets, or one risky asset and many large investors. He also only considers the case in which all large investors have the same risk aversion. Urosevic (2002b) empirically tests the theoretical model in Urosevic (2002a).

Section 5 studies how large investors affect equilibrium trades, asset prices, and market liquidity when there is a distressed seller. An additional contribution of the section is that it studies optimal dynamic liquidations in a framework with endogenous liquidity provision when all market participants are fully optimizing and aware of the distressed sales. Section 6 studies the relationship between liquidity and market structure; a final section concludes.

2 The Basic Model

A basic idea that lies behind many of the results in the paper is that when markets are illiquid, differences in investors' willingness to pay liquidity costs cause them to follow different trading styles. These differences in trading styles affect risk-sharing, and thus have implications for asset pricing and for market illiquidity. The way in which different trading styles emerge and have pricing and liquidity implications is best illustrated in the context of a simple model economy with 2 assets, 3 time periods, and 3 investors.

The economy contains a risk-free, and risky asset. The riskfree asset is in perfectly elastic supply and pays fixed gross return $r > 1$ per period. The risky asset has supply X, and period 3 liquidation value V, where

$$V \sim \mathcal{N}(\bar{V}, \sigma^2).$$

The assets are traded in periods 1 and 2. The assets are liquidated in period 3 and investors consume the liquidation value of the assets. For simplicity, there is no consumption in periods 1 and 2, and the risky asset pays no dividends. These restrictions are relaxed in the general model.

All investors are risk averse, and maximize the CARA utility of their time 3 consumption. Investor 1 represents a continuum of small investors who take prices as given. The small investors have unit density and are indexed by $s \in [0, 1]$. Investors 2 and 3 are large institutional investors who take their price impact into account when trading.

In time periods 1 and 2, trading among investors takes place via a Cournot Stackelberg trading game. In each period of the game, small investors have a demand curve for absorbing orderflow from the large investors. Large investors take this demand curve as given, and choose their trades optimally by playing a Cournot game in which they take their price impacts into account.

To close the basic model, an assumption regarding the information environment is required. To exclusively focus on how large investors affect liquidity and asset pricing, we rule out illiquidity through other sources such as information asymmetry. This means that all investors have the same information sets. They know the setup of the game, as well each each other's utility functions, asset holdings, and trades at all points in time, and that this is common knowledge.[9]

2.1 Model Solution

The model is solved by backwards induction starting from trade in period 2, by first solving the small investors optimization, and then that of the large investors.

Small Investors at time 2

In period 2, each small investor s enters the period with risky asset holdings $Q_s(2)$ and riskfree asset holdings $q_s(2)$, and chooses trades in risky asset holdings and riskfree assets, $\Delta Q_s(2)$ and $\Delta q_s(2)$ to maximize their expected utility from consuming their time 3 wealth.

$$\max_{\Delta Q_s(2), \Delta q_s(2)} - \text{Exp}(-A_s W_s(3)) \tag{1}$$

subject to the budget constraint

$$\Delta q_s(2) = -\Delta Q_s(2) P(2),$$

where

$$W_s(3) = [Q_s(2) + \Delta Q_s(2)]V + r(q_s(2) + \Delta q_s(2)).$$

Note that the second term in the expression for $W_s(3)$ reflects the idea that the riskfree asset grows at rate r between periods.

As is usual in these models, we assume that there are no short-selling or borrowing constraints.[10] Under these conditions, investor s's demand for risky asset purchases takes the usual form:

$$\Delta Q_s(2) = \left(\frac{1}{A_s}\right) \left[\frac{\bar(V) - rP(2)}{\sigma^2}\right] - Q_s(2). \tag{2}$$

Note that because each small investor is infinitesimal, their individual trades do not have price impact. Therefore, equation 2 shows that if trader s experienced an endowment shock

[9] In a world where there are large investors whose trades move prices, private knowledge of one's own utility functions, trading plans, or endowments is itself asymmetric information that could be a source of illiquidity in asset markets as in Vayanos [2001]. To rule out this form of information asymmetry, investors positions must be known at all points in time.

[10] When such constraints exist, they affect liquidity through other channels as in Brunnermeier and Pedersen (2008).

that increased his holdings of Q_s by 1, then his optimal trade would offset this increase by the same amount. This is a key characteristic of investors whose trades do not have price impact.

Integrating risky asset trades across small investors produces ΔQ_1, the net purchases of the small investors as a function of the price.

$$\begin{aligned}
\Delta Q_1(2) &= \int_0^1 \Delta Q_s(2) ds \\
&= \left[\int_0^1 \left(\frac{1}{A_s}\right) ds\right] \times \left[\frac{\bar(V) - rP(2)}{\sigma^2}\right] - \int_0^1 Q_s ds \\
&= \frac{1}{A_1}\left[\frac{\bar(V) - rP(2)}{\sigma^2}\right] - Q_1(2)
\end{aligned} \tag{3}$$

Inverting produces an expression for $P(2)$ as a function of the net orderflow purchased by small investors at time 2:

$$P(2) = \frac{1}{r}\left[\bar{V} - A_1\sigma^2[Q_1(2) + \Delta Q_1(2)]\right] \tag{4}$$

The expression for price of risky assets at time is intuitive; it shows that the equilibrium price is decreasing in the amount of risky assets held or purchased by small investors, and that the magnitude of the decrease depends on the risky of the assets, σ^2, and on A_1 small investors collective risk aversion.[11]

Large Investors at Time 2

Each large investor solves a maximization problem that is comparable to that of the small investors, except that large investors take their price impact into account. More specifically, large investor m where $m = 2, 3$ solves

$$\max_{\Delta Q_m(2), \Delta q_m(2)} - \text{Exp}(-A_m W_m(3)) \tag{5}$$

subject to the budget constraint

$$\Delta q_m(2) = -\Delta Q_m(2) P(2),$$

where

$$W_m(3) = [Q_m(2) + \Delta Q_m(2)]V + r(q_m(2) + \Delta q_m(2)).$$

[11] More formally, A_1 is the inverse of the integrated risk tolerances of the small investors.

Each large investor also takes into account that the price of risky assets must adjust to clear the market, and each large investor accounts for his own impact on the risky asset price while taking the risky asset trades of the other large investors as given. Market clearing requires that the small investors net purchases are equal to the large investors net sales:

$$\Delta Q_1(2) = -[\Delta Q_2(2) + \Delta Q_3(2)].$$

Substituting for $\Delta Q_1(2)$ in small investors demand for risky assets then shows prices in time period 2 are affected by large investors trades:

$$\begin{aligned} P(2) &= \frac{1}{r}\left[\bar{V} - A_1\sigma^2 Q_1(2) + A_1\sigma^2(\Delta Q_2(2) + \Delta Q_3(2))\right] \\ &= \alpha(2) - \beta(2)Q_1(2) + \beta(2)\Delta Q_2(2) + \beta_2\Delta Q_3(2) \end{aligned} \tag{6}$$

The second line of the expression for $P(2)$ shows that price impact of each large investors purchase of an additional unit (taking the other large investors trade as given) have the same value $\beta(2)$ in period 2, which is equal to $A_1\sigma^2/r$.

Given this price impact function, after a bit of algebra the first order condition for investor 2 in period 2, also known as investor 2's reaction function, reduces to:

$$\Delta Q_2(2) = \frac{A_1}{A_2 + 2A_1}Q_1(2) - \frac{A_2}{A_2 + 2A_1}Q_2(2) - \frac{A_1}{A_2 + 2A_1}\Delta Q_3(2) \tag{7}$$

For our purposes, the most interesting feature of investor 2's first order condition is that his optimal trade does not change one for one with a change in Q_2. Instead, if his risky asset holdings are increased by 1, he accounts for the price impact of his trades and only sells a part of it. Moreover, how much he chooses to sell depends on his risk aversion. If investor 2 is very risk averse (high A_2) then his optimal trades move nearly one for one with changes in Q_2 while if his risk aversion is low, then his trading strategy is not very sensitive to his asset holdings. The intution for this result is straightforward. In an illiquid market, relatively risk averse traders are willing to pay a higher liquidity cost to trade toward their optimal risk exposure than are traders who are more tolerant of risk. This reasoning is analogous to Pratt's classic result that absolute risk aversion is proportional to the certainty equivalent wealth that an investor would pay to eliminate risk. In illiquid markets — the certainty equivalent wealth is paid through trading.

An immediate implication of the above reasoning is that when large investors trades move prices, large investors with different risk aversion have different trading styles, and these differences in trading styles have consequences for risk sharing. To illustrate this point, note that the first order condition (reaction function) for large investor 3 in period 2 is analogous to that that of investor 2, and is given by:

$$\Delta Q_3(2) = \frac{A_1}{A_3 + 2A_1} Q_1(2) - \frac{A_3}{A_3 + 2A_1} Q_3(2) - \frac{A_1}{A_3 + 2A_1} \Delta Q_2(2) \tag{8}$$

Suppose $A_3 > A_2$. Then if we give risky asset shares to investor 2, and take them away from investor 3, then because A_3 is greater than A_2 investor 2 will not sell much of the shares that he received, while investor 3 may be anxious to buy more and will have to turn to the small investors for the shares. As a result this redistribution of shares abmong the large investors will change the amount that is purchased from the small investors and will thus affect the equilibrium price.

The above reasoning is incomplete because the optimal trades of investor 2 depend not only on his own holdings, but also on the optimal trades of investor 3. Therefore, the optimal trades of both large investors are determined in equilibrium as the set of large investors trades that simultaneously solve the first order conditions of both investors. With two large investors, the set of equilibirum trades that satisfy their first order conditions is simply where their reaction functions cross. Figure 1 illustrates this graphically when $A_2 < A_3$. In particular, when $A_2 > A_3$, Figure 1 begins with an initial situation in which given the initial endowments of investors 2 and 3, the amount of risky asset that investor 2 wishes to purchase is just equal to the amount that investor 3 wishes to buy, and hence their net trade vector with the small investors is 0. From this inititial point, Figure 2 shows that providing investor 2 with more risky asset and investor 3 with less risky asset results in net purchases of risky assets by the two large investors in period 2. This reduces the amount that is held by small investors and drives up the equilibrium price in period 2, and drives down the equilibrium expected returns between periods 2 and 3.

Formally, this simple example illustrates the general principal that when markets are illiquid, the equilibrium asset prices that prevail in period 2, $P^e(2, Q(2))$ are a function of investors risky asset allocations at the beginning of period 2, and hence depends on $Q(2)$, the stacked vector of risky asset holdings of the investors ($Q(2) = [Q_1(2), Q_2(2), Q_3(2)]'$) at the beginning of period 2. In this case, the relationship is linear, and is given by:

$$P^e(2, Q(2)) = \alpha(2) + \beta(2,1)Q_1(2) + \beta(2,2)Q_2(2) + \beta(2,3)Q_3(2). \tag{9}$$

Although there is only one risky asset in this setting, because its price in period 2 depends on how it is allocated among investors at the beginning of period 2, it suggests risky asset's expected returns will depend on more than their covariance with the market portfolio, and hence the CAPM will not necessarility hold in this setting.

Differences in investors trading styles also affect liquidity. One measure of this liquidity is the slope of the small investors demand curve for absorbing large investors orderflow. To solve for this slope, we need to solve for each small investors demand for assets at time 1, as a function of the trades of large and small investors.

Small Investors at Time 1

To solve for small investors asset demand, note that given the equilibrium price function in period 2 (equation 9), small investor s's value function in period 2 takes the form:

$$V_s(W_s(2); Q(2)) = \psi(Q(2)) \times \mathrm{Exp}[-A_s W_s(2)], \tag{10}$$

where,

$$W_s(2) = (q_s(2) + Q_s(2) P^e(2, Q(2)))$$

Note that the state-vector $Q(2)$ enters the value function in two ways. First, it affects small investors utility by affecting risk sharing between periods 2 and 3 through the function ψ, and it affects wealth at time 2 through the the equilibrium price function $P^e(2, Q(2))$.

To solve the model in period 1, we find small investor s's optimal demand conditional on every possible evolution of the state variables that is consistent with market clearing. This evolution is given by:

$$\begin{aligned} Q_1(2) &= Q_1(1) - (\Delta Q_2(1) + \Delta Q_3(1)) \\ Q_2(2) &= Q_2(1) + \Delta Q_2(1), \\ Q_3(2) &= Q_3(1) + \Delta Q_3(1); \end{aligned}$$

where market clearing is imposed by the condition that the net purchases of the small investors is equal to the the net sales of the large investors.

This condition is written more succinctly as:

$$Q(2) = Q(1) + \Delta Q(1). \tag{11}$$

Given each evolution of the state vector, small investor s's demand solves,

$$\max_{\Delta Q_s(1), \Delta q_s(1)} \mathrm{E}\, V_s[W_s(2); Q(1) + \Delta Q(1)] \tag{12}$$

where,

$$W_s(2) = r[q_s(1) + \Delta q_s(1)] + [Q_s(1) + \Delta Q_s(1)] \times P^e[2, Q(1) + \Delta Q(1)]$$

subject to the budget constraint:

$$\Delta Q_s(1)P(1) = -\Delta q_s(1)$$

Note that in this stylized setting the risky asset is riskless between periods 1 and 2 because conditional on the trades made at time 1, the equilibrium price of the risky asset in period 2 is known. It therefore follows from the small investors first order condition (and no arbitrage) that the price of risky assets between periods 1 and 2 must rise at the risk-free rate. This implies in period 1, the small investors (inverted) demand function for absorbing the orderflow of the large investors satisfies[12]:

$$\begin{aligned} P(1) &= \frac{P^e(2, Q(1) + \Delta Q(1))}{r} \\ &= \frac{\alpha(2)}{r} + \frac{\beta(2,1)}{r}(Q_1(1) - \Delta Q_2(1) - \Delta Q_3(1)) + \frac{\beta(2,2)}{r}(Q_2(1) + \Delta Q_2(1)) \\ &+ \frac{\beta(2,3)}{r}(Q_3(1) + \Delta Q_3(1)) \end{aligned} \tag{13}$$

Grouping the terms that multiply $\Delta Q_2(1)$, or $\Delta Q_3(1)$ produces the slope of the small investors period 1 demand curve with respect to the purchases of investor 2 and 3 respectively. The slope for investor 2 in period 1, denoted $\beta(1,2)$ is given by $\beta(1,2) = [\beta(2,2) - \beta(2,1)]/r$.

The functional form of $\beta(1,2)$ depends on both small investors risk preferences and on both investors reaction functions. Tedious algebra shows it has form:

$$\beta(1,2) = \frac{\sigma^2}{r} \times \frac{A_1^3 + A_1^2 A_3}{3A_1^2 + 2A_1A_2 + 3A_1A_3 + A_2A_3}$$

For simplicity of terminology, hereafter the slope function will be referred to as the price impact function, although it is understood that the price impact of a trade depends on how all investors respond to it. Investor 2's price impact has some of the usual characteristics that one would expect. The price impact is increasing in risk σ^2, and is increasing in A_1 and A_3 the risk aversion of the small investors and of the other large investor. This seems sensible since when risk is transferred in an illiquid market, the risk premium that others will require for taking it on is increasing in the risk and in the risk aversion.

[12]In period 1, each small investor conditions on the trades of the other small investors as a state variable when forming their own demands. For internal consistency, the integrated demand of the small investors, when formed in this way, must be equal to each small investors beliefs about the integrated demand. In this example, this internal consistency condition is satisfied because if risky asset prices rise at the risk-free rate between periods 1 and 2, then small investors are indifferent over how much risky asset they buy in period 1. Therefore, there will be an equilibrium where small investors purchases of the risky asset clear the market and are internally consistent.

The more interesting and unconventional result is that the price impact function for investor 2 is decreasing in investor 2's own risk aversion. This means that if investor 2 is more risk tolerant, he will have a larger price impact and face greater illiquidity in asset markets when he is trading. Similarly, a comparison of the price impact functions for investors 2 and 3 (not shown) reveal that if investor 2 has greater tolerance than investor 3, then investor 2 will have greater price impact than investor 3. This implies that in this symmetric information setting the price impact functions of investors 2 and 3 will be different.

The intuition for why investor 2's price impact in period 1 is increasing in his own risk tolerance is straightforward. The price impact coefficient measures the change in the marginal value of the asset to small investors when investor 2 buys a single share from small investors in period 1. When investor 2 buys a single share in period 1, then the amount that the purchase will change small investors marginal valuation depends on how investor 2 will trade the asset in period 2. If investor 2 is very risk averse, his trading style is such that he will sell most of it in period 2, and hence small investors marginal values' would be little affected in period 2; and by backwards induction, will also be little affected in period 1. Conversely, if investor 2 is more risk tolerant, then when he buys a share in period 1, given his trading style, he will sell little of it in period 2, thus raising small investors marginal values in periods 1 and 2.

This establishes that large investors with different trading styles due to differences in risk preferences will have different equilibrium price impacts. This is both a blessing and a curse for large investors. The downside of having a large price impact is that the market is less liquid for very risk tolerant large investors. At the same time, a large price impact also means that very risk tolerant large investors have very significant market power which they can then expploit to take advantage of distressed traders. Large investors can also use their market power to influence share allocation and price setting at IPO's.

The finding that price impacts depend on large investors risk tolerances also points towards a theory of the market structure of institutional investors since as we will argue below large investors can take actions to change their risk tolerance.

These ideas are further developed in the general model that follows. The general model differs from this example in one very important respect. The risky assets pay risky dividends in all periods; hence they are never riskless. As a result, the equilibrium pattern of returns between periods 1 and 2, which was very simple in this section, will now be more complicated. In particular, the new pattern of returns will now have to be provide returns that are sufficient to compensate small investors for the risk they bear along optimal transition paths of asset holdings, and trades.

3 The General Model

In this section I present a model of the dynamic interaction of large and small investors wihin a stylized economy. The economy contains M market participants, $m = 1, \dots M$, who consume and trade N risky assets and a risk free asset. The stylized economy can be interpreted as representing the markets for all risky assets in the actual economy, or can more realistically be interpreted as representing a subset of the actual markets for which trading is thin. To facilitate solution of the model, the investors in the economy are infinitely lived but only trade risky assets over a large but finite number of time periods $t = 1, \dots T - 1$. In periods T and after, participants do not trade risky assets, but they continue to receive dividends, and borrow and lend at the riskfree rate in order to finance their consumption.[13]

3.1 Participants

Participant 1 represents the net demands of a continuum of infinitesimal ("small") investors, indexed by s, that are uniformly distributed on the unit interval. Each small investor takes asset prices and the state of the economy (including other investors trades) as given. In analogy with the literature on industrial organization, participant 1 is sometimes referred to as the competitive fringe. Participants 2 through M are large investors whose desired orderflow moves prices. Each participant m has CARA utility of per period consumption with coefficient of absolute risk aversion A_m:

$$U_m(C_m(t)) = -e^{-A_m C_m(t)}. \tag{14}$$

Participants choose their asset holdings to maximize their discounted expected future utility of consumption:

$$U_m(C_m(t), \dots, C_m(\infty)) = \mathrm{E}_t \left\{ \sum_{i=0}^{\infty} \delta^i U_m(C_m(t+i)) \right\}, \tag{15}$$

with discount rate δ, $0 < \delta < 1$.

Participants risk tolerances, $1/A_m$, play an important role in the analysis. When markets are competitive, the risk premium for bearing market risk is inversely proportional to the sum total of investors risk tolerances, $\sum_{m=1}^{M} 1/A_m$. I denote this sum as the economy's total risk bearing capacity; and let RBC_m, the risk-bearing capacity of investor m, denote investor

[13]The device of infinitely lived investors and finite number of trading periods was introduced by DeMarzo and Urosevic (2000); it facilitates solution of the model via backwards induction and produces parsimonious solution for asset prices.

m's share of the economy's total risk bearing capacity:

$$RBC_m = \frac{1/A_m}{\sum_{m=1}^{m}(1/A_m)}.$$

In a competitive economy, investors' shares of risk bearing capacity do not influence expected returns, but when markets are imperfectly competitive, whether the risk bearing capacity is concentrated among a small number of large institutional investors, or is more diffusely distributed is important. For this reason, the distribution of risk bearing capacity among investors is referred to as the economy's market structure. For the purposes of this paper, the market structure is exogenous. My preferred interpretation for the origins of the market structure is that all investors in the economy are small; and the market structure reflects how the small investors have chosen to have their assets managed. Those investors who manage their own portfolios are the competitive fringe and are collectively represented by investor 1. In the appendix I show that investor 1's risk tolerance is the sum total of the risk tolerances of all investors who manage their own portfolios. Investors that have similar risk preferences can economize on portfolio management costs by having their portfolios collectively managed by a financial intermediary such as an institutional investor.[14] Because each institutional investor trades on behalf of a positive mass of small investors, the institutional investor is large in asset markets. In the appendix I show that if each large institutional investor makes optimal portfolio decisions on behalf of a mass μ_m of identical small investors that each have risk tolerance $1/A_{m,s}$, then the large investor behaves as if his mass is 1 and his risk tolerance is $1/A_m = \mu_m/A_{m,s}$. Therefore, each large investor's risk bearing capacity depends on the risk tolerance of the base of small investors that he represents, and on the number (measure) of investors who make up that base.[15] Under this interpretation, there is wide scope for differences in the size, and hence risk tolerance of large institutional investors. This is consistent with Gabaix et. al. (2006) who find that differences in the size of institutional investors are described by Zipf's law.

[14]Other reasons why small investors might turn their portfolios over to institutional investors include better management of individual investors liquidity needs (Nanda and Singh, 1998), and reduction in the costs of implementing dynamic trading strategies (Mamaysky and Spiegel, 2002).

[15]If a large investor takes on independent and actuarily fair risk ϵ, and then spreads it among μ_m identical investors that each have risk aversion $A_{m,s}$, then the Arrow-Pratt certainty equivalent that each would require to take on his share of the risk is approximately $.5A_{m,s}\sigma_\epsilon^2/\mu_m^2$. Hence the total risk compensation required by the mass μ_m of investors is $.5A_{m,s}\sigma_\epsilon^2/\mu_m$. This shows absolute risk aversion of the large investor is equal to that of each small investor divided by μ_m; or equivalently that the large investors risk tolerance is equal to $\mu/A_{m,s}$.

3.2 The Assets

The N risky assets are in fixed supply X, and pay perfectly liquid cash dividends that are distributed i.i.d. through time:[16]

$$D(t) \sim \text{ i.i.d. } \mathcal{N}(\bar{D}, \Omega) \tag{16}$$

Participants are endowed with assets at time 1, and then trade them in each period through time $T-1$. Participant m's holdings of risky assets at the beginning of time period t is denoted $Q_m(t)$, and the $NM \times 1$ stacked vector of all participants risky assets holdings is denoted $Q(t)$. Similarly, changes in the risky asset holdings during the period are denoted $\Delta Q_m(t)$ and $\Delta Q(t)$ respectively.[17]

The riskfree asset is in perfectly elastic supply, and grows at gross-rate r between time periods. Within periods the holdings of riskfree assets are exactly equivalent to cash, or to consumption goods (which are normalized to have price 1). Investor m's beginning of period risk-free holdings and the change in his holdings during period t are denoted by $q_m(t)$ and $\Delta q_m(t)$. Hence his end of period holdings are $q_m(t) + \Delta q_m(t)$, and his holdings at the beginning of the next period are:

$$q_m(t+1) = r[q_m(t) + \Delta q_m(t)]. \tag{17}$$

3.3 The Trading Process

At the beginning of each period $t < T$ investors receive dividends $Q_m(t)'D(t)$ on their beginning-of-period risky asset holdings, then they trade and consume, and the period ends. The key features of real-world trading that the model is designed to capture is that some investors are small and take prices as given, while others are large and account for how their trades affect prices. To capture these features, the trading process is modeled as a dynamic Cournot-Stackelberg game of full information. The strategic environment at the beginning of each period t is summarized by the vector of investors risky-asset holdings, $Q(t)$. Given this environment, and the set of all possible trades by the large investors, the small investors formulate their individual asset demands while taking prices and all other

[16]Dividends are i.i.d. for simplicity. When dividends are i.i.d. asset prices are a deterministic function of time and investors asset holdings. In earlier work, dividends followed multivariate AR(1) processes with correlated shocks. This caused prices to be stochastic, but generated the same results on equilibrium expected returns.

[17]Therefore

$$Q_m(t+1) = Q_m(t) + \Delta Q_m(t), \quad \text{and} \quad Q(t+1) = Q(t) + \Delta Q(t).$$

investors orderflow as given; the demands aggregate to form a schedule which describes the market clearing prices at which the competitive fringe is willing to absorb the large investors net orderflow. The large investors take the price schedule as given, and then play a standard Cournot game in which they choose their trades to maximize their value functions subject to their budget constraints while accounting for the price impact of their trades.

One goal in writing this paper is to model illiquidity in a new framework that does not rely on information asymmetry. To ensure that asymmetric information does not drive the results, the trading process is modelled in a symmetric information setting: that which is known by one investor is known by all. This means that at each point in time, investors know the distribution of asset payoffs, each others' utility functions, asset positions, and trades, and all of this is common knowledge. With this framework in mind, the next section outlines how the model is solved.

3.4 Solving the Model

The model is solved through dynamic programming and backwards induction. Dynamic programming with undetermined coefficients is used to find a closed form solution for investors' value functions of entering period T with given holdings of risky and riskfree assets. Value functions for earlier periods are then solved through backwards induction. There are three steps in the induction: 1) for a given state vector $Q(t)$, time $t+1$ value functions, and hypothetical trades by the large investors, I solve for the competitive fringe's period t demand function for the risky assets. Inverting the demand function produces a schedule relating large investors' trades to time t equilibrium asset prices. 2) I solve large participants equilibrium trades when they take the price schedule and each others' trades as given. 3) I solve for the participants' optimal consumption choices. The optimal trade and consumption choices are then used to solve for all investors time t value functions. The same steps are repeated for all earlier periods. Additional information on the backwards induction are provided below. Full details are provided in the appendix.

The Price Schedule Faced by Large Investors

The price schedule maps any possible orderflow of the large investors at time t to the market clearing prices that would result under the assumption that large investors trade optimally in all future periods. To illustrate the price schedule's derivation, imagine that investors enter time period t with risky asset holdings $Q(t)$ and then the large investors submit risky-asset orderflow $\Delta Q_m(t)$, $m = 2, \dots M$. Based on this orderflow, each infinitesimal small investor

$s \in [0, 1]$ solves the maximization problem:

$$\max_{C_s(t),\Delta Q_s(t),\Delta q_s(t)} -e^{-A_s C_s(t)} + \delta E_t\{V_s(q_s(t+1), Q_s(t+1)\ ; Q(t) + \Delta Q(t), t+1)\}, \tag{18}$$

where,

$$q_s(t+1) = r(q_s(t) + \Delta q_s(t),$$

$$Q_s(t+1) = Q_s(t) + \Delta Q_s(t),$$

subject to the budget constraint[18],

$$C_s(t) + \Delta Q_s(t)' P(., t) + \Delta q_s(t) \leq Q_s(t)' D(t).$$

In the budget constraint, $P(., t)$ is the risky assets price vector at time t. In addition, the first two arguments of small investors value functions correspond to their time $t+1$ holdings of riskfree and risky assets. Small investors take the arguments of their value functions that appear after the semicolon as given. These arguments are the economy's state-vector of risky asset holdings at time $t+1$, and time $t+1$.

For the price $P(., t)$ to be market clearing, small investors demands must satisfy equation (18) and prices must be set so that the net orderflow of the small and large investors sums to 0.

$$\int_0^1 \Delta Q_s(t)\ ds + \sum_{m=2}^{M} \Delta Q_m(t) = 0$$

The price schedule must also be consistent with an additional internal consistency condition for small investors orderflow. Recall that small investors take the orderflow of other small investors as given and treat it as a state-variable. For small investors beliefs about the state variable to be internally consistent, $\Delta Q_1(t)$, their beliefs about the net trades of all small investors in equation (18), must be consistent with the optimal behavior of small investors conditional on their beliefs; i.e. internal consistency requires that[19]:

$$\Delta Q_1(t) = \int_0^1 \Delta Q_s(t) ds \tag{19}$$

For any given set of trades by the large investors, I solve for equilibrium prices which satisfy the market clearing and internal consistency conditions. Each such price $P(., t) = P(\Delta Q(t), Q(t), t)$ is one point on the price schedule which is faced by the large investors. The

[18] The budget constraint is standard: it requires that expenditure on period t consumption must be financed by period t dividend income, sales of risky assets, or by running down cash holdings.

[19] $\Delta Q_1(t)$ corresponds to the first N rows of the $Q(t) + \Delta Q(t)$ argument of the small investors value function in equation (18).

full price schedule is found by solving the above problem for all possible $Q(t)$ and all possible $\Delta Q(t)$. The resulting price schedule turns out to a linear function of investors beginning of period risky-asset holdings, $Q(t)$, and of the trades of the large investors, $\Delta Q_m(t)$, $m = 2, \ldots M$:[20]

$$P(.,t) = \frac{1}{r}\left(\beta_0(t) - \beta_Q(t)Q(t) - \sum_{m=2}^{M} \beta_m(t)\Delta Q_m(t)\right), \tag{20}$$

where $-(1/r)\beta_m(t)$ is the slope of the price schedule with respect to ΔQ_m, large investor $m's$ orderflow at time t. It is important to keep in mind that the price schedule is for all N assets. Therefore, $-(1/r)\beta_m(t)$ is an $N \times N$ matrix whose i, j element is the price impact that additional purchases of asset j by investor m have on the price of asset i. The coefficients $\beta_m(t)$ are formally derived in the appendix.[21]

Large Investors Portfolio Problem

Large investors choose their optimal portfolios by solving a maximization problem that is similar to equation (18), but in a Cournot-game framework in which large investors take each others trades as given, and account for their own impact on prices via equation (20). The equilibrium trades are the solutions of a stacked system of linear equations that constitute the first-order conditions (aka reaction functions) for each large investor. The resulting equilibrium within the period is of the Stackelberg-Cournot-Nash type.[22] Given large investors trades, it is possible to solve for optimal consumption and then to use the result to solve for investors value functions at time t as described above. The resulting equilibrium is subgame perfect because investors optimal strategies are solved by backwards induction from period T.

Investors Value Functions

Because there is no trade after period $T-1$, investors value functions in period T and thereafter are the same whether investors are large or small. Because the investors have CARA utility and dividends are normally distributed, the value functions have a very simple form that was found through guessing a solution and then verifying that it satisfies the

[20]In the appendix, see the derivation of equation (A7).

[21]In the appendix, I refer to the matrix vech$[\beta_2(t), \beta_3(t), \ldots, \beta_M(t)]$ as $\beta_{Q_B}(t)$.

[22]The equilbrium trades and prices are a Stackelberg-Cournot-Nash equilibrium if 1) large investors take the price function as given, 2) if each large investors' orderflow is optimal given the price function and given the orderflow of the other large investors, and if 3) the total orderflow is market clearing. Criteria 1 and 2 are satisfied by the derivation of the reaction functions; and criteria 3 is satisfied because the price schedule was constructed so that the competitive fringe would absorb the large investors net orderflow.

Bellman equation. The value function for periods T and thereafter is provided in the next proposition:

Proposition 1 *Let m index small or large investors in periods $t \geq T$. Then, for all investors m with CARA utility and risk aversion A_m, the value of entering period $t \geq T$ with riskfree asset holdings $q_m(t)$ and risky asset holdings $Q_m(t)$ is exponential linear-quadratic in investors own asset holdings, and is given by:*

$$V_m(Q_m(t), q_m, t) = -k_m(t) \exp^{-A_m(t)q_m(t) - A_m(t)Q_m(t)'\bar{V}(t) + .5A_m(t)^2 Q_m(t)'\Omega(t)Q_m(t)} \tag{21}$$

where,

$$\begin{aligned} k_m(T) &= \left(\frac{r}{r-1}\right) \times (\delta r)^{[1/(r-1)]} \\ A_m(t) &= A_m[1 - (1/r)] \\ \bar{V}(t) &= \frac{\bar{D}}{[1 - (1/r)]} \\ \Omega(t) &= \frac{\Omega}{[1 - (1/r)]} \end{aligned}$$

Proof: See section B.2 of the appendix.

For periods $t < T$, investors value functions are also exponential linear quadratic, but have a much more complicated functional form:

Proposition 2 *Small investors value functions for entering period $t < T$ with asset holdings Q_s and q_s when the economy's vector of risky asset holdings at time t is Q is given by:*

$$\begin{aligned} V_s(q_s, Q_s; Q, t) &= -K_s(t)\ F(Q,t)\ e^{-A_s(t)q_s(t) - A_s(t)Q_s'(\bar{D} + P(Q,t)) + .5A_s(t)^2 Q_s'\Omega(t)Q_s}, \\ \text{where } F(Q,t) &= e^{-Q'\bar{v}_s(t) - \frac{1}{2}Q'\theta_s(t)Q} \\ P(Q,t) &= \tfrac{1}{r}(\alpha(t) - \Gamma(t)Q). \end{aligned} \tag{22}$$

where $P(Q,t)$ is the equilibrium price for risky assets that is realized when investors enter period t when the state-vector of risky asset holdings is given by Q.

For large investors m, $m = 2, \ldots M$, the value function for entering period $t < T$ when the state variable is Q and m's holdings of riskfree assets are q_m is given by:

$$V_m(q_m, Q, t) = -K_m(t)e^{-A_m(t)q_m(t) - A_m(t)Q_m'\bar{v}_m(t) + .5A_m(t)^2 Q'\theta_m(t)Q}. \tag{23}$$

Proof: See section B of the appendix.

Investors value functions are a high dimensional function of the state variables $Q(t)$ and t, which are $NM + 1$ dimensional. Nevertheless, because of the CARA-normal framework, the value functions are analytically tractable, and the unknown parameters of the value functions and price function are the solutions of Riccati difference equations that can be solved numerically. Given these solutions, it is straightforward to solve for the equilibrium path of trades and asset prices starting from any initial asset allocation. The properties of asset prices and returns are explored in the next section.

4 Properties of the General Model

To analyze the effects that large, non-price-taking investors have on financial market equilibrium, it is useful to first study market equilibrium for a competitive benchmark economy which contains the same market participants and assets, but where all participants are price-takers.[23]

4.1 Competitive Benchmark

The properties of the competitive economy are presented in the following proposition:

Proposition 3 *When all participants in model described in section 3 take asset prices as given, then assets expected excess returns over the risk free rate satisfy the Capital Asset Pricing Model and are given by the equation:*

$$E_t\{P(t+1) + D - rP(t)\} = \lambda_x \Omega X \tag{24}$$

where λ_x, the price of market risk, is given by:

$$\lambda_x = \frac{1 - (1/r)}{\sum_{m=1}^{M} 1/A_m}. \tag{25}$$

Additionally, risky asset prices are constant for all times t, and equal to:

$$P(t) = \frac{\bar{D}}{r-1} - \frac{\Omega X}{r \sum_{m=1}^{M} (1/A_m)}. \tag{26}$$

[23] One interpretation of the investors in the competitive benchmark economy is that investor $m = 1, \ldots M$ is a representative investor for a continuum of price taking investors whose integrated risk tolerance is the same as the risk tolerance of investor m.

The vector of investors optimal risky asset holdings is also constant through time and denoted by Q^W. The risky asset holdings of investor m are denoted by Q^W_m and given by:

$$Q^W_m = \frac{(1/A_m)X}{\sum_{m=1}^{M}(1/A_m)}. \tag{27}$$

Proof: See section B.9 of the appendix.

The results in proposition 3 are well known and are a special case of the CAPM for CARA utility that is presented in Stapleton and Subrahmanyam (1978). Note that with CARA utility, if expected dividends are very low, or risk aversion is very high, then asset prices can be zero or negative. To avoid dividing by a zero or negative price, assets excess returns take the form that is given in the proposition. Assuming sufficient regularity for prices to be positive, let $R_i(t+1) - r$ and $R_m(t+1) - r$ denote the excess return over the risk-free rate for asset i and and for the market portfolio. Then, excess returns also have the more usual CAPM representation:

Corollary 1 *Assuming sufficient regularity to ensure that all risky-asset prices are positive at time t, then*

$$\mathrm{E}_t[R_i(t+1) - r] = \beta_i \,\mathrm{E}_t[R_m(t+1) - r], \tag{28}$$

where $\beta_i = \frac{\mathrm{Cov}_t[R_i(t+1), R_m(t+1)]}{\mathrm{Var}_t[R_m(t+1)]}$.

Proof*: See appendix B.9.*

Note that in the current setting, when market are competitive, they are effectively complete and hence equilibrium asset holdings are pareto optimal, and risk sharing is efficient. When risk sharing is efficient the percentage of risky assets that each investor owns is equal to his risk bearing capacity; and the equilibrium asset prices and expected returns only depend on the risk bearing capacity of the economy, and not on the market structure as measured by the distribution of risk bearing capacity across investors.

4.2 Imperfect Competition and Asset Pricing

Given the features of the competitive benchmark model, I now turn to the properties of the imperfect competition model. Because the utility functions and assets in both economies are the same, the imperfect competition model inherits many of the the properties of the perfect competition benchmark model, as detailed in the next proposition:

Proposition 4 *When asset markets are imperfectly competitive as specified in section 3 of the text, then if market participants initial asset holdings are Q^W, then investors will hold*

Q^W *forever, and asset prices and expected returns will be the same as when there is perfect competition.*

Proof: When investors risky asset holdings are Q^W, then investors asset holdings are pareto optimal in all time periods. Hence there is no basis for trade among the investors and their asset holdings will remain at Q^W. Because Q^W is the vector of asset holdings from a competitive equilibrium, the resulting prices and expected returns which support Q^W are the same as in the competitive equilibrium. $\square$

Proposition 4 shows that even when markets are illiquid, when risk sharing is optimal, assets can be priced as if illiquidity does not matter, and in the case of the model, CAPM pricing can result. The main consequences of the model for asset pricing occur when investors initial asset holdings are not pareto optimal. In such circumstances, investors trade until they reach a pareto optimal allocation of risky assets.[24] The process for reaching efficient allocations depends on market liquidity. Large investors can reduce their liquidity costs by trading at a slower rate and breaking up their trades through time. The speed with which large investors trade toward a pareto optimal allocation is also influenced by their risk aversions. In a one period setting with perfect liquidity, investors risk aversions' measure the certainty equivalent wealth that they are willing to pay to eliminate risk from their portfolio. In a dynamic setting, the certainty equivalents translate into the liquidity costs that investors are willing to pay to eliminate or take on additional risks. Investors that have high coefficients of absolute risk aversion are expected to be willing to pay a higher liquidity cost in order to quickly eliminate undesirable risk from their portfolio, while investors that are more risk-tolerant are expected to trade more slowly because they are not willing to pay a high price to eliminate risk. This reasoning suggests that when markets are illiquid, how risky assets are allocated among investors with different risk aversions will influence asset prices because it will influence equilibrium risk-sharing. This intuition is confirmed below:

Proposition 5 *When investors asset holdings are not Pareto Optimal, equilibrium expected asset returns satisfy a linear factor model in which one factor is the market portfolio, and the other factors correspond to the deviation of large investors asset holdings from pareto optimal asset holdings.*

$$P(t+1) + \bar{D} - rP(t) = \lambda_X \Omega X + \sum_{j=2}^{M} \lambda(m,t)\Omega(Q_m(t) - Q_m^W) \tag{29}$$

[24] I do not have a formal proof of convergence to pareto optimal asset holdings when investors vary in their risk aversion, but it occurs in simulations. Moreover, Urosevic (2002a) has a proof for convergence when all large investors have the same risk aversion. To be the best of my knowledge he does not have a proof for the more general case.

Proof: See section B.4 of the appendix.

In analogy with corollary 1, define $P_F(t)$ as the stacked price vector of portfolios that consists of the market portfolio, and of the prices of portfolios that consist of the deviation of each large investors risky asset holdings from those associated with a pareto optimum at time t. Assuming the prices of these portfolios are nonzero, define $R_F(t+1)$ as the vector of return on these portfolios, $R_i(t+1)$ as the return on the i'th risky asset, and β_{iF} as the true vector of beta's in an OLS regression of $R_i(t+1)$ on $R_F(t+1)$. Under these circumstances, assets excess returns have an equivalent more conventional representation in which their excess expected returns depend on their loadings on the returns of the factor mimicking portfolios:

Corollary 2 *Assuming sufficient regularity so that $P_F(t)$ is not zero, then under the condition of proposition 5,*

$$\mathrm{E}_t[R_i(t+1) - r] = \beta'_{iF} E_t[R_F(t+1) - \iota r], \tag{30}$$

where ι is an $M \times 1$ vector of ones.
Proof: *See the appendix.*

Proposition 5 and corollary 2 establish that when markets are illiquid and asset holdings are not pareto optimal, then assets 1-period expected returns have a factor-like structure in which the market portfolio and the deviation of each large investors asset holdings from his pareto optimal asset holdings is a priced factor. The effect of the additional factors on expected returns is transient, and the persistence of their effects depends on the speed with which each investor trades back toward pareto-optimal holdings. A measure of persistence over a single time period is provided by the prices of risk, $\lambda_m(t)$ $(m = 2, \ldots M)$, which together with Ω, scale the effect that a deviation from imperfect risk sharing by large investor m at time t has on expected excess returns between time periods t and $t+1$. As hinted at earlier, the risk prices vary by large investors risk-aversion, and are larger for more risk-tolerant investors since those investors have a stronger preference for avoiding liquidity costs by trading slowly. A measure of persistence over longer periods is the effect that imperfect risk sharing at time t has on one-period expected excess returns at arbitrary future time periods $t + \tau$. It turns out that this relationship has a factor structure analogous to that in proposition 5:

Corollary 3 *When asset holdings at time t are not pareto-optimal, then asset returns at time $t + \tau$ follow a factor model in which one factor is the excess returns on the market portfolio, and the other factors correspond to the deviations of large investors asset holdings from those associated with perfect risk sharing.*

$$P(t+\tau+1)+\bar{D}-rP(t+\tau)=\lambda_X\Omega X+\sum_{m=2}^{M}\lambda_m(t,\tau)\Omega(Q_m(t)-Q_m^W), \tag{31}$$

where $\lambda_m(t,\tau)$ are τ period ahead risk factor prices. In analogy with corollary 2, there is a corresponding relationship between assets' excess expected returns at time $t+\tau$ and their loadings on factor mimicking portfolios that are constructed based on imperfect risk-sharing at time t.

In equation (31), if a factor price remains large for high τ, then the time t deviation from efficient risk sharing for the corresponding large investor, has a persistent effect on expected excess returns at a τ period horizon. To study persistence, I numerically solved for risk prices and for τ-period ahead risk prices in an example that has 2000 trading periods that each last 1 day, when there are 6 investors who strongly differ in their risk aversion. The large investors risk aversion increases with investor number so that investor 2 is the least risk averse and investor 6 is the most risk averse.[25] In the example, the market price of risk is positive as expected, and the other prices of risk are all negative (Figure 3). Intuition for the latter result comes from noting that if a large investor initially holds more risky asset than is pareto optimal, then he will sell the assets back slowly to avoid liquidity costs, and hence the marginal investor (the competitive fringe) will hold less of the risky assets than is efficient in the transition. Consequently, the marginal investors will require a lower rate of return on its risky asset holdings during the transition, which is manifested in the negative price of factor risk. Consistent with my earlier reasoning on risk aversion, the market prices of risk are the more negative for large investors with the greater risk tolerances because they prefer to sell more slowly to avoid liquidity costs.[26] This reasoning suggests that the persistence of shocks to investors asset holdings depends on the risk tolerance of the investors who are shocked. The τ period ahead risk prices as of period $1,000$ confirm this reasoning within the example: shocks to the asset holdings of investors 3-6 have an effect on one-period expected excess returns that is significant for a period of at most 5 days (Figure 4, Panel A). By contrast, shocks to investor 2, the most risk tolerant large investor, have an effect on equilibrium excess returns that persists for more than 500 days.

The analysis in this subsection sheds light on the breakdown of asset pricing models following large shocks. It shows that if markets are illiquid, then after a large shock to investors

[25] Additional details on this example are provided in section C of the appendix.

[26] One puzzling aspect of the risk prices is that they increase through time and eventually converge as the number of remaining trading periods becomes small. The convergence occurs because the risk prices measure how changes in current risky asset allocations affect the competitive fringe's future holdings. When there are very few, or in the limit no trading periods remaining, then the risk prices become the same because there is no time over which the deviations from pareto optimal asset holdings can be reversed. Similarly, the risk prices grow because investors require more risk compensation when deviations from pareto optimal holdings cannot be reversed.

asset holdings, additional transient pricing factors will appear that proxy for imperfect risk-sharing. Because the factors are transient, the number of additional risk factors that are needed to explain returns will be maximal at the time of the initial shocks, and will then diminish through time. Finally, the analysis shows that the persistence of shocks depends on the "size" of the large investors that are shocked. In particular, the model predicts that if an institutional investor with a very large portfolio (which corresponds to high risk tolerance) is shocked, then mispricing will be more long-lived because it will take a a much longer time to adjust to the shock.

4.3 Contagion

The previous section focused on how transient pricing factors can appear following large shocks to investors asset holdings. The same shocks have an interpretation within the contagion literature. The contagion literature attempts to understand how shocks that originate within one asset class or market then spread to others. It is important to emphasize that this subsection is labelled contagion only because it pertains to shock propogation. The spread of shocks in the model is fully rational, and has none of the negative connotations that are associated with the word contagion. To model how shocks spread across assets or markets, suppose asset holdings are pareto optimal and then a shock occurs for asset j which causes the reallocation of investors holdings of that asset only. Any such reallocation can be expressed as a linear combination of "basis shocks" that cause each large investors asset holdings to differ from his pareto optimal asset holdings. Because the trajectory of asset holdings and asset return dynamics are also linear in the basis shocks (see the appendix), the propagation of shocks to the holdings of asset j can be studied through basis shocks alone. Also because of linearity, it is sufficient to study the first derivative of assets' excess return response to the basis shocks. Other assets price responses to basis shocks for asset j are summarized below:

Proposition 6 *When investors asset holdings are initially pareto optimal, a perturbation which drives large investor $m's$ holdings of asset j away from a pareto optimum in period t while other large investors holdings are held fixed alters time $t+1$ expected excess returns by*

$$\frac{\partial\, \mathrm{E}_t(P(t+1) + D - rP(t))}{\partial(Q_m(j,t) - Q_m(j)^W)} = \lambda_{m,t}\Omega[.,j], \tag{32}$$

and alters time $t+\tau$ asset returns by

$$\frac{\partial\, \mathrm{E}_t\, P(t+\tau+1) + D - rP(t+\tau)}{\partial(Q_m(j,t) - Q_m(j)^W)} = \lambda_m(t,\tau)\Omega[.,j], \tag{33}$$

where $Q_m(j,t)$ is the j'th element of $Q_m(t)$ and $\Omega[.,j]$ is the j'th column of Ω.

Proof: Straightforward by differentiating equations (29) and (31).

Because the effects of the shocks are proportional to the j'th column of Ω, the model suggests that the price effects of the shocks have a "beta" representation. This is confirmed below:

Corollary 4 *Let $\beta[i,j] = \Omega[i,j]/\Omega[j,j]$, and let $\Delta Z[i,t+\tau]$ and $\Delta Z[j,t+\tau]$ denote the change in expected excess return effects for assets i and j that result from a shock or shocks which disrupt pareto optimal risk sharing for asset j. Then, for all $0 < \tau < T - \tau$,*

$$\Delta Z[i,t+\tau] = \beta[i,j]\Delta Z[j,t+\tau] \tag{34}$$

Proof: Straightforward from proposition 6.

Proposition 6 shows that the effect of basis shocks to investor m in market j has a proportional effect on all other markets in future time periods. Therefore when a shock to an investor has a long-lived effect on asset returns in market j it also has a long-lived effect for all other assets. Because the persistence of shocks varies by investor, the model shows that one reason that contagious shocks vary in the persistence of their effects on prices and returns is differences in the risk preferences of the investors who were initially affected by the shock.

The model makes strong predictions about the price and trading effects of shocks to investors holdings of asset j. Corollary 4 shows that such shocks only affect the returns of assets with correlated dividends. The trading volume implications are even stronger, and in fact unrealistic. In the appendix I show that the equilibrium trades for each asset only depend on whether holdings for that asset are pareto optimal, irrespective of the holdings of other assets. This implies that a deviation from pareto optimal asset holdings for asset j has no effect on trading volume for asset i.

It is important to note that the contagion analysis is limited because it only focuses on shocks that take the form of reshuffling investors' asset holdings. If instead a cashflow shock occurs that requires an investor to optimally sell assets in order to raise cash, then price and trade effects will occur for many assets irrespective of dividend correlations. Cashflow shocks are too complicated to model in the current setting. The next section focuses on simple shocks that require a distressed investor to sell his holdings of a single risky asset. The contagious effects of such shocks are straightforward from corollary 4, therefore the next section only focuses on the single asset case.

5 Distressed Asset Sales

This section studies the dynamics of asset prices and trades when there are distressed sales. The analysis extends the example from section 4.2 of the text and section C of the appendix. The modified example has a single risky asset, as well as a distressed large seller, represented by investor 7. Investors trades are modelled over 2200 periods, and the time scale is renormalized so that time starts from period -200. At time -200, investors initial risky asset holdings are Pareto efficient, which implies from proposition 4, that asset returns will satisfy the CAPM. At time 0 there is an informational shock: all investors learn that investor 7 will be forced to sell out his risky-asset position and then exit the market. It is assumed the sales will occur at a constant rate during time periods 390 to 400, and that investor 7 is locked in and cannot trade before period 390. Both assumptions will be relaxed eventually. The constant rate assumption is made initially to provide comparability with BP.[27] The lock-in assumption is made for expositional purposes: the trading patterns that emerge due to the distressed sales are more easily illustrated when stretched out over time.[28] In addition, the lock-in assumption also provides ideal circumstances for other investors to front-run by selling ahead of the distressed investor. The interesting question is how much front-running the model can generate under the ideal circumstances.

The distressed asset sales in the example can alternatively be interpreted as the announcement of a seasoned equity issuance followed later by issuance at the time of the distressed sales. The resulting price dynamics below are qualitatively similar to Newman and Rierson's (2003) study of how bond spreads respond to additional issuance in the European Telecommunications market.[29]

To isolate the effects that large investors have on asset price dynamics when there are distressed sales, it is useful to first study a competitive benchmark case with the same investors but in which all investors are price-takers.

5.1 Competitive Benchmark

In the competitive benchmark economy, before time period 0 and after time period 400, investors believe they will hold their risky asset allocations forever. Therefore, equilibrium

[27]BP assume that there is a maximal rate at which the distressed seller can sell, and show it is optimal for him to sell at that rate.

[28]The price and trade dynamics are qualitatively similar when the lock-in time span is shortened.

[29]Newman and Rierson focus on new issuance of telecom bonds that are close substitutes to outstanding telecom bonds. The price dynamics reported below are similar to those for the outstanding telecom bonds when a new issue occurs. This is because a new issue is a sufficiently close substitute to existing bonds, that it is almost like a seasoned issue of the existing bonds from the standpoint of an investor.

asset prices will be set as in equation (26), but with 7 investors before time period 0, and with 6 investors after time period 400:

$$P(t) = \frac{\bar{D}}{r-1} - \frac{\Omega X}{r \sum_{m=1}^{7} 1/A_m}, \qquad t \leq 0; \tag{35}$$

and after investor 7 sells all of his assets and exits the market, prices are set so that:

$$P(t) = \frac{\bar{D}}{r-1} - \frac{\Omega X}{r \sum_{m=1}^{6} 1/A_m}, \qquad t \geq 400 < T, \tag{36}$$

where X is the outstanding supply of risky assets.

Between periods 0 and 399, investors 1-6 anticipate and plan to absorb the distressed sales, and asset returns must provide them sufficient sufficient risk compensation per period for their dynamic asset holdings. This requires that prices between times 0 and 399 satisfy the difference equation:

$$P(t+1) + \bar{D} - rP(t) = \frac{[1-(1/r)]\Omega X[1:6,t]}{\sum_{m=1}^{6}(1/A_m)}, \qquad 0 \leq t \leq 399, \tag{37}$$

subject to the boundary condition (36), where $X[1:6,t]$ is the net risky asset holdings that investors 1-6 hold from time t to time $t+1$.

Solving for the competitive price path in the example, shows that news of the future distressed sales causes prices to jump down at time 0, and then to decline at a barely discernible rate until they stabilize in period 400 (Figure 5). Equilibrium excess returns correspondingly rise along the transition path; and asset returns satisfy a CAPM-like pricing relationship period by period, where the market portfolio is the float of outstanding assets $X[1:6,t]$ at each time t. Because markets are perfectly liquid in the competitive equilibrium, the future distressed sales do not create a basis for trade among investors 1-6, therefore trading only occurs in periods 390-400 when the distressed investor is selling.

5.2 Imperfect Competition

Imperfectly competitive markets generate different patterns of trades and prices. News of the distressed sales cause an immediate price drop that initially overshoots the competitive price path. Prices then drift down until the distressed sales have been completed, and then slowly recover towards the competitive price path (Figure 5). Price overshooting occurs because large agents pare back their demands in order to receive a better price per unit. This is most evident for investor 2. In the competitive equilibrium, where risk-sharing is efficient, investor 2 buys more risky asset than other large investors because he is the most risk-tolerant. When markets are imperfectly competitive, investor 2 instead buys little at the time of the distressed sales. He instead sits back from purchasing shares and shifts the burden of the share purchases onto more risk-averse investors. These investors in turn receive a high risk premium because they buy the assets at a low price. Investor 2 then buys slowly through time from the other investors at low prices after the distressed sales have occurred (Figure 6).

If investors asset holdings are initially pareto-optimal, as they are in the example, then all of the large and small investors benefit as a result of the distressed investor — even though the distressed investor's action reduces asset prices. To show that the other investors benefit — note that when investors have pareto optimal asset holdings in the model, they don't trade away from them — and they just consume the income produced by the assets. When there is a distressed investor, each investor retains the option of "refusing to trade" and just consuming the income from their initial asset holdings. The fact that the large and small investors trade away from this possibility, means that relative to their initial pareto optimal asset holdings, the distressed investor makes all other investors better off.

In terms of asset pricing effects, the imperfect risk-sharing that occurs with imperfect competition causes the original CAPM pricing relationship to temporarily breakdown after time 0. Between times 1 and 399, it is not clear whether returns instead follow a multi-factor model (it may be multi-factor with non-zero alpha), but after time 400 returns will temporarily follow a multi-factor model as in proposition 5, and then these transient factors will eventually disappear. In addition to a breakdown in pricing models, the distressed sales generate abnormal trading volume relative to the competitive benchmark. When markets are perfectly competitive, and hence perfectly liquid, the assets are sold to the investors that plan to hold them, causing the number of periods with trading volume to equal the number of periods with distressed sales. By contrast, when markets are imperfectly competitive, the trading volume at the time of the distressed sales is the same as in the competitive benchmark, but because risk-sharing is inefficient, thousands of periods of retrade are generated long after the distressed sales are over (Figure 6). Thus, imperfect competition generates significant trading volume and persistence in trading volume.

Part of the trading activity that is generated by the distressed sales occurs beforehand, and can be interpreted as front-running since investors 1, 2, and 4-6 sell ahead of the distressed sales once they learn about them at time 0 (Figure 6). The front-running in the example is somewhat non-standard. Traditional front-running involves a trader using *private knowledge* of pending sales to sell ahead of the orderflow to uninformed buyers at high prices. By doing this, the front-runners take advantage of uninformed buyers, and depress the prices received by the seller. Front-running differs here because informed sellers cannot exploit uninformed buyers since all investors in the model are symmetrically informed. Consequently, there is much less basis for trade in the present setting, and the amount of front-running trades are small relative to the amount of distressed sales.

Because the front-running here is non-traditional, it is ex ante unclear whether it is motivated by sharing the risks of the future distressed sales (since the sharing of risks must involve one party selling to another) or if it also reduces the seller's revenue. To examine this question, I solved for the price received by the distressed seller as a function of the warning time that other investors had before the distressed sales took place under the assumption that more warning time should lead to lower revenues.[30] In the example, I found that earlier knowledge of the distressed sales leads to a lower price for the distressed seller. If the advance knowledge is interpreted as preannouncement as in the Sunshine Trading literature (Admati and Pfleiderer, 1991), then the analysis shows that when trading is strategic, all investors are rational, and information is symmetric, then the preannouncement of trades in advance of when they occur can potentially harm the party that announces the trades even if those trades are known to be uninformed. That said, the losses due to preannouncement and front-running are relatively small: the advance warning of 400 time periods costs the distressed seller only 4 cents per share, with less cost for shorter warning times. The more important cost is due to illiquidity, and can be measured as the difference in the price received by the distressed seller and the price he would have received if markets were competitive. Examination of figure 5 shows that this cost is about 85 cents per share.

5.3 Endowment Shocks

One of the unrealistic aspects of the distressed sales analysis is that the behavior of the distressed seller is mechanical; the distressed seller does not choose an optimal trading strategy given the liquidity that other investors in the market make available. A simple method to examine distressed sales when distressed investors follow an optimal trading strategy is to consider how one of the investors whose behavior is formally modelled respond to an endowment shock that increases their holdings of risky assets. Following such a shock, the investor will follow an optimal trading strategy in transferring part of their position to the other investors in the model. For purposes of comparison, the size of the endowment shock is equal to the quantity of distressed sales used to generate figures 5 and 6; and the shocks are applied to large investor 2, the most risk tolerant large investor, and investor 6, the least risk tolerant large investor. A comparison of price and trade dynamics reveals very significant differences based on the identity of the investor who receives the shock. When

[30] For simplicity I assume that there is only one distressed trade at time 400.

investor 6 is shocked, because he is very risk averse, he very rapidly sells off the risky assets to other investors, eliminating most of his holdings within 4 trading periods (Figure 7, panel F). When instead investor 2 is shocked, because he is less risk averse than investor 6, he instead sells slowly through time to minimize the price impact of his trades (Figure 9, panel B). Because investor 6 sells rapidly, prices overshoot their competitive equilibrium values (not shown). By contrast, because investor 2 only transfers risk to other investors slowly through time, prices *undershoot* the competitive price path (Figure 8).

The endowment shocks in the example do not generate front-running, but predatory trading occurs some of the time: there is none when investor 6 is shocked, and there is a short-lived amount after investor 2 is shocked. This takes the form that some investors initially sell small amounts when investor 2 is selling, and then later buy while investor 2 is still selling. I do not yet have strong intuition for why predatory trading emerges when some investors are shocked, but not others.[31]

5.4 Optimal Liquidations

An unrealistic aspect of modeling distressed sales with endowment shocks is that there is not necessarily an urgency to liquidate on the part of shocked investors. This is evident in the case of investor 2, who chooses to sell over more than a thousand time periods. To reinstate a sense of urgency, I revert here to the modeling in section 5.2 in which all investors learn at time 0 that a distressed investor (investor 7) must liquidate his risky asset position by date 400, but now require that he follows an optimal strategy when doing so, and then exits the risky-asset market forever. To maintain comparability with the earlier analysis, investor 7 has CARA utility of consumption, is infinitely lived, and has the same initial risky-asset holdings as in in section 5.2.

The distressed investors' problem of maximizing utility subject to liquidating his position by a specified date is the subject of the optimal liquidation literature [Bertsimas and Lo (1998), Almgren and Chriss (2000), Subramanian and Jarrow (2001), Subramanian (2000), and others]. To the best of my knowledge, all papers in that literature model liquidations in a partial equilibrium setting in which the price impact function for the liquidating investors trades is exogenously specified. The unique aspect of this analysis of optimal liquidations

[31] As partial intuition for the results when there is an endowment shock to investor 2, note that the investors who initially sell and then later buy do so because returns are initially low, and then later rise, which seems rational. Of course, not all investors can initially sell—someone must take the other side. Investor 3, who initially and subsequently buys, is the most risk tolerant of the remaining large investors and has the most utility to gain from eventually acquiring more risky assets. Acquiring more risky assets when returns are initially dropping, as he does, can be sensible in equilibrium because if he acquires less initially, because of future illiquidity it may cost him more to acquire them later; additionally because there is no trading at period $T-1$, if he puts off buying early he may not be able to acquire the additional assets at all.

is that the price impact function is endogenously determined by the behavior of all other investors in the market, and their knowledge that optimal liquidations will take place. I do not yet have theorems about how the model behaves when there are optimal distressed sales; all results are based on simulations.

When the distressed investor follows an optimal liquidation policy, the resulting path of prices and trades is qualitatively different than when his sales are concentrated near time 400.[32] The main difference is that prices overshoot the competitive price-path by less when distressed sales are optimal than when they are concentrated (Figure 10). Furthermore, the minimal front-running that was present when distressed trades were concentrated vanishes, and unlike section 5.3, predatory trading does not emerge; instead, the distressed seller begins selling immediately upon news that he must liquidate and all of the other investors purchase assets immediately and along nearly the entire price path (Figure 12).[33]

The trades of the liquidating investor are of independent interest. When there is a distressed investor and markets are illiquid, intuition suggests that the distressed investor should break up his sales to minimize price impact, but doesn't specify how to do so; for example should liquidations occur through a large number of small trades, or a small number of large trades? In the example, the optimal liquidation strategy involves selling large amounts of risky assets at times 0 and 400 and dribbling out small additional amounts of risky asset during the periods in between (Figure 11). Additional simulations reveal that the optimal liquidation strategy depends on market structure as measured by how the risk bearing capacity of the other investors in the economy is distributed among the remaining investors. Intuitively, when the economy's risk bearing capacity is evenly dispersed among many large investors the market will be more competitive and liquid than when risk bearing capacity is concentrated among a smaller number of investors. This high liquidity should give the distressed investor an incentive to wait before selling. Consistent with this intuition, at one extreme when markets are very competitive, the liquidating investor holds all of his assets until time 400 and then liquidates all at once (not shown). In the other extreme, when markets are highly uncompetitive, the liquidating investor sells a lot at time 0, and the rest at time 400, but sells little or nothing in between (not shown). Much more analysis can be done on optimal liquidations in this framework; I hope to purse this topic in future work.

[32] For this subsection, figures 12 and 10 actually contrast optimal distressed sales against an alternative in which distressed sales are concentrated at time 400, and not against the alternative in section 5.2 where the distressed sales are evenly spread from periods 390-400. The distinction among these alternatives is essentially immaterial because the paths of prices and trades using both alternatives to optimal liquidations are qualitatively very similar (see figures 5 and 6).

[33] For reasons that are not yet clear, the distressed seller purchases a small quantity of risky asset just before liquidating his position. I am investigating this further.

5.5 The Distressed Investor Literature

To close this section, I would like to compare the results on front-running and predatory trading with the three main papers in the distressed investor literature (BP, CLV, and AMR). My main critique of these papers is that all three rely on a long-term investor with exogenously given demand curve. Market dynamics could be different if the long-term investor optimizes. Additionally, because classes of investors are risk-neutral in these models, other assumptions are required to solve these models. In AMR, strategic traders have a short horizon for holding positions. In BP, there is an exogenous transactions cost technology, constraints on the size of investors positions, and constraints on their maximal trading speed. In CLV, trading has temporary and permanent effects on asset price dynamics, and these effects are imposed exogenously. By contrast, all investors here are risk averse. This allows me to examine the patterns of risk sharing when there is a distressed investor. In addition, outside of assuming there is a final period of trade, no other forms of exogenous transaction costs or limits on trading are imposed. As a result, I can study how relaxing these restrictions affects the results in this literature.

Two findings emerge from the analysis. The first concerns how shocks are absorbed, and risks are shared when there are distressed sales. In the predatory trading and distressed investor literature — these shocks are to a great extent initially absorbed by the exogenously specified long run investor — and then other strategic investors purchase assets back from this long run investor through time. Here, all investors focus on the long run and short-run, but the equilibrium trades and prices are nevertheless similar. In particular, the "long run investors" in the current setting are relatively risk averse large and small investors. These investors inefficiently absorb the distressed sales first at a very low price that is partially compensation for the inefficient risk sharing. The "predators" more closely resemble the large investors with high levels of risk tolerance. In equilibrium they have price impact, and they exploit this market power, by purchasing at a very slow rate during and after the distressed sales have occurred. This inefficient risk sharing guarantees them and the other large investors a high profit.

Our second finding is that the results in the other papers, in particular BP, are qualitatively robust to a different setting. In this paper front-running and predatory trading sometimes emerge without the strong assumptions in BP, and in a setting in which all of the investors optimize and are aware of the distressed sales. The main qualification is that in the setting considered here front-running and predatory trading have relatively small effects on trades and prices when other participants are aware of the distressed sales and optimize to account for the them. I suspect the reason that front-running and predatory trading have small effects here is because there is little basis for strategic investors to engage in these activities when all investors know about the distressed sales.

Based on the above, my interpretation of BP and CLV is that their model describes equilibrium when knowledge of the distressed sales is not widely known, and in particular is not known by the buyers who take the other side of the strategic investors' trades. My results also suggest that the primary reason that distressed sellers do not want knowledge of their distress to be widely known is not because they fear front-running or predatory trading, but rather that they fear losing the opportunity to sell at a high price to buyers who are unaware of their distress.

When all investors are aware of the distressed seller, the losses suffered by the distressed seller are related to the severity of the imperfect competition in asset markets, which in turn depends on market structure, as measured by the distribution of the market's risk bearing capacity across the investors in the market, and on market liquidity. These subjects are addressed in the next section.

6 Liquidity and Market Structure

Recall that the economy's market structure is defined (in section 3.1) as the distribution of risk bearing capacity across investors, and that when when markets are imperfectly competitive, section 4.2 shows that market structure can influence the pattern of equilibrium asset returns. The purpose of this section is to study liquidity in the model and its relationship to market structure.

There are many possible ways to define and measure liquidity. I consider three simple measures. The first measure is the price discount that a distressed seller pays by selling into an imperfectly competitive (illiquid) market instead of into a perfectly competitive liquid market. The second is the liquidity that large investors receive from the competitive fringe of small investors. The second measure is the slope of the price schedule that large investors face when choosing their risky asset holdings (see equation (20)). Liquidity is lower when the slope measure is larger. The slope measure is in some sense comparable to the λ coefficient of Kyle (1985) and to to the bid-ask spread. A deficiency of slope as a liquidity measure is that it is only based on the fringe's willingness to absorb orderflow from a large investor: it holds the behavior of other large investors as fixed. The third measure remedies this deficiency by measuring the immediate price impact that occurs when one large investor has to (for unmodeled exogenous reasons) immediately sell 1 share of his asset holdings to the competitive fringe and other large investors. This price impact measure is most sensible when investors have no other basis for trade that might confound the price impact computations.

Therefore, I compute the third measure when investors asset holdings are pareto optimal.[34]

The distressed seller's liquidity was studied for the example in section 5; for simplicity all asset sales occurred at time 400. To study the role of market structure, the economy's annualized total risk bearing capacity was normalized to 1, and 10 percent was allocated to the competitive fringe and the other 90 percent to the large investors.[35] The large investors risk bearing capacity was further parameterized so that it was geometrically declining in investor number, i.e. $(1/A_{m+1}) = \rho(1/A_m)$ where $\rho \in (0, 1)$. By altering ρ, this parameterization allows as limiting cases for the large investors risk bearing capacity to be concentrated with one large investor, or to be shared equally by all large investors. Additionally, in this set-up, the fraction of risk bearing capacity held by the most risk tolerant large investor is a sufficient statistic for the entire market structure.

The results show that differences in market structure have a very significant effect on the liquidity cost to the distressed investor. When 60% of the large investors' risk bearing capacity is held by investor 2, as it is in figure 5, then the liquidity cost is about 90 cents per share. If instead 90% is held by investor 2, then the liquidity cost moves dramatically higher to $7.75 per share. This figure is 150% greater than the total price impact of the distressed trades in a competitive environment. On the other hand, if investor 2 holds 50% of the large investors risk bearing capacity, the liquidity cost is reduced to 23 cents, and grows smaller yet as the distribution of risk bearing capacity becomes flatter. The risk bearing capacity of the competitive fringe also matters. If the fringe's capacity grows and that of the large investors shrinks (while the economy's total risk tolerance is held constant) then liquidity increases for the distressed investor. The risk-free rate also matters: simulations show the price discount for illiquidity shrinks when real interest rates rise to 4 percent from 2 percent, and the discount for illiquidity grows when interest rates decline from 2 percent to 1 percent.[36] Interest rates are important because large investors with relatively high risk tolerances excercise their market power over a distressed seller by paring back their purchases from the distressed seller in order to acquire the assets from other investors in the future at depressed prices. At higher levels of interest rates, the revenues from following this strategy are discounted at a greater rate, which reduces the benefit from delaying purchases; this erodes large investors market power, causing the price discount to shrink.

The second and third liquidity measures are investor-specific: they measure the amount of liquidity that is available to each large investor. An interesting feature of the general

[34] See section B.8 of the appendix for details.

[35] The normalization of aggregate risk tolerances to 1 scales equilibrium risk premia, which are homogeneous of degree -1 in investors risk tolerances, but has no effect on equilibrium trades which are homogeneous of degree 0 in investors risk tolerances.

[36] These results are based on fixing the real rate at various levels and assuming it will stay there forever.

model, which is consistent with the basic model, is that liquidity varies by large investor. For both liquidity measures, in almost all time periods of the model, liquidity is monotone decreasing in investors risk tolerance: the more risk tolerant a large investor the less liquidity that he receives (Figure 13 for measure 2, not shown for measure 3). Investor 2, the most risk tolerant of the large investors receives far less liquidity than the other investors. Intuition for why the slope of the fringe's demand curve varies by large investor comes from noting that the slope at a point in time measures the change in the competitive fringe's valuation of the risky asset when some of that asset is transferred from the fringe to a large investor. In a dynamic setting, the change in the fringe's marginal valuation depends on the future trading strategy of the large investor who buys the asset. If the large investor is very risk averse, one might expect the large investor to quickly sell the asset back to the fringe again. In this case a point in time purchase by a very risk averse investor has only a small effect on the fringe's marginal valuation. Conversely, if the investor is very risk tolerant, then he will hold the asset for a longer time, which means a sale to a very risk tolerant large investor has a greater effect on the fringe's marginal valuation. The interesting aspect of these results is that because more risk tolerant investors are less willing to pay liquidity costs, in the equilibrium of the model, they face a price function with a steeper slope and hence actually receive less liquidity.

The most risk-tolerant investors receive less liquidity for an additional reason that helps to explain the results for the third liquidity-measure. This measure is the price impact from transferring risk from one investor to the other investors in the model. When the most risk tolerant large investor transfers risk to other less risk tolerant large investors, one should expect a bigger price move because the less risk tolerant investors should require a bigger premium to temporarily take on the additional risk.

It is important to emphasize that the patterns of liquidity that are generated in the model are extremely complicated, and the intuition, while it points towards determinants of liquidity, is not consistent with all of the results. For example, the explanation for the results on liquidity for individual investors (liquidity measures 2 and 3) is incomplete because the monotone relationship between risk tolerances and large investors' liquidity breaks down after period 1600 (not shown). Although I suspect the breakdown is related to approaching the last period of trade in the model, the reason for the breakdown is unknown.

The relationship between liquidity and uncertainty is also extremely complicated. Intermediate steps in the proof of proposition 9 show that the slope measure of liquidity for each large investor has form:

$$\beta_m(t) = f_m(A_1, \ldots, A_m, t) \times \Omega,$$

where $f_m(.)$ is a scalar function of market structure and time. The expression for slope shows that liquidity measure 2 depends on market structure and uncertainty about dividends, and that uncertainty about dividends reduces liquidity. The relationship between liquidity for large investors (liquidity measure 2) and uncertainty over the amount of distressed sales at a future point in time was also studied (see section B.6 of the appendix). Intuition suggests uncertainty over future orderflow, should, like uncertainty over dividends, make returns more uncertain, and hence increase slope and reduce liquidity. Simulations show that for some parameterizations orderflow uncertainty does reduce liquidity for a period of time before the distressed sales; however, for other parameterizations I find the reverse to be true: future orderflow uncertainty can sometimes increase current liquidity. This result is contrary to intuition and suggests that the relationship between liquidity and market conditions is extremely complicated, even within this stylized model.[37]

7 Summary and Conclusions

A growing share of financial assets are owned or managed by large institutional investors whose desired trades are large enough to move prices, and who account for their price impact when trading. In this paper, I have shown that when investors account for their price impact, it slows the adjustment to optimal risk-sharing following large shocks, and these deviations from optimal risk-sharing are temporarily priced in asset markets. This insight provides an economic explanation for time variation in liquidity factor-loadings, and rewards for illiquidity risk, and it helps to explain the breakdown in asset pricing relationships during periods of market turbulence.

The paper also makes several contributions to the literature on distressed investors. This includes extending the literature to a setting in which all investors are fully rational, optimizing, and aware of the distressed sales. It also shows that front-running and predatory trading can emerge in this setting, although the magnitude of such activities is small. In addition, this is perhaps the first paper to solve for optimal liquidations in a dynamic setting with endogenous liquidity provision when all liquidity providers are fully rational and optimizing.

The paper also contributes to the literature on liquidity by introducing the concept of market structure of risk-bearing capacity, which is measured by the cross-sectional distribution of large investors capacity to absorb risk. In the paper, market structure is shown to affect the price impact of trades, the persistence of shocks to asset pricing relationships, and the pattern of optimal liquidations by a distressed investor. Although the paper shows that

[37] This result is *not* contrary to intuition in noisy rational expectations models that have both information asymmetry and noise trading, since greater uncertainty, as measured by greater volatility of noise traders demands, can reduce adverse selection and thereby increase liquidity. However, it is unintuitive here because there is no information asymmetry in the present model.

market structure of risk-bearing is important, only a very limited set of market structures is examined. The additional implications of various market structures for risk-bearing capacity is a topic for future research.

Appendix

A Notation

There are M investors and N risky assets. $Q(t)$ is the stacked $NM \times 1$ vector of investors risky asset holdings at the beginning of time t:

$$Q(t) = \begin{pmatrix} Q_1(t) \\ \vdots \\ Q_M(t) \end{pmatrix},$$

and $Q_1(t)$ represents the net asset holdings of a continuum of infinitesimal "small" investors indexed by s, with measures $\mu(s)$:

$$Q_1(t) = \int_0^1 Q_s(t)\mu(s)ds.$$

The small investors are often referred to as the competitive fringe. The large investors risky asset holdings are $Q_2(t)$ through $Q_M(t)$, and their stacked vector is $Q_B(t)$. $\Delta Q(t) = Q(t+1) - Q(t)$ is the change in investors risky asset holdings during period t. $\Delta Q_1(t)$ and $\Delta Q_B(t)$ have analogous meanings.

The algebra requires summations of the elements of stacked vectors. The matrix $S = \iota_M' \otimes I_N$ is used to perform these summations where ι_M is an $M \times 1$ vector of ones, and I_N is the $N \times N$ identity matrix.[38] The matrix S_i is used for selecting submatrices of stacked vectors. S_i has form $S_i = \iota_{i,M}' \otimes I_N$, where $\iota_{i,M}$ is an M vector has a 1 in its i'th element, and has zeros elsewhere.[39] S and S_i will always have N rows but might have different column dimensions to conform to the vectors that they multiply.

$P(Q,t) : (Q(t), t) \to \mathcal{R}$ denotes the function that maps asset holdings into equilibrium risky-asset prices during time t. The distribution of dividends is i.i.d. $D(t) \sim \mathcal{N}(\bar{D}, \Omega)$, but to simplify the proofs, they are represented with the more general form: $D(t) \sim \mathcal{N}[\bar{D}, \Omega(t)]$ where, $\Omega(t)$ is a deterministic function of time.

In the rest of the exposition, I will occasionally suppress time subscripts to save space.

[38] For example, $SQ(t) = \sum_{m=1}^{M} Q_m(t)$

[39] To illustrate the use of the selection matrix, $Q_m(t) = S_m Q(t)$.

B Proofs

The proof of proposition 1 is given in section B.2. Proposition 3.4 appears in the text, and is restated below: **Proposition 2**: *Small investors value functions for entering period t with asset holdings Q_s and q_s when the economy's vector of risky asset holdings at time t is Q is given by:*

$$\begin{aligned} V_s(q_s, Q_s, Q, t) &= -K_s(t)\ F(Q,t)\ e^{-A_s(t)q_s(t)-A_s(t)Q_s'(\bar{D}+P(Q,t))+.5A_s(t)^2Q_s'\Omega(t)Q_s}, \\ where\ F(Q,t) &= e^{-Q'\bar{v}_1(t)-\frac{1}{2}Q'\theta_1(t)Q} \qquad (22) \\ P(Q,t) &= \tfrac{1}{r}(\alpha(t)-\Gamma(t)Q). \end{aligned}$$

Additionally, large investor m's value function for entering period t when the state variable is Q and his holdings of riskfree assets are q_m is given by:

$$V_m(q_m, Q, t) = -K_m(t)e^{-A_m(t)q_m(t)-A_m(t)Q_m'\bar{v}_m(t)+.5A_m(t)^2Q'\theta_m(t)Q}. \qquad (23)$$

Proof: The proof is by induction. Part I establishes that if value functions have this form at time t, then they have the same form at time $t-1$. Part II establishes the result for time T, the first period in which trade cannot occur.

B.1 Part I:

Suppose the form of the value function is correct for time t. To solve for value functions at time $t-1$, I first solve for the competitive fringe's demand curve for absorbing the large investors net order flow, and then solve the large investors and competitive fringe's equilbrium portfolio and consumption choices, and then solve for the value function at time $t-1$.

The competitive fringe's demand curve

The competitive fringe is a continuum of infinitesimal investors that are distributed uniformly on the unit interval with total measure 1, i.e. $\mu(s) = 1$ for $s \in [0,1]$. At time $t-1$, each participant s of the competitive fringe solves:

$$\max_{\substack{C_s(t-1),\\ \Delta Q_s(t-1),\\ \Delta q_s(t-1)}} \quad -e^{-A_s C_s(t-1)} - \delta V_s(q_s(t), Q_s(t), Q(t), t) \tag{A1}$$

subject to the budget constraint:

$$C_s(t-1) + \Delta Q_s' P(., t-1) + \Delta q_s(t-1) = Q_s(t-1)' D(t-1) \tag{A2}$$

where,

$$\begin{aligned} q_s(t) &= r(q_s(t-1) + \Delta q_s(t-1)), \\ Q_s(t) &= Q_s(t-1) + \Delta Q_s(t-1), \\ Q(t) &= Q(t-1) + \Delta Q(t-1), \end{aligned} \tag{A3}$$

and $P(., t-1)$ represents the equilibrium price schedule for the risky assets at time $t-1$. Small investors take the schedule as given. Using the budget constraint to solve for $\Delta q_s(t-1)$, and then plugging the results into equation (A3), and then substituting in the value function produces the unconstrained problem:

$$\begin{aligned} \max_{\substack{C_s(t-1),\\ \Delta Q_s(t-1)}} \quad & -e^{-A_s C_s(t-1)} - \Big(\delta K_1(t)\ F(Q,t)\ e^{-A_s(t) r[q_s(t-1) + Q_s(t-1)'(D(t-1) + P(.,t-1)) - C_s(t-1)]} \\ & \times e^{A_s(t) r[Q_s(t-1) + \Delta Q_s(t-1)]' P(.,t-1)} \\ & \times\ e^{-A_s(t)(Q_s(t-1)+\Delta Q_s(t-1))'(\bar{D}+P(Q,t)) + .5 A_s(t)^2 (Q_s(t-1)+\Delta Q_s(t-1))' \Omega(t) (Q_s(t-1)+\Delta Q_s(t-1))'}\Big) \end{aligned} \tag{A4}$$

Examination shows that optimal portfolio trades $\Delta Q_s(t-1)$ can be chosen independently of consumption. Solving for $\Delta Q_s(t-1)$ shows:

$$\Delta Q_s(t-1) = \frac{1}{A_s(t)} \Omega(t)^{-1} \left(\bar{D} + P(Q,t) - rP(., t-1)\right) - Q_s(t-1) \tag{A5}$$

Integrating both side of the above equation with respect to $\mu(s)$ generates the net demand of the competitive fringe:

$$\begin{aligned} \Delta Q_s(t-1) &= \textstyle\int_0^1 \Delta Q_s(t-1) \mu(s) ds \\ &= \tfrac{1}{A_1(t)} \Omega(t)^{-1} \left(\bar{D} + P(Q,t) - rP(., t-1)\right) - Q_1(t-1) \end{aligned} \tag{A6}$$

where,

$$\frac{1}{A_1(t)} = \int_0^1 \frac{1}{A_s(t)}\mu(s)ds \quad \text{and} \quad Q_1(t-1) = \int_0^1 Q_s(t-1)\mu(s)ds.$$

The Price Schedule Faced by Large Investors

The price schedule faced by large investors maps large investors desired net orderflow to the prices the competitive fringe requires to absorb it. This requires solving for $P(.,t-1)$ in (A6) such that when large investors choose risky asset trade vector $\Delta Q_B(t-1)$ during period $t-1$ then the competitive fringe chooses trade $-S\Delta Q_B(t-1)$. Rearranging, equation (A6) shows:

$$\begin{aligned} P(.,t-1) &= \frac{1}{r}\left(\bar{D} + P(t,Q(t)) - A_1(t)\Omega(t)[Q_1(t-1) + \Delta Q_1(t-1)]\right) \\ &= \frac{1}{r}\left(\bar{D} + P(t,Q(t-1) + \Delta Q(t-1)) - A_1(t)\Omega(t)[Q_1(t-1) + \Delta Q_1(t-1)]\right) \end{aligned}$$

Substituting $-S\Delta Q_B(t-1)$ for $\Delta Q_1(t-1)$, and recalling that $P(t,Q(t)) = \frac{1}{r}\left(\alpha(t) - \Gamma(t)Q(t)\right)$, by assumption, then reveals that the price schedule has form:

$$P(.,t-1) = \frac{1}{r}\left(\beta_0(t-1) - \beta_Q(t-1)Q(t-1) - \beta_{Q_B}(t-1)\Delta Q_B(t-1)\right), \tag{A7}$$

where,

$$\beta_0(t-1) = \bar{D} + (1/r)\alpha(t) \tag{A8}$$

$$\beta_Q(t-1) = (1/r)(\Gamma(t) + rA_1(t)\Omega(t)S_1) \tag{A9}$$

$$\beta_{Q_B}(t-1) = (1/r)\Gamma(t)\begin{pmatrix} -S \\ I \end{pmatrix} - A_1(t)\Omega(t)S \tag{A10}$$

Given the price schedule in equation (A7), large investors at time $t-1$ solve the maximization problem:

Large Investors Maximization Problem

$$\max_{\substack{C_m(t-1), \\ \Delta Q_m(t-1), \\ \Delta q_m(t-1)}} \quad -e^{-A_m C_m(t-1)} - \delta V_m(q_m(t), Q(t), t) \tag{A11}$$

subject to the budget constraint

$$C_m(t-1) + \Delta Q_m(t-1)'P(.,t-1) + \Delta q_m(t-1) = Q_m(t-1)'D(t-1). \tag{A12}$$

Substituting the budget constraint into the value function, and writing $Q(t)$, and $Q_m(t-1)$ as $Q + \Delta Q$, and Q_m, investor m faces the unconstrained problem:

$$\max_{C_m(t-1),\Delta Q_m} -e^{-A_m C_m(t-1)} - \delta \left\{k_m(t) \times \exp\left(-rA_m(t)[q_m(t-1) + Q_m' D(t-1) - C_m(t-1)]\right)\right.$$
$$\left. \times \ \exp\left(-A_m(t)(Q+\Delta Q)'\bar{v}_m(t) + .5A_m(t)^2(Q+\Delta Q)'\theta_m(t)(Q+\Delta Q) + rA_m(t)\Delta Q_m' P(.,t-1)\right)\right\}. \tag{A13}$$

Examination of the maximand shows that ΔQ_m can be chosen before optimal consumption. Each large investor chooses ΔQ_m while taking the choices of the other large investors as given, while accounting for the effect of his own choices on prices and on the asset holdings of investor 1 since by construction $\Delta Q_1 = -S\Delta Q_m$. The first order condition for large investor m has form:

$$\begin{aligned} 0 = -A_m(t)[(-S_1 + S_m)\bar{v}_m(t)] + A_m(t)^2(-S_1 + S_m)[(\theta_m(t) + \theta_m(t)')/2](Q + \Delta Q) \\ + A_m(t)\left[rP(.,t-1) - S_m\beta_{Q_B}(t-1)'S_m\Delta Q_B\right], \end{aligned} \tag{A14}$$

Substituting for $P(.,t-1)$ from equation (A7), writing $Q + \Delta Q$ as $Q + \begin{pmatrix} -S\Delta Q_B \\ \Delta Q_B \end{pmatrix}$ and simplifying, produces the following reaction function for large investor m:

$$\pi_m(t-1)\Delta Q_B = \chi_m(t-1) + \xi_m(t-1)Q, \tag{A15}$$

where,

$$\begin{aligned} \pi_m(t-1) = & A_m(t)(-S_1 + S_m)[(\theta_m(t) + \theta_m(t)')/2]\begin{pmatrix} -S \\ I \end{pmatrix} \\ & - \beta_{Q_B}(t-1) - S_m\beta_{Q_B}(t-1)'S_m \end{aligned} \tag{A16}$$

$$\chi_m(t-1) = (-S_1 + S_m)\bar{v}_m(t) - \beta_0(t-1) \tag{A17}$$

$$\xi_m(t-1) = \beta_Q(t-1) - A_m(t)(-S_1 + S_m)[(\theta_m(t) + \theta_m(t)')/2] \tag{A18}$$

Stacking the (M-1) reaction functions produces a system of $(M-1)N$ linear equations in $(M-1)N$ unknowns:

$$\Pi(t-1)\Delta Q_B(t-1) = \chi(t-1) + \xi(t-1)Q(t-1) \tag{A19}$$

Assume that $\Pi(t-1)$ is invertible. Then the solution for $\Delta Q_B(t-1)$ is unique, and given by

$$\Delta Q_B(t-1) = \Pi(t-1)^{-1}\chi(t-1) + \Pi(t-1)^{-1}\xi(t-1)Q(t-1) \tag{A20}$$

Equilibrium Asset Holdings

The solution for $\Delta Q_1(t-1)$ is $-S\Delta Q_B(t-1)$. Therefore, the solution for $\Delta Q(t-1) = (\Delta Q_1(t-1)', \Delta Q_B(t-1)')'$ can be written as:

$$\Delta Q(t-1) = H_0(t-1) + H_1(t-1)Q(t-1). \tag{A21}$$

where,

$$H_0(t-1) = \left(\begin{array}{c} -S\Pi(t-1)^{-1}\chi(t-1) \\ \Pi(t-1)^{-1}\chi(t-1) \end{array} \right), \quad \text{and} \quad H_1(t-1) = \left(\begin{array}{c} -S\Pi(t-1)^{-1}\xi(t-1) \\ \Pi(t-1)^{-1}\xi(t-1) \end{array} \right). \tag{A22}$$

With the above notation, the equilibrium purchases by large participant m in period $t-1$ are given by

$$\Delta Q_m(t-1) = S_m[H_0(t-1) + H_1(t-1)Q(t-1)] \tag{A23}$$

Additionally, the equilibrium transition dynamics for beginning of period risky asset holdings are given by:

$$Q(t) = G_0(t-1) + G_1(t-1)Q(t-1) \tag{A24}$$

where $G_0(t-1) = H_0(t-1)$ and $G_1(t-1) = H_1(t-1) + I$.

Equilibrium Price Function

The equilibrium price function for period $t-1$ is found by plugging the solution for large investors equilibrium trades from equation (A20) into the price schedule faced by large investors (equation (A7)). The price function for period $t-1$ has form:

$$P(t-1, Q) = \frac{1}{r}\left(\alpha(t-1) - \Gamma(t-1)Q\right) \tag{A25}$$

where,

$$\alpha(t-1) = \beta_0(t-1) - \beta_{Q_B}(t-1)\pi(t-1)^{-1}\chi(t-1) \tag{A26}$$

$$\Gamma(t-1) = \beta_Q(t-1) + \beta_{Q_B}(t-1)\pi(t-1)^{-1}\xi(t-1) \tag{A27}$$

Large Investors Consumption

Large investors optimal time $t-1$ consumption depends on optimal time $t-1$ trades. Plugging the expressions for equilibrium prices and trades [equations (A24), (A25), and (A21)] into equation (A13), shows large investors consumption choice problem has form:

$$\max_{C_m(t-1)} -e^{-A_m C_m(t-1)} - \delta k_m(t) e^{rA_m(t)C_m(t-1)} \times \psi_m(Q(t-1), q_m(t-1), D(t-1), t-1), \tag{A28}$$

where

$$\begin{aligned}\psi_m(Q, q_m, D(t-1), &t-1) = \\ &e^{-A_m(t)r(q_m(t-1)+Q_m(t-1)'D(t-1))} \\ &\times e^{-A_m(t)r[S_m(H_0(t)+H_1(t)Q)(t-1)]'(\alpha(t-1)-\Gamma(t-1)Q(t-1))/r} \\ &\times e^{-A_m(t)(G_0(t)+G_1(t)Q(t-1))'\bar{v}_m(t)+.5A_m(t)^2(G_0(t)+G_1(t)Q(t-1))'\theta_m(t)(G_0(t)+G_1(t)Q(t-1))}\end{aligned} \tag{A29}$$

Solving the first order condition for optimal consumption shows

$$C_m(t-1) = \frac{-1}{A_m(t)r + A_m} \ln\left(\frac{\delta k_m(t) A_m(t) r \psi_m(Q(t-1), q_m(t-1), D(t-1), t-1)}{A_m}\right). \tag{A30}$$

Large investors value function at time $t-1$

Define $V_m(t-1, D(t-1), Q, q_m)$ as the value function to large investor m from entering period $t-1$ when the vector of risky asset holdings is Q, and his riskless asset holdings are q_m, and the dividend realization at time $t-1$ is $D(t-1)$. After substituting the optimal consumption choice in (A30) into equation (A28), this value function is given by:

$$V_m(q_m, Q, t-1, D(t-1)) = -\left[\frac{1+r_m^*(t)}{r_m^*(t)}\right]\left[\delta k_m(t) r_m^*(t) \psi_m(Q, q_m, D(t-1), t-1)\right]^{\frac{1}{1+r_m^*(t)}} \tag{A31}$$

where,

$$r_m^*(t) = A_m(t)r/A_m \tag{A32}$$

Taking the expectation of the value function in equation (A31) with respect to the distribution of $D(t-1)$ produces $V_m(q_m, Q, t-1)$, which is the large investors m's value function at time $t-1$:

$$V_m(q_m, Q, t-1) = -k_m(t-1) \times e^{-A_m(t-1)q_m - A_m(t-1)Q'\bar{v}_m(t-1) + .5A_m(t-1)^2 Q'\theta_m(t-1)Q} \tag{A33}$$

The parameters of the value function satisfy the nonlinear Riccati difference equations:

$$A_m(t-1) = A_m(t)r/(1 + r_m^*(t)) \tag{A34}$$

$$k_m(t-1) = \left[\frac{r_m^*(t)+1}{r_m^*(t)}\right] [\delta k_m(t) r_m^*(t)]^{\frac{1}{1+r_m^*(t)}} \\ \times e^{A_m(t-1)H_0(t-1)'S_m'\alpha(t-1)/r - A_m(t-1)G_0(t-1)'\bar{v}_m(t)/r + .5A_m(t-1)^2((1+r_m^*(t))/r^2)(G_0(t-1)'\theta_m(t)G_0(t-1))} \tag{A35}$$

$$\bar{v}_m(t-1) = S_m'\bar{D} - H_1(t-1)'S_m'\alpha(t-1)/r + \Gamma(t-1)'S_m H_0(t-1)/r + G_1(t-1)'\bar{v}_m(t)/r \\ - A_m(t-1)(1 + r_m^*(t))G_1(t-1)'\left(\frac{\theta_m(t) + \theta_m(t)'}{2}\right) G_0(t-1)/r^2 \tag{A36}$$

$$\theta_m(t-1) = \frac{-2H_1(t-1)'S_m'\Gamma(t-1)}{rA_m(t-1)} + (1 + r_m^*(t))G_1(t-1)'\theta_m(t)G_1(t-1)/r^2 + S_m'\Omega(t-1)S_m \tag{A37}$$

Small investors optimal consumption

Small investors consumption depend on small investors optimal trades (equation (A5)) which can be found by substituting equilibrium prices $P(Q, t-1)$ for the price schedule $P(., t-1)$. Optimal consumption also depends on post-trade asset holdings, $Q(t-1) + \Delta Q(t-1)$, and equilibrium prices [equations (A24) and (A25)]. Substituting these in equation (A4), shows small investors consumption problem has form:

$$\max_{C_s(t-1)} -e^{-A_s C_s(t-1)} - \delta k_s(t) e^{rA_s(t)C_s(t-1)} \times \psi_s(Q(t-1), Q_s(t-1), q_s(t-1), D(t-1), t-1), \tag{A38}$$

where,

$$\psi_s(Q(t-1), Q_s(t-1), q_s(t-1), D(t-1), t-1) =$$
$$e^{-A_s(t)r\{q_s(t-1)+Q_s(t-1)'[P(t-1,Q(t-1))+D(t-1)]\}}$$
$$\times e^{-.5\{\bar{D}+P[t,Q(t-1)+\Delta Q(t-1)]-rP[t-1,Q(t-1)]\}'\Omega(t)^{-1}\{\bar{D}+P[t,Q(t-1)+\Delta Q(t-1)]-rP[t-1,Q(t-1)]\}}$$
$$\times e^{-[G_0(t-1)+G_1(t-1)Q(t-1)]'\bar{v}_s(t)-.5[G_0(t-1)+G_1(t-1)Q(t-1)]'\theta_s(t)[G_0(t-1)+G_1(t-1)Q(t-1)]} \tag{A39}$$

The first order condition shows optimal consumption is given by:

$$C_s(t-1) = \frac{-1}{A_s(t)r + A_s}\ln\left(\frac{\delta k_s(t)A_s(t)r\psi_s(Q(t-1), Q_s(t-1), q_s, D(t-1), t-1)}{A_s}\right) \tag{A40}$$

Small investors value function at time $t-1$

Define $V_s(q_s(t-1), Q_s(t-1), t-1, D(t-1))$ as the value function to small investor s from entering period $t-1$ when the vector of risky asset holdings is $Q(t-1)$, and his risky and riskless asset holdings are $Q_s(t-1)$ and $q_s(t-1)$, and the dividend realization at time $t-1$ is $D(t-1)$. After substituting the optimal consumption choice in (A40) into equation (A38), the value function is given by:

$$V_s(q_s(t-1), Q_s(t-1), Q(t-1), t-1, D(t-1)) =$$
$$-\left[\frac{1+r_s^*(t)}{r_s^*(t)}\right][\delta k_s(t)r_s^*(t)\psi_s(Q(t-1), Q_s(t-1), q_s(t-1), D(t-1), t-1)]^{\frac{1}{1+r_s^*(t)}} \tag{A41}$$

where,

$$r_s^*(t) = A_s(t)r/A_s \tag{A42}$$

Taking the expectation of the value function function with respect to the distribution of $D(t-1)$ produces the small investor's value function at time $t-1$:

$$\begin{aligned} V_s(q_s, Q_s, Q, t-1) &= -K_s(t-1)\ F(Q, t-1)\ e^{-A_s(t-1)q_s(t-1)-A_s(t-1)Q_s'(\bar{D}+P(Q,t-1))+.5A_s(t-1)^2Q_s'\Omega(t-1)Q_s}, \\ \text{where } F(Q, t-1) &= e^{-Q'\bar{v}_s(t-1)-\frac{1}{2}Q'\theta_s(t-1)Q} \\ P(Q, t-1) &= \tfrac{1}{r}(\alpha(t-1) - \Gamma(t-1)Q). \end{aligned} \tag{A43}$$

The parameters in the small investors value functions satisfy the Riccati equations:

$$A_s(t-1) = \frac{rA_s(t)}{1+r_s^*(t)} \tag{A44}$$

$$k_s(t-1) = \left[\frac{r_s^*(t)+1}{r_s^*(t)}\right]\left[\delta k_s(t-1)r_s^*(t)e^{-.5a_0(t-1)'\Omega(t)^{-1}a_0(t-1)-G_0(t-1)'\bar{v}_s(t)-.5G_0(t-1)'\theta_s(t)G_0(t-1)}\right]^{\frac{1}{1+r_s^*(t)}}, \tag{A45}$$

where,

$$a_0(t-1) = \bar{D} + \frac{1}{r}\left[\alpha(t) - \Gamma(t)G_0(t-1)\right] - \alpha(t-1). \tag{A46}$$

$$\bar{v}_s(t-1) = \frac{a_1(t-1)'\Omega(t)^{-1}a_0(t-1) + G_1(t-1)'\bar{v}_s(t) + G_1(t-1)'[(\theta_s(t)+\theta_s(t)')/2]G_0(t-1)}{1+r_s^*(t)}, \tag{A47}$$

where,

$$a_1(t-1) = \Gamma(t-1) - \frac{1}{r}\Gamma(t)G_1(t-1). \tag{A48}$$

$$\theta_s(t-1) = \frac{a_1(t-1)'\Omega(t)^{-1}a_1(t-1) + G_1(t-1)'\theta_s(t)G_1(t-1)}{1+r_s^*(t)} \tag{A49}$$

This completes part I of the proof because equations (A33) and (A43) verify that the value functions at time $t-1$ have the same form as at time t.

B.2 Part II.

To establish part II of the proof, I need to show that investors value functions for entering entering period T has the same functional form as given in the proposition. The functional form is given below.

Investors Value Functions at Time T

Recall that investors are infinitely lived but that from time T onwards they cannot alter their holdings of risky assets, but they can continue to alter their consumption, and their holdings of riskless assets. Because investors cannot trade after period T, the distinction between small and large investors after this period is irrelevant. Hence, the index m used below could be for either a large or small investor. Using the Bellman principle, the value function of entering period T with risky asset holdings Q_m and risk-free holdings q_m, conditional on time T dividends $D(T)$ satisfies the system of equations:

$$V_m(Q_m(T), q_m(T), t|D(T)) = \max_{C_m(T)} -\exp^{-A_m C_m(T)} + \delta \, \mathrm{E} \, V_m(Q_m(t+1), q_m(t+1), t+1), \tag{A50}$$

subject to the constraints:

$$q_m(T+1) = r[q_m(T) + Q_m(T)' D(T) - C(T)] \tag{A51}$$

$V_m(Q_m(T), q_m(T), T)$, the unconditional value of entering period T with holdings $Q_m(T)$ and $q_m(T)$, is given by:

$$V_m(Q_m(T), q_m(T), T) = \int_{D(T)} V_m(Q_m(T), q_m(T), T|D(T)) f(D(T)) dD(T) \tag{A52}$$

where, $f(D(T))$ is the probability density function for $D(T)$.

Equation (A52) is the Bellman equation for this optimization problem. Inspection shows that the function,

$$V_m(Q_m(T), q_m, T) = -k_m(T) \exp^{-A_m(1-(1/r))q_m(T) - A_m(1-(1/r))\frac{Q_m(T)'\bar{D}}{1-(1/r)} + .5A_m^2(1-(1/r))^2 \frac{Q_m(T)'\Omega Q_m(T)}{1-(1/r)}} \tag{A53}$$

solves the Bellman equation, where

$$k_m(T) = \left(\frac{r}{r-1}\right) \times (\delta r)^{\frac{1}{r-1}} \tag{A54}$$

For large investors, this equation can be rewritten as:

$$V_m(Q_m(T), q_m, T) = -k_m(T) \exp^{-A_m(T) q_m(T) - A_m(T) Q' \bar{v}_m(T) + .5 A_m(T)^2 Q' \theta_m(T) Q} \tag{A55}$$

where,

$$\begin{aligned} A_m(T) &= A_m[1 - (1/r)] && \text{(A56)} \\ \bar{v}_m(T) &= \frac{S_m' \bar{D}}{1 - (1/r)} && \text{(A57)} \\ \theta_m(T) &= \frac{S_m' \Omega S_m}{1 - (1/r)} && \text{(A58)} \end{aligned}$$

For small investors, the value function in equation (A53) can be rewritten in the form:

$$V_s(Q_s(T), q_s, T) = -k_s(T) F(Q, T) \exp^{-A_s(T) q_s(T) - A_s(T) Q_s(T)'[\bar{D} + P(Q,T)] + .5 A_s(T)^2 Q_s(T)' \Omega(T) Q_s(T)} \tag{A59}$$

solves the Bellman equation, where

$$\begin{aligned}
A_s(T) &= A_s[1-(1/r)] && \text{(A60)}\\
\Gamma(T) &= 0 && \text{(A61)}\\
k_s(T) &= \left(\frac{r}{r-1}\right) \times (\delta r)^{\frac{1}{r-1}} && \text{(A62)}\\
F(Q,T) &= -e^{Q'\bar{v}_s(T)-.5Q'\theta_s(T)Q} && \text{(A63)}\\
\bar{v}_s(T) &= 0 && \text{(A64)}\\
\theta_s(T) &= 0 && \text{(A65)}\\
P(Q,t) &= \frac{1}{r}(\alpha(T)-\Gamma(T)Q) && \text{(A66)}\\
\alpha(T) &= \bar{D}/[1-(1/r)] && \text{(A67)}\\
\Gamma(T) &= 0 && \text{(A68)}\\
\Omega(T) &= \Omega/[1-(1/r)] && \text{(A69)}
\end{aligned}$$

Since these value functions have the same form as given in the proposition, this completes the induction and the proof of proposition 2. Inspection of the value functions in period T show they have the form given in proposition 1 hence completing the proof of that proposition. □

Interpretation of Large Investors Risk Aversion

In this subsection, the portfolio problem of large investors is recast into a related problem in which the demands of each large investor represent the portfolio choices of a mass of identical small investors that pool their investment management decisions to economize on the costs of monitoring the market. Specifically, the demands and consumption choices of each large investor m, represents the demands of a mass μ_m of small investors that have identical risk aversion $A_{m,s}$, identical endowments of risky and risk free assets ($Q_{m,s}(T-1)$ and $q_{m,s}(T-1)$), and choose identical consumption $C_{m,s}(T-1)$. Given investors time T value functions (equation (A53)), at time $T-1$ each large investor maximizes the utility of a representative small investor by solving the maximization problem:

$$\max_{C_{m,s},\Delta Q_{m,s}} -e^{-A_{m,s}C_{m,s}} - \delta V_{m,s}(Q_{m,s}(T), q_{m,s}(T), T) \tag{A70}$$

subject to the standard budget constraint (A12) and subject to the constraint that the large investors total trading activity and consumption choices are split evenly among the mass μ_m of small investors that he represents. For example, if he wants to buy 1 share of stock for each small investor, and there are 100 small investors then he has to buy 100 shares of stock. More generally, the large investors choices are related to the small investors choices as follows:

$$
\begin{aligned}
Q_m + \Delta Q_m &= (Q_{m,s} + \Delta Q_{m,s}) \times \mu_m \\
q_m + \Delta q_m &= (q_{m,s} + \Delta q_{m,s}) \times \mu_m \\
C_m &= C_{m,s} \times \mu_m
\end{aligned} \tag{A71}
$$

Using these to substitute for $C_{m,s}$, $Q_{m,s}+\Delta Q_{m,s}$ and $q_{m,s}+\Delta q_{m,s}$ in the objective function then shows that the maximization problem solved for the representative small investor has the exact same form as the maximization problem for a large investor with risk aversion $A_{m,s}/\mu_m$. If each large investor is interpreted as a mutual fund, this means that the risk tolerance of the fund is equal to the risk tolerance of the typical investor in the fund times the number of investors in the fund as measured by μ_m. Given this relationship is true in the last period, it is straightforward to establish it in earlier periods.

B.3 Solutions for Value Function Parameters

This section provides further information on the parameters of investors value functions when there are not distressed sales. The results in this section are used to establish the main results on asset pricing in section B.4. Distressed sales alter the Riccati difference equations of investors value functions; this topic is dealt with separately in section B.5.

The next proposition provides more information about the parameters of the value functions for the large investors when there are not distressed sales:

Proposition 7 *For all time periods $t = 1, \ldots, T-1$, and for large investors $m = 2, \ldots M$:*

$$\bar{v}_m(t) = \frac{S'_m \bar{D}}{1-(1/r)} \tag{A72}$$

$$\alpha(t) = \bar{D}/[1-(1/r)] \tag{A73}$$

$$A_m(t) = A_m[1-(1/r)] \tag{A74}$$

$$r^*(t) = r-1 \tag{A75}$$

$$k_m(t) = \left(\frac{r}{r-1}\right) \times (\delta r)^{\frac{1}{r-1}} \tag{A76}$$

Proof:

For $\bar{v}_m(t)$ and $\alpha(t)$:

The proof is by induction. First, suppose the results for $\bar{v}_m(t)$ and $\alpha(t)$ are true at time t. Then, $\alpha(t) = \bar{D}/[1-(1/r)]$, implies from equation (A8) that $\beta_0(t-1) = \bar{D}/[1-(1/r)]$. Then equation (A17) implies $(-S_1 + S_m)\bar{v}_m(t) - \beta_0(t-1) = 0$. As a result $\chi(t-1) = 0$, which implies from equation (A26) that $\alpha(t-1) = \beta_0(t-1) = \bar{D}/[1-(1/r)]$ and from equations (A22) and (A24) that $H_0(t-1) = G_0(t-1) = 0$. Substituting for $H_0(t-1)$ and $G_0(t-1)$ in equation (A36) and simplifying then shows:

$$\bar{v}_m(t-1) = S'_m\bar{D} - H_1(t-1)'S'_m\alpha(t-1)/r + G_1(t-1)'\bar{v}_m(t)/r \tag{A77}$$

Recalling that $G_1(t-1) = H_1(t-1) + I$, then plugging in the expressions for $\alpha(t-1)$ and $\bar{v}_m(t)$ into the expression for $\bar{v}_m(t-1)$ confirms the result for time $T-1$. To complete the induction, it suffices to note that the results hold at time T as shown in equations (A67) and (A57). This completes the proof for $\bar{v}_m(t)$ and $\alpha_m(t)$.

For $A_m(t)$ and $r^*(t)$:

The proof is by induction. Suppose the results are true at time t, then applying the solutions for $A_m(t)$ and $r^*_m(t)$ in equation (A34) produces the hypothesized expression for $A_m(t-1)$, and applying the solution for $A_m(t-1)$ in equation (A32) produces the hypothesized expression for $r^*_m(t-1)$. To complete the induction note that the result is true for $A_m(T)$ and $r^*(T)$.

For $k_m(t)$:

The proof is by induction. Assume it is true for time t. Then plugging the hypothesized solution into equation (A35) and plugging in the solutions for $r^*_m(t)$ and plugging in $H_0(t-1) = G_0(t-1) = 0$ (established in the proof for $\bar{v}_m(t)$) and then simplifying shows the result holds for $k_m(t-1)$. To complete the induction note that $k_m(T)$ has the hypothesized form.
$\square$

The next proposition provides information on the value functions of the small investors:

Proposition 8 *For all time periods $t = 1, \ldots, T-1$, and for each small investor s*

$$a_0(t) = 0 \tag{A78}$$
$$\bar{v}_s(t) = 0 \tag{A79}$$
$$A_s(t) = A_s[1-(1/r)] \tag{A80}$$
$$r^*_s(t) = r-1 \tag{A81}$$
$$k_s(t) = \left(\frac{r}{r-1}\right) \times (\delta r)^{\frac{1}{r-1}} \tag{A82}$$

Proof:

For $a_0(t)$ and $\bar{v}_s(t)$: Plugging the solutions for $\alpha(t)$ and $G_0(t-1)$ from proposition 7 into equation (A46) shows that $a_0(t) = 0$ for all times t. Since $G_0(t-1) = 0$ for all times t, it then follows from equation (A47) that if $\bar{v}_s(t) = 0$, then so does $\bar{v}_s(t-1)$. To complete the induction, note that $\bar{v}_s(T) = 0$ (equation (A64)).

For $A_s(t)$, $r_s^*(t)$, and $k_s(t)$:

The form of the proof is identical to that given in proposition 7.□

Proposition 9 *Assume that for $t < T$, conditional on state variable $Q(t)$ the Nash Equilibrium trades of the large investors exists and is unique. Then for all $m = 2, \ldots, M$ and $t = 1, \ldots, T$, $\theta_m(t)$ has form:*

$$\vartheta_m(t) \otimes \Omega, \tag{A83}$$

where, $\vartheta_m(t)$ is $M \times M$; and

$$\Gamma(t) = \gamma(t) \otimes \Omega, \tag{A84}$$

where, $\gamma(t)$ is $1 \times M$.

Proof: The proof is by induction. First, assume that the theorem is true at time t. Then, from equations (A10) and (A9) $\beta_{Q_B}(t-1) = B_{Q_B}(t-1) \otimes \Omega$, and $\beta_Q(t-1) = B_Q(t-1) \otimes \Omega$, where $B_{Q_B}(t-1)$ is $1 \times M-1$ and $\beta_Q(t-1)$ is $1 \times M$. Applying these substitutions in large investors reaction functions and then stacking the results reveals that in equation (A19), $\pi(t-1) = \mathcal{P}(t-1) \otimes \Omega$ and $\xi(t-1) = Z(t-1) \otimes \Omega$. The assumption that the Nash Equilibrium trades in each period are unique implies that $\mathcal{P}(t-1)$ is invertible. Solving for $H_0(t-1)$ and $H_1(t-1)$ then shows that $H_0(t-1) = 0$ and

$$\begin{aligned}
H_1(t-1) &= \begin{pmatrix} -S[P(t-1)^{-1}Z(t-1)] \otimes I_N \\ (P(t-1)^{-1}Z(t-1)) \otimes I_N \end{pmatrix} && \text{(A85)} \\
&= \begin{pmatrix} [-\iota_M' \mathcal{P}(t-1)^{-1}Z(t-1)] \otimes I_N \\ (\mathcal{P}(t-1)^{-1}Z(t-1)) \otimes I_N \end{pmatrix} && \text{(A86)} \\
&= \mathcal{H}_1(t-1) \otimes I_N && \text{(A87)}
\end{aligned}$$

where ι_M is a $1 \times M$ vector of ones, and $\mathcal{H}_1(t-1)$ is $M \times M$. Since $G_1(t-1) = H_1(t-1) + I_{NM}$, it follows that $G_1(t-1) = \mathcal{G}_1(t-1) \otimes I_N$ for $\mathcal{G}_1(t-1) = \mathcal{H}_1(t-1) + I_M$. From here, substitution in equation (A27) shows that $\Gamma(t-1) = \gamma(t-1) \otimes \Omega$ and substitution in equation (A37) shows that $\theta_m(t-1) = \vartheta_m(t-1) \otimes \Omega$. To complete the induction, note that the conditions of the proposition are satisfied at time T. □.

Corollary 5 *For each small investors, and for each time period $t = 1, \ldots T$,*

$$\theta_s(t) = \vartheta_s(t) \otimes \Omega,$$

where $\vartheta_s(t)$ is $M \times M$.

Proof: Straightforward induction involving application of the results from proposition 9.

Corollary 6 *Let $\Delta Q_{.,n}(t)$ and $Q_{.,n}(t)$ denote the $M \times 1$ vector of time t asset holdings and trades of asset n by investors 1 through M. Then,under the assumptions of proposition 9 for all $n = 1, \ldots N$, $\Delta Q_{.,n}(t) = \mathcal{H}_1(t) Q_{.,n}(t)$.*

Proof: By definition, $\Delta Q_{.,n}(t) = (I_M \otimes e_n') \Delta Q(t)$, where e_n is an $N \times 1$ vector which has 1 for its n'th element and zeroes elsewhere. Since $H_0(t) = 0$, we also know

$$\Delta Q(t) = H_1(t) Q(t) \tag{A88}$$

$$= [\mathcal{H}_1(t) \otimes I_N] Q(t). \tag{A89}$$

Multiplying both sides of equation (A89) by $(I_M \otimes e_n')$ and then simplifying establishes the result $\square$.

Corollary 6 shows that the equilbrium trades for each asset n only depend on market participants holdings of that asset. They do not depend on market participants holdings of other assets. Clearly, the result in the corollary is very special, not general.

B.4 Proofs of Asset Pricing Propositions

Proposition 5: *When investors asset holdings are not Pareto Optimal, equilibrium expected asset returns satisfy a linear factor model in which one factor is the market portfolio, and the other factors correspond to the deviation of large investors asset holdings from pareto optimal asset holdings.*

Proof: Let Q^W denote the vector of pareto optimal holdings of risky assets. Manipulation of the equation for equilibrium prices given in proposition 2, and substitution of $G_0(t)+G(t)Q(t)$ for $Q(t+1)$ shows:

$$P(t+1) + \bar{D} - rP(t) = [\frac{1}{r}\alpha(t+1) + \bar{D} - \alpha(t)] - [\frac{1}{r}\Gamma(t+1)G_0(t)] + [\Gamma(t) - \frac{1}{r}\Gamma(t+1)G_1(t)]Q(t)$$

Plugging in the solution for $\alpha(t) = \alpha(t-1) = \bar{D}/[1-(1/r)]$ shows the first term in braces is zero. The second term in braces is zero since proposition 7 shows that $G_0(t) = 0$. Adding and subtracting Q^W to $Q(t)$, the above equation can be rewritten as:

$$P(t+1)+\bar{D}-rP(t) = [\Gamma(t)-\frac{1}{r}\Gamma(t+1)G_1(t)](Q(t)-Q^W)+[\Gamma(t)-\frac{1}{r}\Gamma(t+1)G_1(t)]Q^W \tag{A90}$$

Using the fact that $Q_1 = X - SQ_B$, the vector $Q(t) - Q^W$ can be expressed in terms of the deviations of large investors asset holdings from pareto optimal asset holdings:

$$\begin{aligned} Q(t) - Q_W &= \begin{bmatrix} (X - SQ_B) - (X - SQ_B^W) \\ Q_B - Q_B^W \end{bmatrix} \\ &= \begin{bmatrix} -S \\ I \end{bmatrix} (Q_B - Q_B^W) \end{aligned}$$

Additionally, an implication of proposition 4 and the expression for equilibrium $P(t)$ with perfect competition, is that

$$[\Gamma(t) - \frac{1}{r}\Gamma(t+1)G_1(t)]Q^W = \lambda_X \Omega X.$$

Making both of these substitutions in equation (A90) shows

$$P(t+1) + \bar{D} - rP(t) = \lambda_X \Omega X + [\Gamma(t) - \frac{1}{r}\Gamma(t+1)G_1(t)] \begin{pmatrix} -S \\ I \end{pmatrix} (Q_B(t) - Q_B^W)$$

Finally, applying the algebra in proposition 9 shows

$$[\Gamma(t) - \frac{1}{r}\Gamma(t+1)G_1(t)] \begin{pmatrix} -S \\ I \end{pmatrix} = \lambda(t) \otimes \Omega \tag{A91}$$

where $\lambda(t)$ is $1 \times M - 1$. Making this substitution then shows:

$$\begin{aligned} P(t+1) + \bar{D} - rP(t) &= \lambda_X \Omega X + [\lambda(t) \otimes \Omega](Q_B(t) - Q_B^W) && \text{(A92)} \\ &= \lambda_X \Omega X + \sum_{j=2}^{M} \lambda(m,t)\Omega(Q_m(t) - Q_m^W) && \text{(A93)} \end{aligned}$$

where $\lambda(m,t) = \lambda(t)s'_{m-1}$. $\square$.

Proposition 5 implies that assets expected excess returns over the riskfree rate can alternatively be expressed in terms of assets loadings on factor mimicking portfolios and on the excess expected returns on those portfolios as shown in the following proposition.

Corollary 2: *Assuming sufficient regularity so that no elements of $P_F(t)$ are equal to zero*[40], *then under the condition of proposition 5,*

$$\mathrm{E}_t[R_i(t+1) - r] = \beta'_{iF} E_t[R_F(t+1) - \iota r], \tag{A94}$$

[40]The elements of $P_F(t)$ such that for $m \geq 2$, $Q_m(t) = Q_m^W$, will be equal to zero. Modified versions of the corollary will still be true in this case; the details are left to the reader.

where ι is an $M \times 1$ vector of ones.

Proof: Let $\psi(t) = [X, Q_2(t) - Q_2^W, \ldots, Q_M(t) - Q_M^W]$ denote the share-holdings in the factor portfolios at time t. Let $P_F(t) = \psi(t)'P(t)$, denote the M-vector of prices of the factor portfolios at time t; let $P_F^D(t)$ denote the diagonalized matrix of these prices, and let $R_F(t+1) = [P_F^D(t)]^{-1}\psi(t)'[P(t+1) + D(t+1) - rP(t)]$ denote the M-vector of returns of the factor portfolios. Because prices are deterministic, $V_F(t+1)$, the variance-covariance matrix of the factor returns is given by, $V_F(t+1) = [P_F^D(t)]^{-1}\psi(t)'\Omega\psi(t)\ [P_F^D(t)]^{-1}$, and the vector of covariances of the returns on asset i with the factor returns is given by $C_{iF}(t+1) = \frac{\iota'_{i,N}}{P_i(t)}\Omega\psi(t)'[P_F^D(t)]^{-1}$. Let $\lambda(t) = [\lambda_X, \lambda(2,t), \ldots, \lambda(M,t)]'$ denote the stacked vector of market prices of risk from equation (29) at time t, then it follows from equation (29), that $\mathrm{E}_t(R_F(t+1) - \iota r) = V_F(t+1)[P_F^D(t)]\lambda(t)$, which implies $\lambda(t) = [P_F^D(t)]^{-1}[V_F(t+1)]^{-1}\,\mathrm{E}_t(R_F(t+1) - \iota r)$. An additional implication of (29) is that $\mathrm{E}_t(R_i(t+1) - r) = C_{iF}(t+1)[P_F^D(t)]\lambda(t)$. Substituting in for $\lambda(t)$ then shows that

$$
\begin{aligned}
\mathrm{E}_t[R_i(t+1) - r] &= C_{iF}(t+1)[P_F^D(t)]\lambda(t) \\
&= C_{iF}(t+1)[P_F^D(t)][P_F^D(t)]^{-1}[V_F(t+1)]^{-1}\,\mathrm{E}_t(R_F(t+1) - \iota r) \\
&= C_{iF}(t+1)[V_F(t+1)]^{-1}\,\mathrm{E}_t(R_F(t+1) - \iota r) \\
&= \beta'_{iF}\,\mathrm{E}_t(R_F(t+1) - \iota r).\Box
\end{aligned}
$$

Corollary 7 *When asset holdings are not pareto optimal, then asset returns between time t and time $t+1$ have an alternative M-factor model representation in which one factor is the market portfolio, and the other factors are the returns on large investors risky asset holdings.*

Proof: From proposition 5, we know that:

$$P(t+1) + \bar{D} - rP(t) = \lambda_X \Omega X + \sum_{j=2}^{M} \lambda(m,t)\Omega(Q_m(t) - Q_m^W)$$

Making the substitution $Q_m^W = \frac{(1/A_m)X}{\sum_{m=1}^{M} 1/A_m}$, and then simplifying shows

$$P(t+1) + \bar{D} - rP(t) = \lambda_X(t)\Omega X + \sum_{j=2}^{M} \lambda(m,t)\Omega Q_m(t)$$

where

$$\lambda_X(t) = \lambda_X - \sum_{m=2}^{M} \left(\frac{\lambda(m,t)/A_m}{\sum_{j=1}^{M} 1/A_j} \right) \quad \Box.$$

Corollary 3: *When asset holdings at time t are not pareto-optimal, then asset returns at time $t+\tau$ follow a factor model in which the market portfolio and the deviation of large investors asset holdings from pareto-optimal asset holdings at time t are factors.*

Proof: Iterating equation (A90), by τ periods shows:

$$P(t+\tau+1)+\bar{D}-rP(t+\tau) = [\Gamma(t+\tau)-\frac{1}{r}\Gamma(t+1+\tau)G_1(t+\tau)](Q(t+\tau)-Q^W) + [\Gamma(t+\tau)-\frac{1}{r}\Gamma(t+\tau+1)G_1(t+\tau)]Q^W, \tag{A95}$$

which implies:

$$P(t+\tau+1)+\bar{D}-rP(t+\tau) = \lambda_X\Omega X + [\lambda(t,\tau)\otimes\Omega](Q(t)-Q^W) \tag{A96}$$

$$= \lambda_X\Omega X + \sum_{m=2}^{M}\lambda_m(t,\tau)\Omega(Q_m(t)-Q_m^W) \tag{A97}$$

where,

$$\lambda(t,\tau)\otimes\Omega = [\Gamma(t+\tau)-\frac{1}{r}\Gamma(t+1+\tau)G_1(t+\tau)]\prod_{j=0}^{\tau-1}G_1(t+j),$$

and $\lambda_m(t,\tau)=\lambda(t,\tau)S'_{m-1}$ $\square$.

Assume that asset returns are generated by the large investor model, and that investors asset holdings are pareto optimal. In the absence of any shocks which perturb investors asset holdings, asset returns will be indistinguishable from those associated with the CAPM. If there is a small 1-time perturbation in asset holdings, then because the deviation in asset prices is proportional to the size of the perturbation, for small enough perturbations, the behavior of the model is statistically indistinguishable from the CAPM for a given span of data. This (obvious) point is made formally in the next proposition:

Proposition 10 *In a sample of τ 1-period returns, let ϕ denote the size of a Chi-Square test that Jensen's α is not equal to 0, and let ρ denote the power of the test. Then for every $\rho>\phi$ there are perturbations from pareto optimal asset holdings for which the power of the test is less than ρ.*

Proof: For simplicity, assume that λ_X, Ω, and X are known. Then an estimate of Jensen's α is given by

$$\hat{\alpha} = \frac{1}{\tau}\sum_{i=1}^{\tau}P(t+i)+D(t+i)-rP(t+i-1)-\lambda_X\Omega.$$

Under the null hypothesis that assets are priced by the CAPM,

$$\sqrt{\tau}\hat{\alpha} \sim \mathcal{N}(0, \Omega),$$

and

$$\tau\hat{\alpha}'\Omega^{-1}\hat{\alpha} \sim \chi^2(0, N),$$

where the non-centrality parameter is 0. Under the alternative, assets are not priced by the CAPM. If there is an initial perturbation of asset holdings under the alternative, without loss of generality parameterize the direction of the perturbation for each $m = 2, \ldots M$ by $\bigtriangledown_m$ and the size of the perturbation by scalar δ so that $(Q_m(t) - Q_m^W) = \delta\bigtriangledown_m$. Then, $\sqrt{\tau}\hat{\alpha} \sim N(\mu, \Omega)$ where, $\mu = \delta\frac{1}{\sqrt{\tau}}\sum_{m=2}^{M}\left(\sum_{i=1}^{\tau}\lambda_m(t, t+i)\bigtriangledown_m\right)$. Therefore, $\tau\hat{\alpha}'\Omega^{-1}\hat{\alpha} \sim \chi^2(\psi, N)$ where $\psi = \mu'\Omega^{-1}\mu = O(\delta^2)$. The power of the test is continuously increasing in the non-centrality parameter, and hence in δ; and when δ is 0, the power of the test is equal to its size. Therefore, by the continuity of the power function, for small enough δ (small enough perturbations) the power of the test is less than ϕ. $\square$

B.5 Distressed Sales

Suppose one of the large investors is forced to sell their holdings of risky assets at time τ_s, and then exit the market. In this section, I model how such distressed sales affect equilibrium trades and prices when investors learn of the future sales at time τ_R, but the sales do not occur until time τ_S. For simplicity, it is assumed that the distressed seller is not allowed to trade between times τ_R and τ_S, and that no market participants (including the distressed seller) are aware of the distress before time τ_R.

Time τ_S

After investors enter time τ_S, and receive their dividend and interest payments, I assume they learn that one large investor will sell ΔQ_D units of risky assets during trade in time period τ_S. To solve for how this affects investors value functions, I first solve for how it affects the prices at which the competitive fringe is willing to absorb the large investors orderflow. I then solve for the large investors equilibrium orderflow, and then I solve for equilibrium risky asset trades, prices, and consumption at time τ_S. Finally, I solve for the equilibrium value function for entering time τ_S.

For all of the analysis in this section, I assume there are $M + 1$ investors whose risky asset holdings before time τ_R are equal to Q^W. Without loss of generality, I assume that the $M + 1$'st investor will have to sell ΔQ_D at time τ_S. I further assume that this investor cannot trade until time τ_S. With these assumptions, the $M + 1$'st investor is a modelling device for distressed sales at time τ_S.

The primary focus is on how the other M investors behave as a result of the distressed sales. Let $Q(\tau_S)$ denote the risky asset holdings of investors 1 through M at time τ_S. At time τ_S, for any given purchases of risky assets ΔQ_B by large investors 2 through M, prices at time τ_S and $\tau_S + 1$ must equilibrate so that the competitive fringe fringe is willing to absorb $\Delta Q_D - S\Delta Q_B$, which is the distressed sales less the net purchases of the other large investors. From the equilibrium price function at time $\tau_S + 1$, we know that given the distressed sales and hypothesized large investors purchases, that

$$Q(\tau_S + 1) = \begin{pmatrix} Q_1(\tau_S) + \Delta Q_D - S\Delta Q_B(\tau_S) \\ Q_B(\tau_S) + \Delta Q_B(\tau_S) \end{pmatrix},$$

and that

$$P(\tau_S + 1, Q(\tau_S + 1)) = \frac{1}{r}\left(\alpha(\tau_S + 1) - \Gamma(\tau_S + 1)Q(\tau_S + 1)\right)$$

To solve for the price schedule faced by large investors, I follow the steps beginning from equation (A5) to derive a price schedule which is analogous to that in equation (A7):

$$P(., \tau_S) = \frac{1}{r}\left(\beta_0(\tau_S) - \beta_Q(\tau_S)Q(\tau_S) - \beta_{Q_B}(\tau_S)\Delta Q_B(\tau_S)\right), \tag{A98}$$

where,

$$\beta_0(\tau_S) = \bar{D} + (1/r)\alpha(\tau_S + 1) - (1/r)\Gamma(\tau_S + 1)S_1'\Delta Q_D - A_1(\tau_S + 1)\Omega(\tau_S + 1)\Delta Q_D \tag{A99}$$

$$\beta_Q(\tau_S) = (1/r)(\Gamma(\tau_S + 1) + rA_1(\tau_S + 1)\Omega(\tau_S + 1)S_1) \tag{A100}$$

$$\beta_{Q_B}(\tau_S) = (1/r)\Gamma(\tau_S + 1)\begin{pmatrix} -S \\ I \end{pmatrix} - A_1(\tau_S + 1)\Omega(\tau_S + 1)S \tag{A101}$$

Notice, that the price schedule faced by large investors in equation (A98) differs from that in equation (A7) because $\beta_0(\tau_S)$ contains additional terms that cause the price to change to compensate the fringe for absorbing a portion of the distressed asset sales.

Given the price schedule, large investors will choose their net purchases $\Delta Q_B(\tau_S)$ while accounting for the fact that for a given purchase of risky assets on their behalf, the amount of risky assets that will be held by the fringe has increased by ΔQ_D. The resulting reaction function for large investor m is given by:

$$\pi_m(\tau_S)\Delta Q_B = \chi_m(\tau_S) + \xi_m(\tau_S)Q, \tag{A102}$$

where,

$$
\begin{aligned}
\pi_m(\tau_S) = A_m(\tau_S+1)(-S_1+S_m)[(\theta_m(\tau_S+1)+\theta_m(\tau_S+1)')/2] \begin{pmatrix} -S \\ I \end{pmatrix} \\
- \beta_{Q_B}(\tau_S) - S_m \beta_{Q_B}(\tau_S)' S_m
\end{aligned} \tag{A103}
$$

$$
\begin{aligned}
\chi_m(\tau_S) =& (-S_1+S_m)\bar{v}_m(\tau_S+1) - \beta_0(\tau_S) \\
& - A_m(\tau_S+1)(-S_1+S_m)\left(\frac{\theta_m(\tau_S+1)+\theta_m(\tau_S+1)'}{2}\right) S_1' \Delta Q_D
\end{aligned} \tag{A104}
$$

$$
\xi_m(\tau_S) = \beta_Q(\tau_S) - A_m(t)(-S_1+S_m)[(\theta_m(\tau_S+1)+\theta_m(\tau_S+1)')/2] \tag{A105}
$$

The distressed sales cause the χ_m term of large investors reaction function to change through a price schedule effect because $\beta_0(\tau_S)$ changes, and through a risk-sharing effect because the distressed sales that are not absorbed by large investors will be absorbed by the competitive fringe.

Stacking the reaction functions together and solving for the equilibrium trades of the large investors, produces a solution for large investors trades which is analogous to that in equation (A20). The competitive fringes equilibrium trades are then given by $\Delta Q_D - S\Delta Q_B$. Following the development in equation (A21), we have

$$
\Delta Q(\tau_S) = H_0(\tau_S) + H_1(\tau_S) Q(\tau_S). \tag{A106}
$$

where,

$$
H_0(\tau_S) = \begin{pmatrix} -S\Pi(\tau_S)^{-1}\chi(\tau_S) + \Delta Q_D \\ \Pi(\tau_S)^{-1}\chi(\tau_S) \end{pmatrix}, \quad \text{and} \quad H_1(\tau_S) = \begin{pmatrix} -S\Pi(\tau_S)^{-1}\xi(\tau_S) \\ \Pi(\tau_S)^{-1}\xi(\tau_S) \end{pmatrix}. \tag{A107}
$$

From the proof of proposition 7, we know that $\chi_m(\tau_s) = 0$ when $\Delta Q_D = 0$. It then follows that $\chi(\tau_S)$ and $H_0(\tau_S)$ are linear functions of ΔQ_D. For convenience, I write these relationships as:

$$
\chi(\tau_S) = \chi(\tau_S)^* \Delta Q_D, \tag{A108}
$$

$$
H_0(\tau_S) = H_0(\tau_S)^* \Delta Q_D. \tag{A109}
$$

Then, substituting the solution for ΔQ_B into equation (A98)shows that solutions for the coefficients in the equilibrium price function in period τ_S take the form:

$$\alpha(\tau_S) = \frac{\bar{D}}{1-(1/r)} - \alpha(\tau_S)^*(\Delta Q_D) \tag{A110}$$

where

$$\alpha(\tau_S)^* = (1/r)\Gamma(\tau_S+1)S_1' + A_1(\tau_S+1)\Omega(\tau_S+1) + \beta_{Q_B}(\tau_S)\pi(\tau_S)^{-1}\chi(\tau_S)^*.$$

The expression for $\Gamma(\tau_S)$ is not changed by the distressed sales.

Also, the path of equilibrium asset holdings in moving from period τ_S to period τ_S+1 is given by:

$$Q(\tau_S+1) = G_0(\tau_S) + G_1(\tau_S)Q(\tau_S),$$

where $G_0(\tau_S) = H_0(\tau_S)$ and $G_1(\tau_S) = H_1(\tau_S) + I$.

In the last two expressions, the difference with the expressions in equations (A22) and (A24) is that $H_0(\tau_S)$ contains an additional term which reflects the fact that the distressed sales increase the amount of assets that are collectively held by all other market participants. This implies that for $t \leq \tau_S$, when there are distressed sales, then $H_0(t)$ does not equal 0.

Given the solutions for the equilibrium price schedule, reaction functions, and asset holding transitions, the solution for the value functions in period τ_S proceeds as in any period t and has the same functional form. For all periods t such that $\tau_R \leq t \leq \tau_S$, the value functions are solved by backwards induction using the same approach as used earlier. Finally, before period τ_R investors are not aware of the distressed sales, so the value function has the same form that was solved for in the undistressed sales case. Put differently, after trade in period $\tau_R - 1$ is over, when investors learn about the future distressed sales, their value functions jump, and the future distressed sales create a basis for trade among the investors.

An important question is whether the distressed sales affect market liquidity after market participants learn of the sales, but before the sales occur. If liquidity is measured by the slope of the price function faced by large investors, then the answer is no. An examination of equations (A9) and (A10), and equation (A27) shows that the "slope" measure of market liquidity is determined by parameters that are invariant to the distressed sales when the quantity of distressed sales is known. Therefore, knowledge of the impending sales do not alter liquidity. Intuition for this result is that liquidity is based on market strucuture, and on the riskiness of each share of stock. Because the quantity of impending sales is known, it does not alter the riskiness of holding a share, and hence it has no effect on liquidity.[41] However, if the quantity of distressed sales at time τ_S is random instead, then this creates price risk at the time of the sales (prices are no longer deterministic) that will have liquidity effects. The effects of random sales on liquidity is examined further in what follows.

B.6 Random Distressed Sales

For simplicity, assume that distressed sales are normally distributed:

$$\Delta Q_D \sim \mathcal{N}(\mu_D, \Sigma_D) \tag{A111}$$

where $\mu > 0$ is interpreted as distressed purchases.

To solve for the effect that distressed sales have on investors value functions, I substitute the expressions for $H_0(\tau_S)$ and $\alpha(\tau_S)$ into large and small investors value functions and then take expectations with respect to the distribution of distressed sales. When doing so, I use the results from proposition 7 and 8 to simplify the analysis.

Large Investors Value Function of Entering Period τ_S

After substituting the new expressions for $H_0(\tau_S)$ and $\alpha(\tau_s)$ into large investors value functions, the value function conditional on ΔQ_D is an exponential linear quadratic function of ΔQ_D with the following form:

$$\begin{aligned} V_m(q_m, Q, \tau_s|\Delta Q_D) = &\left[\frac{r^*_m(t)+1}{r^*_m(t)}\right] [\delta k_m(t) r^*_m(t)]^{\frac{1}{1+r^*_m(t)}} \\ &\times e^{-.5\Delta Q_D' \Phi^*_m \Delta Q_D - A_m(\tau_s) Q'(\bar{v}_m(\tau_s+1) + L^*_m \Delta Q_D)} \times e^{.5 A_m(\tau_s)^2 Q' \theta^*_m(\tau_s) Q} \end{aligned} \tag{A112}$$

where, Φ^*_m is given by[42]:

$$\Phi^*_m = \text{Symm}\left(2A_m(\tau_s) H_0^{*'} S_m' \alpha^*/r - A_m(\tau_s)^2 H_0^{*'} \theta_m(\tau_s+1) H_0^*/r\right),$$

$$\theta^*_m(\tau_s) = \frac{-2H_1(\tau_S)' S_m' \Gamma(\tau_S)}{r A_m(\tau_S)} + G_1(\tau_S)' \theta_m(\tau_S+1) G_1(\tau_S)/r + S_m' \Omega(\tau_S) S_m$$

[41] In Kyle and Xiong (2002) additional share sales do alter liquidity because the sales alter investors wealth, and thus increase investors absolute risk aversion.

[42] The operator Symm() operates on squares matrices and returns the symmetric version of the matrix. For example, $\text{Symm}(X) = (X + X')/2$.

and

$$L_m^* = H_1(\tau_s)'S_m'\alpha^*/r + \Gamma(\tau_s)'S_m H_0^*/r - A_m(\tau_s)G_1(\tau_s)'\left(\frac{\theta_m(\tau_s+1)+\theta_m(\tau_s+1)'}{2}\right)G_0^*/r$$

A sufficient condition for $E_{\Delta Q_D}\{V_m(q_m, Q, \tau_s|\Delta Q_D)\}$ to be bounded is that Φ_m^* is positive semidefinite. There is no guarantee that this condition will be satisfied. An alternative sufficient condition for the expectation to exist is that $\Sigma_D^{-1} + \Phi_m^*$ is positive definite. If Σ_D is a scalar multiple of some nonsingular matrix M then, it is clear that when the scalar multiple is close enough to zero, positive definiteness is guaranteed. I will assume that Σ_D is small enough to so that positive definiteness is guaranteed.

Taking expectations with respect to ΔQ_D then shows that:

$$V_m(q_m, Q, \tau_s) = k_m(\tau_S) \times e^{-A_m(\tau_S)q_m - A_m(\tau_s)Q'\bar{v}_m(\tau_S) + .5A_m(\tau_S)^2 Q'\theta_m(\tau_S)Q} \tag{A113}$$

where,

$$\begin{aligned} k_m(\tau_S) &= |I + \Sigma_D\Phi_m^*|^{-.5} \times \left(\tfrac{r}{r-1}\right) \times (\delta r)^{\frac{1}{r-1}} e^{.5\mu_D'\Sigma_D^{-1}\left[(\Sigma_D^{-1}+\Phi_m^*)^{-1}-\Sigma_D\right]\Sigma_D^{-1}\mu_D}, \\ \bar{v}_m(\tau_S) &= \bar{v}_m(\tau_S+1) + L_m^*(\Sigma_D^{-1}+\Phi_m^*)^{-1}\Sigma_D^{-1}\mu_D, \\ \theta_m(\tau_S) &= \theta_m^*(\tau_S) + L_m^*(\Sigma_D^{-1}+\Phi_m^*)^{-1}L_m^{*\prime} \end{aligned} \tag{A114}$$

The random asset sales significantly change the parameters of the large investors value functions at time τ_S. $\bar{v}_m(\tau_s)$ is equal to its value without distressed sales $((\bar{v}_m(\tau_s+1))$ plus an additional term that reflects the random distressed sales. Similarly, $\theta_m(\tau_s)$ is equal to its value without distressed sales $(\theta_m^*(\tau_s))$ plus an additional term that reflects the distressed sales. The most significant change is that tedious calculations show that the $NM \times NM$ matrix $\theta_m(\tau_s)$ cannot be written as the kronecker product of an $M \times M$ matrix with Ω. This means that the asset pricing relationships between time periods τ_R (the time of the rumor) and τ_S (the time of the random distressed sales)are different than earlier periods. Even without the risk, pricing relationships are still affected by distressed sales as shown by taking limits of the value function as $\Sigma_d \to 0$. In this limit, $\theta_m(\tau_S)$ approaches its original value, but the asset sales affect $\bar{v}_m(\tau_s)$ by the amount $-L_m^*\mu_D$.This is sufficient to cause the asset pricing relationships to break down in the period between τ_R and τ_S.

Although the asset sales during time τ_S are random, large investors value function for time τ_S remains an exponential linear quadratic function of large investors risky asset holdings Q. This property will be convenient for solving the model backwards.

Small Investors Value Function of Entering Period τ_S

Following the same basic approach as for large investors, small investors value function conditional on ΔQ_D is exponential linear quadratic in ΔQ_D and takes the form:

$$V_s(q_s, Q_s, Q, \tau_S|\Delta Q_D) = \left(\frac{r}{r-1}\right) \times (\delta r)^{\frac{1}{r-1}} \times e^{-.5\Delta Q_D' \Phi_s^* \Delta Q_D - Q'\bar{v}_s^* \Delta Q_D - .5Q'\theta_s^* Q} \\ \times e^{-A_s(\tau_s)q_s(\tau_s) - A_s(\tau_s)Q_s'\left[\frac{\bar{D}}{1-(1/r)} - \frac{\alpha^*}{r}\Delta Q_D - \frac{\Gamma(\tau_s)}{r}Q\right] + .5A_s(\tau_S)^2 Q_s' \Omega(\tau_S) Q_s}, \tag{A115}$$

where,

$$\begin{aligned} \Phi_s^* &= \text{Symm}\left(\frac{a_0^{*\prime}\Omega(\tau_S+1)^{-1}a_0^* + G_0^{*\prime}\theta_s(\tau_s+1)G_0^*}{r}\right) \\ a_0^* &= \alpha^* - \frac{\Gamma(\tau_s+1)G_0^*}{r} \\ \bar{v}_s^* &= \frac{a_1(\tau_s)'\Omega(\tau_s+1)^{-1}a_0^* + G_1(\tau_s)'[(\theta_s(\tau_s+1)+\theta_s(\tau_s+1)')/2]G_0^*}{r} \\ \theta_s^* &= \frac{a_1(\tau_s)'\Omega(\tau_s+1)^{-1}a_1(\tau_s) + G_1(\tau_s)'\theta_s(\tau_s+1)G_1(\tau_s)}{r} \end{aligned} \tag{A116}$$

Then, taking expectations with respect to ΔQ_D shows that:

$$V_s(q_s, Q_s, Q, \tau_S) = -K_s(\tau_S)F(Q, \tau_S)e^{-A_s(\tau_s)q_s - A_s(\tau_s)Q_s'[\bar{D} + \hat{P}(Q,\tau_s)] + .5A_s(\tau_s)^2 Q_s' \Omega^*(\tau_s) Q_s}, \tag{A117}$$

where,

$$K_s(\tau_s) = |I + \Sigma_D \Phi_s^*|^{-.5} \times \left(\frac{r}{r-1}\right) \times (\delta r)^{\frac{1}{r-1}} e^{.5\mu_D' \Sigma_D^{-1}\left[(\Sigma_D^{-1} + \Phi_s^*)^{-1} - \Sigma_D\right]\Sigma_D^{-1}\mu_D}, \tag{A118}$$

$$\begin{aligned} F(Q, \tau_s) &= e^{-Q'\bar{v}_s(\tau_s) - \frac{1}{2}Q'\theta_s(\tau_s)Q}, \\ \bar{v}_s(\tau_s) &= v_s^*(\Sigma_D^{-1} + \Phi_S^*)^{-1}\Sigma_D^{-1}\mu_D, \\ \theta_s(\tau_s) &= \theta_s^* - \bar{v}_s^*(\Phi_s^* + \Sigma_D^{-1})^{-1}\bar{v}_s^{*\prime}; \end{aligned} \tag{A119}$$

$$\begin{aligned} \hat{P}(Q, \tau_s) &= \tfrac{1}{r}\left(\alpha(\tau_s+1) - \Gamma(\tau_s)Q\right) - \tfrac{\alpha^*}{r}(\Sigma_D^{-1} + \Phi_s^*)^{-1}\Sigma_D^{-1}\mu_D + \tfrac{\alpha^*}{r}(\Sigma_D^{-1} + \Phi_s^*)^{-1}\bar{v}_s^{*\prime}Q \\ &= \tfrac{1}{r}\left(\left[\alpha(\tau_s+1) - \alpha^*(\Sigma_D^{-1} + \Phi_s^*)^{-1}\Sigma_D^{-1}\mu_D\right] - \left[\Gamma(\tau_s) - \tfrac{\alpha^*}{r}(\Sigma_D^{-1} + \Phi_s^*)^{-1}\bar{v}_s^{*\prime}\right]Q\right); \end{aligned} \tag{A120}$$

and,

$$\Omega^*(\tau_s) = \Omega(\tau_S) + \frac{\alpha^{*\prime}}{r}(\Phi_s^* + \Sigma_D^{-1})^{-1}\frac{\alpha^*}{r}. \tag{A121}$$

It is clear that the random distressed sales significantly affect the value function of the small investors. The variance of asset prices increases the variance of excess returns $[\Omega^*(\tau_s)]$ beyond the amount that they would be in the absence of distressed sales $[\Omega[\tau_s)]$. Additionally, because asset prices in period τ_S are random, and correlated with changes in the state variable Q, the correlation of asset prices with the state variable affects small investors demand for the risky assets. The correlations show up when deriving the pseudo-price function $\hat{P}(Q, \tau_S)$, which appears in small investors value functions. It should be noted that $\hat{P}(Q, \tau_S)$ is not the equilibrium price function in period τ_S and it is not the expected value of the equilibrium price function; instead it is a grouping of terms in the value function so that the small investors value function has the same form as in periods $t > \tau_S$. Because large investors value functions also have the same form as in periods $t > \tau_s$, the model can be solved backwards from this point using the same sets of Riccatti equations that were used in the earlier analysis.

More specifically, to analyze how random distressed sales affect the liquidity received by large investors, I use the pseudo price function in period τ_S to solve for $\beta_{Q_B}(\tau_s - 1)$, which is the slope of the price schedule faced by large investors in the period before the distressed sales take place. Applying the analysis in equations (A7) through (A10) shows that

$$\begin{aligned} \beta_{Q_B}(\tau_s - 1) = & \left\{ (1/r)\Gamma(\tau_s) \begin{pmatrix} -S \\ I \end{pmatrix} - A_1(\tau_s)\Omega(\tau_s)S \right\} \\ & - \frac{\alpha^*}{r}(\Sigma_d^{-1} + \Phi_s^*)^{-1}\overline{v}_s^{*\prime} \begin{pmatrix} -S \\ I \end{pmatrix} - A_1(\tau_s)\frac{\alpha^{*\prime}}{r}(\Sigma_d^{-1} + \Phi_s^*)^{-1}\frac{\alpha^*}{r}S \end{aligned} \tag{A122}$$

Equation (A122) shows that in time period $\tau_s - 1$, the liquidity received by large investors is equal to the liquidity without the rumor (the term in braces) plus a term which reflects the covariance of prices with large investors asset holdings (the second term) plus a term which reflects an increased variance of prices (the third term). Examination of the third term shows that the distressed sales affect the liquidity received by all large investors by the same amount. But, the distressed sales affect the liquidity received by large investors by potentially different amounts because the covariance between prices and the state variables depends on each large investors willingess to absorb the distressed asset sales, and this varies across large investors who differ in their risk aversion.

To for how distressed sales affect liquidity and trades in earlier periods, it is necessary to use the analysis in this section to solve the model numerically.

B.7 Optimal Liquidations

Suppose there is a large investor who learns at time τ_R that he must liquidate his portfolio of risky assets by time τ_S. What is the optimal liquidation strategy that the investor should follow? To solve for the optimal liquidation strategy, I simply solve for the value functions of the liquidator, and the other investors in the period that the remainder of the position is liquidated. I then backwards induct to solve for investors value functions in earlier periods. Since the parameters of the value functions determine liquidity conditions, liquidity is endogenous: i.e. the liquidating investors price impact function and all other large investors price impact functions depend on the fact that the liquidating investor plans to liquidate by a certain date. Given the liquidity conditions, the trading path is also endogenous. The value functions of the investors are derived below. With these value functions in hand, it is straightforward to solve for the path of optimal liquidations.

The liquidator

The liquidating investor is assumed to be investor $M+1$ in the model. He maximizes the discounted expected utility of future consumption. Like other investors in the model, he has discount rate δ and CARA utility of per-period consumption with coefficient of absolute risk aversion A_{M+1}. At the time that he liquidates his portfolio of risky assets, he is assumed to choose his future consumption path subject to the constraint that he only holds the riskless asset in his portfolio. Under this assumption, straightforward dynamic programming show that after the risky asset portion of his portfolio has been totally liquidated, the value of the liquidators remaining wealth has form:

$$V_{M+1}(W) = \left(\frac{r}{r-1}\right)(r\delta)^{\frac{1}{r-1}} \times e^{-A(1-(1/r))W} \tag{A123}$$

If the liquidating investor enters period τ_S with risky asset holdings Q_{M+1}, and the other investors enter the period with the $NM \times 1$ vector of risky asset holdings Q, then from equation (A110) and proposition 2 we know that when the liquidating investor sells off his remaining assets in period τ_S, then equilibrium prices have the form:

$$P(Q, Q_{M+1}, \tau_S) = \frac{1}{r}\left(\frac{\bar{D}}{1-(1/r)} - \alpha(\tau_S)^* Q_{M+1} - \Gamma(\tau_s)Q\right). \tag{A124}$$

Therefore, the liquidating investors after liquidation wealth, W, is equal to

$$W = q_{M+1} + Q'_{M+1}[D(\tau_s) + P(Q, Q_{M+1}, \tau_S)],$$

where q_{M+1} is the cash carried into the period, and the following two terms are revenues from dividends within the period and revenues from liquidating the risky asset position. Substituting the expression for equilibrium price into wealth, and then substituting the result into the liquidating investors value function and taking expectations over the distribution of dividends, shows that to the liquidating investor, the value of entering period τ_S when the vector of risky asset holdings in the economy is given by $Q^* = (Q', Q'_{M+1})'$, is exponential linear quadratic in the state variables and has form:

$$V_{M+1}(q_m, Q^*, \tau_S) = -K_{M+1}(\tau_s)e^{-A_{M+1}(\tau_S)Q^{*'}\bar{v}_{M+1}(\tau_s)+.5A_{M+1}(\tau_s)Q^{*'}\theta_{M+1}(\tau_S)Q^*-A_{M+1}(\tau_S)q_{m+1}}, \tag{A125}$$

where,

$$A_{M+1}(\tau_s) = A_{M+1}(1-(1/r)) \tag{A126}$$
$$K_{M+1}(\tau_s) = \left(\frac{r}{r-1}\right) \times (\delta r)^{\frac{1}{r-1}} \tag{A127}$$
$$\bar{v}_{M+1}(\tau_S) = S'_{M+1}\bar{D}/[1-(1/r)], \tag{A128}$$

and,

$$\theta_{M+1}(\tau_S) = \begin{bmatrix} 0_{[NM\times NM]} & \Gamma(\tau_S)'/A_{M+1}(\tau_S) \\ \Gamma(\tau_S)/A_{M+1}(\tau_S) & [\alpha^*(\tau_S)/A_{M+1}(\tau_S)] + \Omega \end{bmatrix}. \tag{A129}$$

The value functions for the large and small investors can be derived similarly.

Large Investors Value Function at Time τ_S

The expression for large investors value function conditional on distressed sales of amount ΔQ_D at time τ_S is provided in equation (A112). Making the subsitution $\Delta Q_D = Q_{M+1}$, and simplifying then shows that large investors value function (investors 2 through M) in time period τ_S has form:

$$V_m(q_m, Q^*, \tau_S) = -K_m(\tau_s)e^{-A_m(\tau_S)Q^{*'}\bar{v}_m(\tau_s)+.5A_m(\tau_s)Q^{*'}\theta_m(\tau_S)Q^*-A_m(\tau_S)q_m}, \tag{A130}$$

where,

$$A_m(\tau_s) = A_m(1-(1/r)) \tag{A131}$$
$$k_m(\tau_s) = \left(\frac{r}{r-1}\right) \times (\delta r)^{\frac{1}{r-1}} \tag{A132}$$
$$\bar{v}_m(\tau_S) = S'_m\bar{D}/[1-(1/r)], \tag{A133}$$

and,

$$\theta_m(\tau_S) = \begin{bmatrix} \theta_m(\tau_s)^* & -L_m^{*'}/A_m(\tau_S) \\ -L_m^*/A_m(\tau_S) & -\frac{\Phi_m^*(\tau_S)}{A_m(\tau_S)^2} \end{bmatrix}. \tag{A134}$$

Small Investors Value Function at Time τ_S

Small investors value function conditional on distressed sales ΔQ_D at time τ_S is provided in equation (A115). Simplifying this expression using the same steps as for large investors shows that small investors value function is given by:

$$V_s(q_s, Q_s, Q^*, \tau_S) = K_s(\tau_s)F(Q^*, \tau_s)e^{-A_s(\tau_s)q_s - A_s(\tau_s)Q_s'[\bar{D}+P(Q^*,\tau_s)]+.5A_s(\tau_s)^2 Q_s'\Omega^*(\tau_s)Q_s}, \quad \text{(A135)}$$

where,

$$\begin{aligned}
A_s(\tau_s) &= A_s[1-(1/r)] && \text{(A136)}\\
K_s(\tau_s) &= \left(\frac{r}{r-1}\right) \times (\delta r)^{\frac{1}{r-1}} && \text{(A137)}\\
F(Q^*, \tau_S) &= e^{-Q^{*\prime}\bar{v}_s(\tau_S) - \frac{1}{2}Q^{*\prime}\theta_s(\tau_s)Q^*} && \text{(A138)}\\
P(Q^*, \tau_S) &= \frac{1}{r}\left(\frac{\bar{D}}{1-(1/r)} - \Gamma^*(\tau_S)Q^*\right) && \text{(A139)}
\end{aligned}$$

and,

$$\begin{aligned}
\Gamma^*(\tau_S) &= [\Gamma(\tau_s), \alpha^*(\tau_S)]\\
\bar{v}_s(\tau_S) &= 0_{[N(M+1)\times 1]}\\
\theta_s(\tau_S) &= \begin{pmatrix} \theta_s^* & \bar{v}_s^* \\ \bar{v}_s^{*\prime} & \Phi_s^* \end{pmatrix}.
\end{aligned} \quad \text{(A140)}$$

Because all three types of investors at time τ_S have value functions that have the same form as in proposition 2, it is possible to solve for their value functions and trades using the same approach as in section B.1 of the appendix.

B.8 Liquidity Received by the Large Investors

The paper uses two measures of the liquidity received by large investors, the first is the slope of the price schedule with respect to large investors trades, or β_{Q_B} from equation (A7). The second is the price response when a large investor has to sell a share of stock for exogenous reasons to the other large investors. I evaluate this price effect when all investors asset holdings are initially Q_W. To model this effect, I assume that the reaction function for the large investor with the exogenous shock takes the form $\Delta Q_m = -\iota_N$, and N-vector of ones. I substitute this reaction function for that of investor m in equation (A19) to solve for ΔQ_B. The resulting price effect comes from equation (A7) and is equal to $-\beta_{Q_B}\Delta Q_B$.

B.9 Competitive Benchmark Model

It is useful to contrast the behavior in the multi-market model with large investors with the behavior of asset prices and trades in the same model when all investors are price takers. The derivation of prices and trades in this case is a special case of the derivation with large investors. It is also a special case of the derivation in Stapleton and Subrahmanyam (1978). Therefore, I will not provide a detailed derivation, but will instead just provide results.

There are two cases to consider. The first is that investors are infinitely lived and trade risky assets forever. The second is that investors live forever, but that after period $T-1$ trade in risky assets ceases, and investors consume their dividends and invest in the risk-free asset.

Case 1: Infinitely lived investors / Risky Asset Trade in All Periods.

In this infinite period set-up with competitive markets, the equilibrium risk-premium should be time invariant. Denote this risk premium by ρ, where,

$$\rho = P(t+1) + \bar{D} - rP(t) \tag{A141}$$

Solving this equation forward while imposing the transversality condition $\lim_{t\to\infty} r^{-t}P(t) = 0$, shows that

$$P(t) = \frac{\bar{D} - \rho}{r-1}$$

for all time periods t.

Given the hypothesized behavior of prices, it remains to solve for ρ and then to show that the hypothesized behavior of prices is consistent with equilibrium.

The function,

$$V_m(W,t) = -\frac{r}{r-1}(A_m\ r\ \delta)^{\frac{-1}{r-1}} \exp^{-A_m(1-(1/r))W - \frac{.5\rho'\Omega^{-1}\rho}{r-1}}$$

where $\rho = \frac{(1-(1/r))\Omega X}{\sum_{m=1}^{M}(1/A_m)}$
satisfies the Bellman equation,

$$V_m(W,t) = \max_{\substack{C_m(t),\\ Q_m(t),\\ B_m(t+1)}} \quad -e^{-A_m C_m(t)} + E_t\{\delta V_m(W(t+1), t+1)\},$$

such that,

$$W(t+1) = Q_m(t)'(P(t+1) + D(t+1)) + B_m(t+1),$$

and

$$B_m(t+1) = r(W(t) - Q_m(t)'P(t) - C_m(t)).$$

In addition, agents optimal choices of $Q_m(t)$ at each time t satisfy the market clearing condition for the hypothesized ρ. Substituting the hypothesized ρ into the expression for equilibrium price, it follows that in a competitive equilibrium, the equilibrium price is given by

$$P(t) = \frac{\bar{D}}{r-1} - \frac{\Omega X}{r \sum_{m=1}^{M} \frac{1}{A_m}} \tag{A142}$$

Case 2: Infinite Lived Agents, but no Risky Asset Trade After Period T-1

In this case, the value of carrying risky assets into period T is given in equation (A53). At time $T-1$, when investors have this value function at time T, it turns out that the equilibrium price of risky assets at time T_1 is

$$P(T-1) = \frac{\bar{D}}{r-1} - \frac{\Omega X}{r \sum_{m=1}^{M} \frac{1}{A_m}}, \tag{A143}$$

which is the same as the price at time $T-1$ in case 2. Moreover, the price at all time periods before $T-1$ is the same as at period $T-1$.

From the equations for price, straightforward algebra shows assets excess returns are as given in proposition 3. In addition, provided that the prices of all assets are positive, the following corollary is also satisfied:

Corollary 1: *Assuming sufficient regularity to ensure that all risky-asset prices are positive at time t, then*

$$\mathrm{E}_t[R_i(t+1) - r] = \beta_i \, \mathrm{E}_t[R_m(t+1) - r],$$

where $\beta_i = \frac{\mathrm{Cov}_t[R_i(t+1), R_m(t+1)]}{\mathrm{Var}_t[R_m(t+1)]}$.

Proof: Let $P_m(t) = X'P(t)$ denote the price of the market portfolio at time t. Then, assuming $P(t) > 0$, $R_m(t) = \frac{X'[P(t+1)+D(t+1)-rP(t)]}{P_m(t)}$, and from equation (26), $E_t[R_m(t) - r] = \frac{\lambda_x X'\Omega X}{P_m(t)}$ and $\mathrm{Var}_t(R_m(t)) = \frac{\lambda_x X'\Omega X}{P_m(t)^2}$. Using analogous reasoning, $(R_i(t+1) - r) = \frac{P_i(t+1)+D_i(t+1)-r}{P_i(t)}$, and from equation (26) $\mathrm{E}_t[R_i(t+1) - r] = \frac{\lambda_X \iota'_{i,N}\Omega X}{P_i(t)}$, and $\mathrm{Cov}_t[R_i(t+1), R_m(t+1)] = \frac{\iota'_{i,N}\Omega X}{P_i(t)\ P_m(t)}$. It then follows that:

$$\begin{aligned}
\mathrm{E}_t[R_i(t+1) - r] &= \frac{\lambda_X \iota'_{i,N}\Omega X}{P_i(t)} \\
&= \frac{\mathrm{Cov}_t[R_i(t+1), R_m(t+1)]}{Var_t[R_m(t+1)]}\ Var_t[R_m(t+1)]\ \lambda_X\ P_m(t) \\
&= \beta_i E_t[R_m(t+1) - r]\Box.
\end{aligned}$$

C Details on The Simulations

For the simulations reported in section 4.2, and for the analysis on distressed sales, the instantaneous risk aversions of investors 1 through 6 are 10, 1.8328889, 4.5822222, 11.455556, 28.638889, and 71.597222, respectively. This corresponds to the risk tolerances of large investor $m+1$ having a risk tolerance equal to 0.4 times the risk tolerance of large investor m for $m = 2$ to $m = 5$. The annualized risk free rate is $r = 1.02$; the annualized discount rate is $\delta = 0.9$; annualized dividends are normally distributed with mean$\bar{D} = 1$ and variance $\Omega = 1$. These parameters have to be scaled based on the trading frequency. I assume trades occur once a day and that there are 250 trading days per year, which corresponds to a period length of $h = 1/250$. The appropriate scaling of interest rates and discount rates are r^h and δ^h. For daily dividends, the mean and variance are $\bar{D}h$ and Ωh. Finally, each participants daily risk aversion is scaled to be A/h. The number of outstanding shares of assets only affects the level of prices, but not price dynamics or risk premia. For simplicity, the supply of assets is normalized to 1. When there are distressed sales, the total amount of distressed sales is 5.5 shares.

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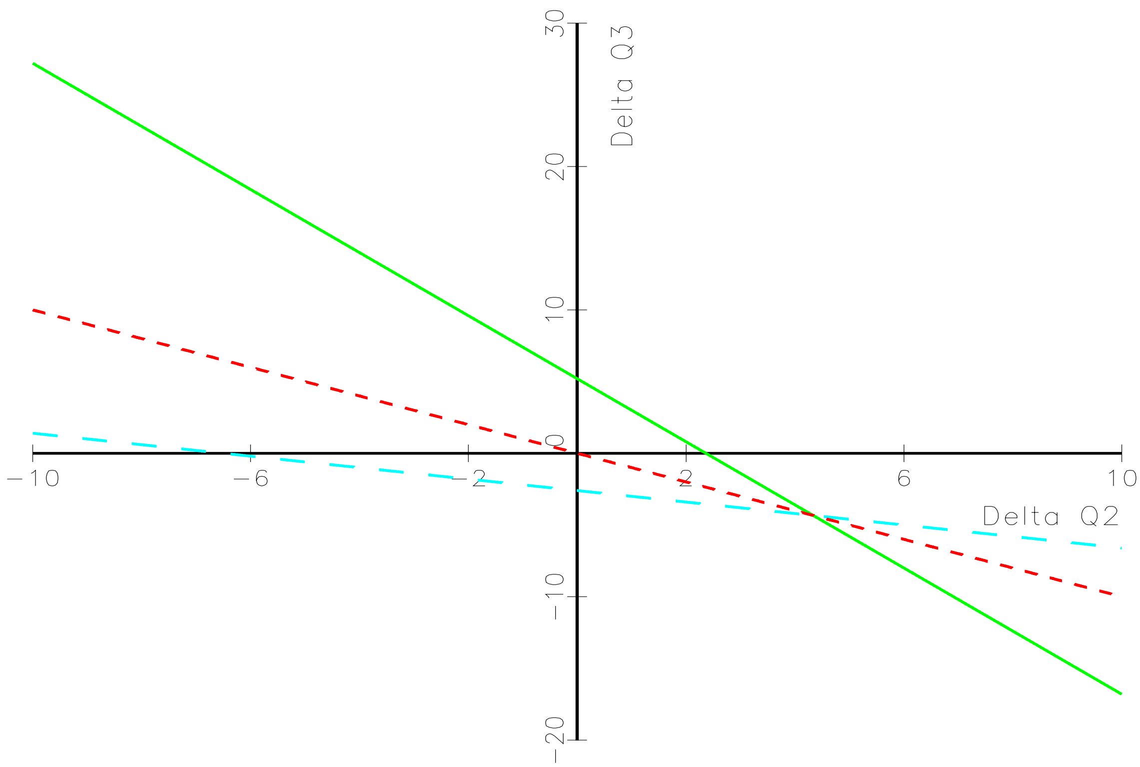
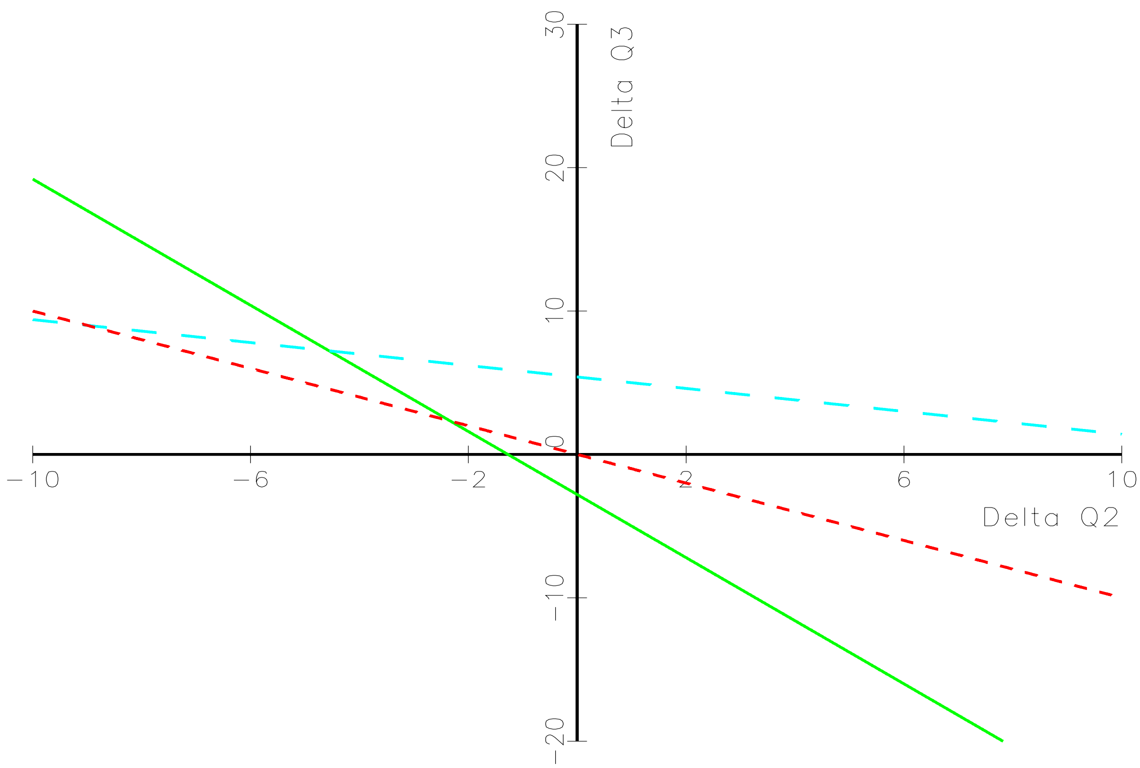
Figure 1: **Reaction Functions when $A_2 = 0.2$ and $A_3 = 0.5$, $Q_2 = -11$, and $Q_3 = 19$**



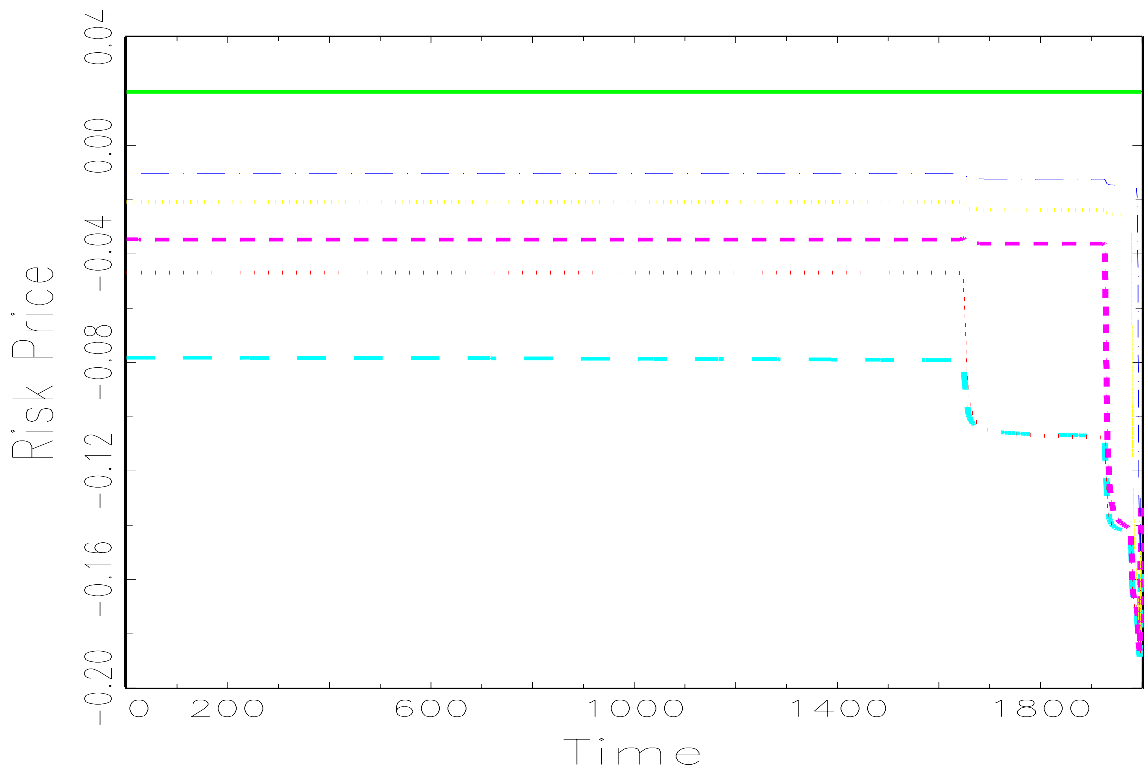
Notes: For the basic model in section 2, the figure presents reaction functions for the optimal trades (delta q2 and delta q3) for large investors 2 and 3 (solid and line with long dashes). The line through the orgin (short dashes) is the line delta q2 + delta q3 = 0, and denotes the situation where the net trade between the large and small investors is 0. If the equilibrium trades among large investors is to the north east of this line, then the net purchases from small investors is positive. In the figure, given large investors initial endowments (q2 = -11, q3 = 19) the optimal net trade that the large investors make with the small investors is 0.

Figure 2: **Reaction Functions when $A_2 = 0.2$ and $A_3 = 0.5$, $Q_2 = 29$ and $Q_3 = -21$**



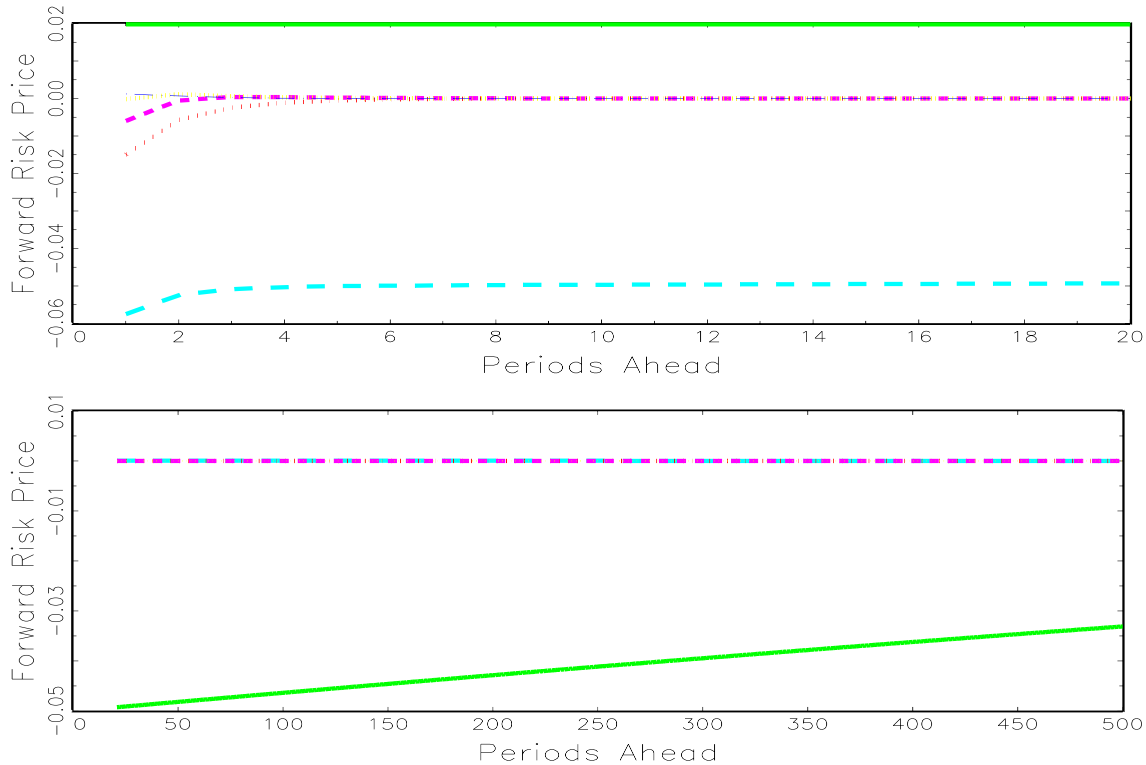
Notes: For the basic model in section 2, the figure presents reaction functions for the optimal trades (delta q2 and delta q3) for large investors 2 and 3 (solid and line with long dashes). The line through the orgin (short dashes) is the line delta q2 + delta q3 = 0, and denotes the situation where the net trade between the large and small investors is 0. If the equilibrium trades among large investors is to the north east of this line, then the net purchases from small investors is positive. Relative to the asset holdings of investors 2 and 3 in Figure 1, in this figure risky assets have been transferred from investor 3 to investor 2, and as a result the net trade that the large investors make with the small investors is positive.

Figure 3: **Prices of Risk**



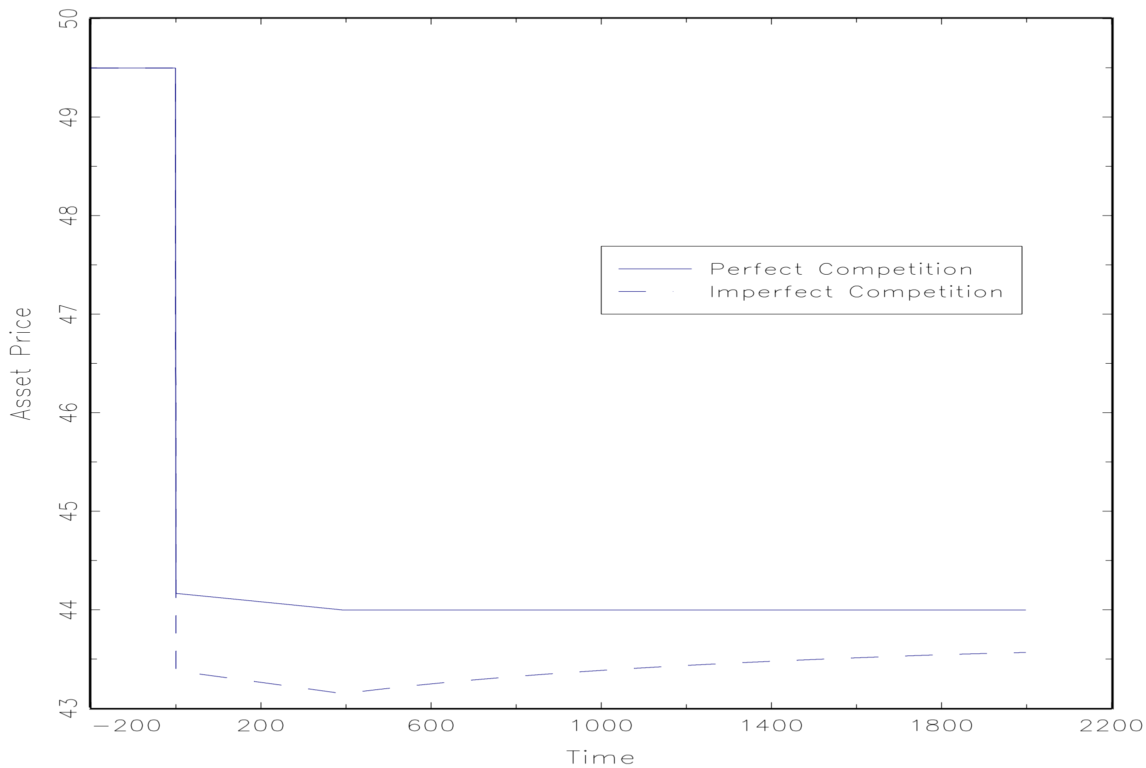
Notes: The figure presents the time-series behavior of equilibrium prices for factor risk, where the factors are the market portfolio, and deviations of each large investors risky asset holdings from those associated with perfect risksharing. The price for the market factor is denoted by the solid line. The large investors vary in their risk aversion, investor 2 is the least risk averse and investor 6 is the most risk averse. The risk prices also vary by risk aversion. The lowest risk prices (most negative) correspond to investor 2 while the highest nonpositive risk price corresponds to investor 6.

Figure 4: **Forward Prices of Risk**



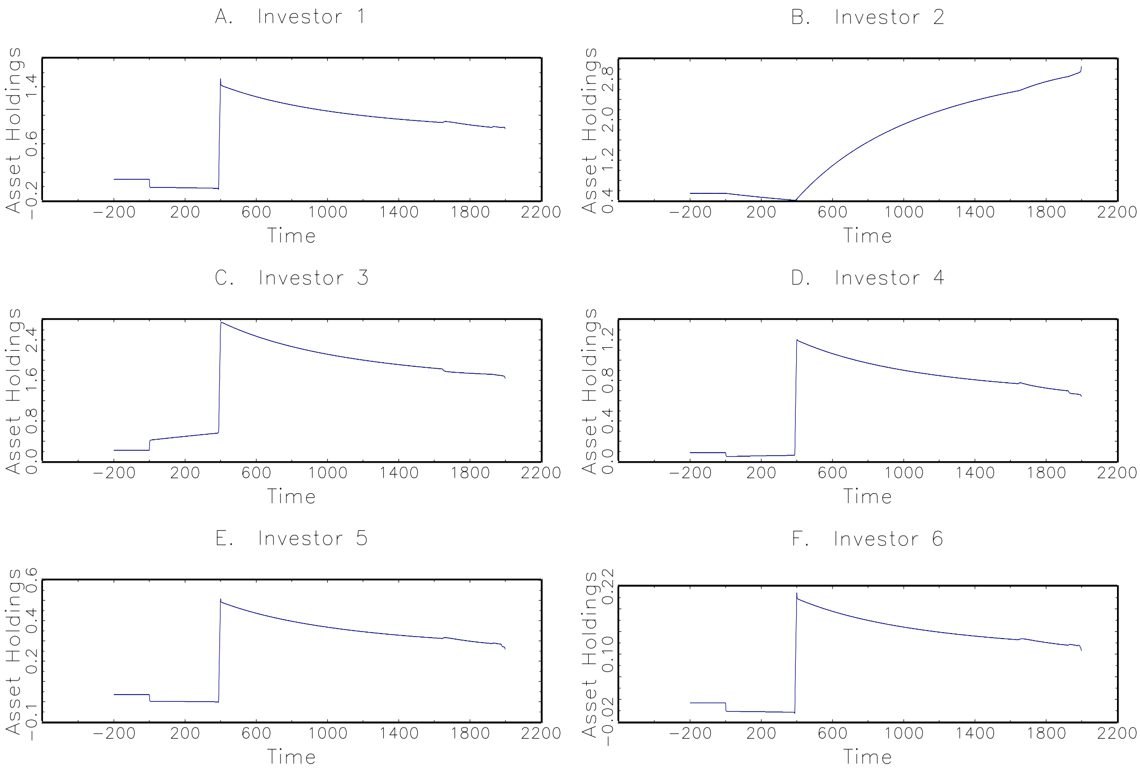
Notes: The figure presents the time-series behavior of equilibrium forward prices for factor risk, where the forward price measures the effect that deviations from optimal risk sharing at time t have on asset returns at forward time $t + \tau$. In the figure $t = 1000$. The deviations from optimal risk sharing correspond to risk factors as discussed in figure 3. The forward price for the market factor is denoted by the solid line. The large investors vary in their risk aversion, investor 2 is the least risk averse and investor 6 is the most risk averse. The forward risk prices also vary by risk aversion. The lowest forward risk price in each panel (most negative) correspond to investor 2 while the highest nonpositive forward risk price corresponds to investor 6.

Figure 5: **Equilibrium Price Paths**



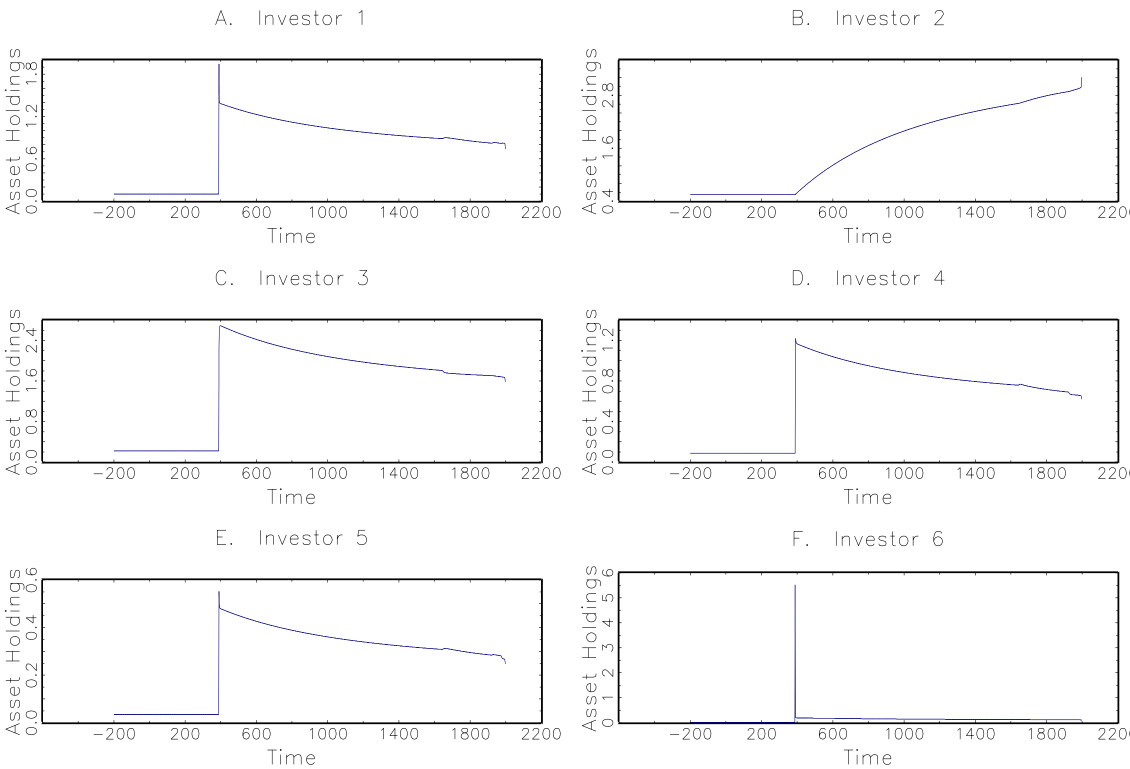
Notes: The figure illustrates equilibrium price paths when investors hear a rumor at time 0 (which they know to be true) that future distressed sales will occur during time periods $390 - 400$. Price dynamics are presented when asset markets are competitive and when they are imperfectly competitive. The imperfectly competitive price path assumes that large investors vary sharply in their risk aversion. The associated trades are provided in figure 6. Further details are in section 5 in the text.

Figure 6: **Equilibrium Trades when there is a Distressed Investor**



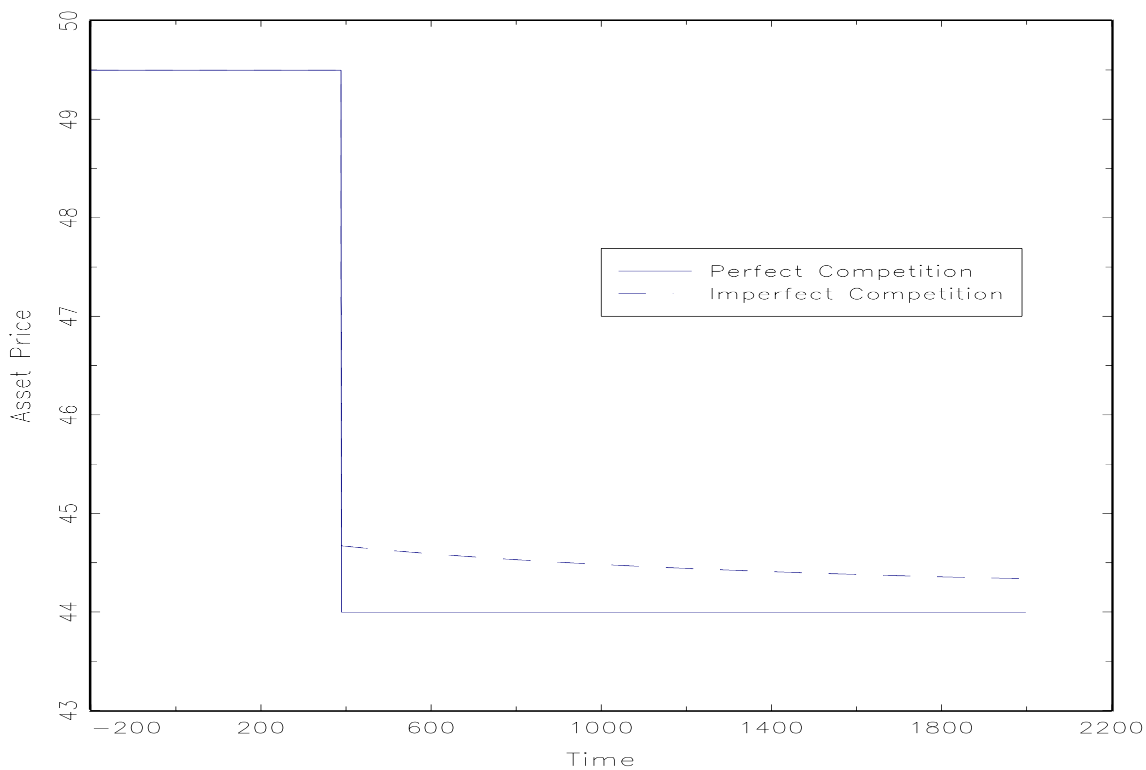
Notes: When markets are imperfectly competitive, the figure presents the equilibrium trade patterns that emerge in response to the rumor of distressed sales at time 0 followed by the distressed sales from time periods 390-400. Investor 1 takes prices as given; investors 2-6 are non-price taking investors who differ in their risk aversion. Large investors risk aversion is increasing in investor number. Details on the trade patterns are provided in section 5 of the text.

Figure 7: **Trade Response: Endowment Shock to Investor 6**



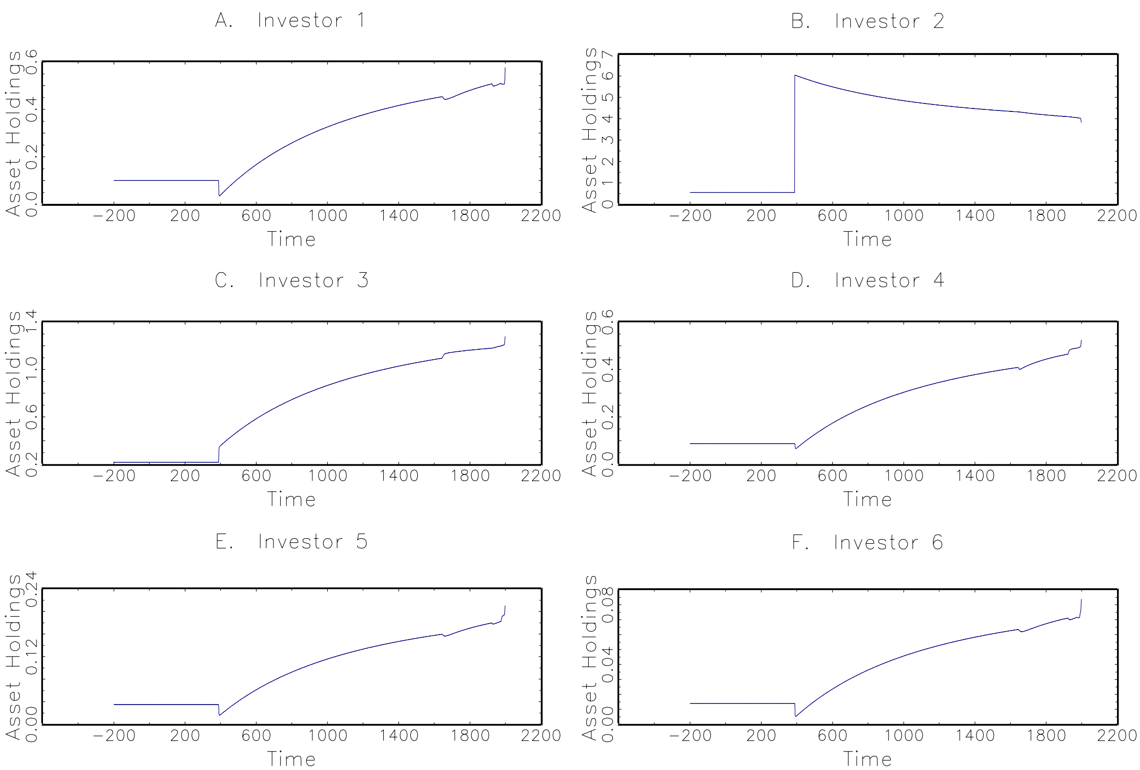
Notes: The figure presents the equilibrium trade response when large investor 6 receives a positive endowment shock at time 400, and is then free to sell part of her endowment through time to other investors.

Figure 8: **Price Response: Endowment Shock to Investor 2**



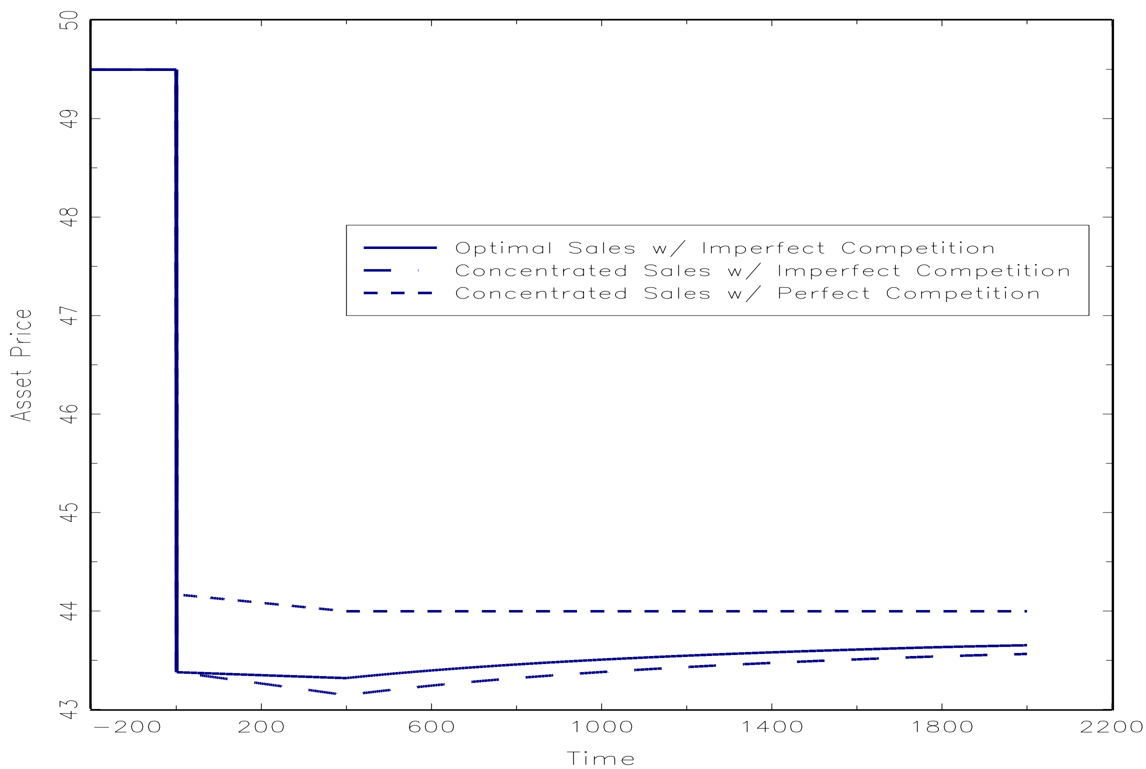
Notes: The figure presents the equilibrium price response when large investor 2 receives a positive endowment shock at time 400, and is then free to sell part of her endowment through time to other investors. Equilibrium trades are presented in figure 9.

Figure 9: **Trade Response: Endowment Shock to Investor 2**



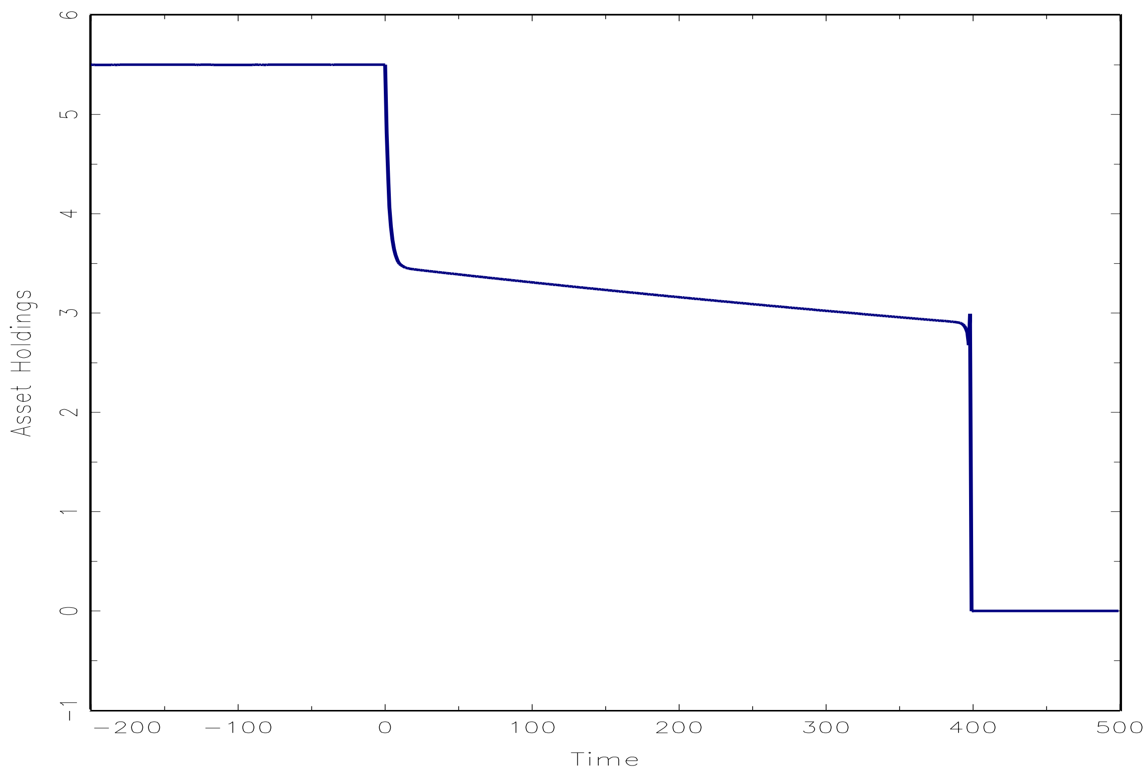
Notes: The figure presents the equilibrium trade response when large investor 2 receives a positive endowment shock at time 400, and is then free to sell part of her endowment through time to other investors. Equilibrium prices are presented in figure 8.

Figure 10: **Price Path with Optimal Liquidation**



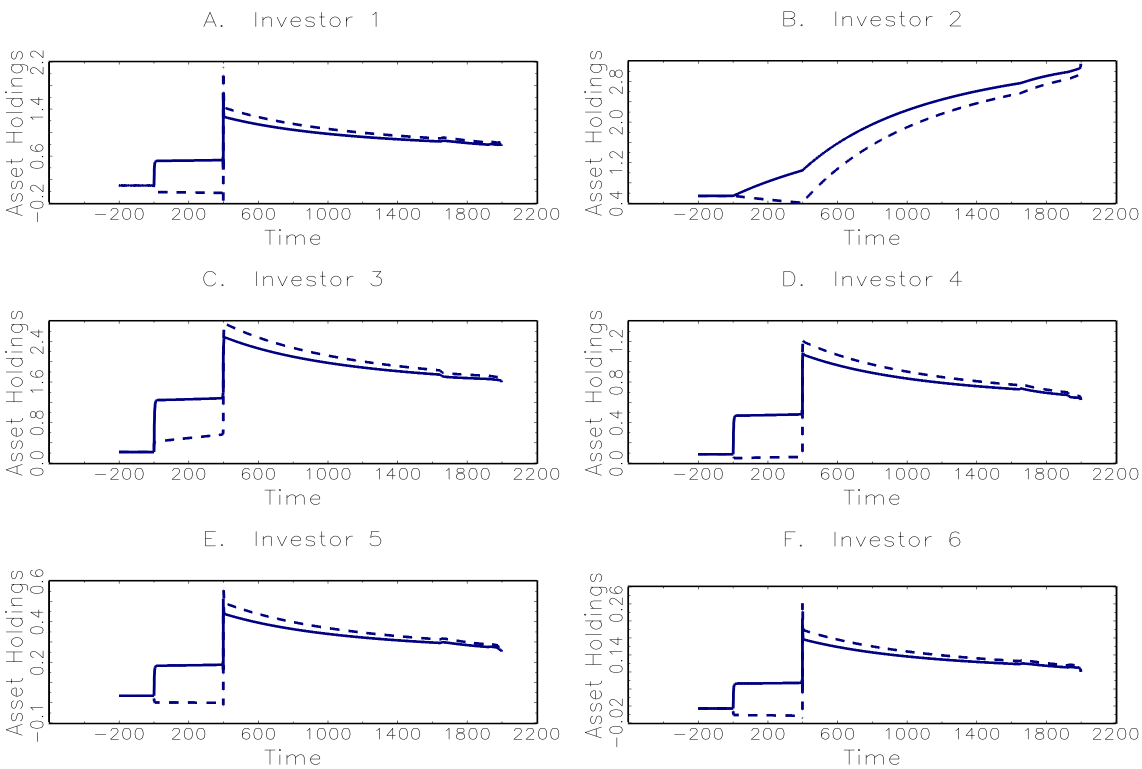
Notes: The figure presents price paths when large investors learn at time 0 that one large investor must liquidate his risky asset position by time 400 and then exit the market. The liquidation scenarios are optimal sales into illiquid, imperfectly competitive, markets (solid line); sales into illiquid markets that are concentrated at time 400 (dashed line); and sales into liquid, perfectly competitive, markets that are concentrated at time 400 (short dashed line). The asset holdings for the optimally liquidating large investor are presented in figure 11, and the asset holdings for the other investors are presented in figure 12.

Figure 11: **Trades by Optimally Liquidating Distressed Investor**



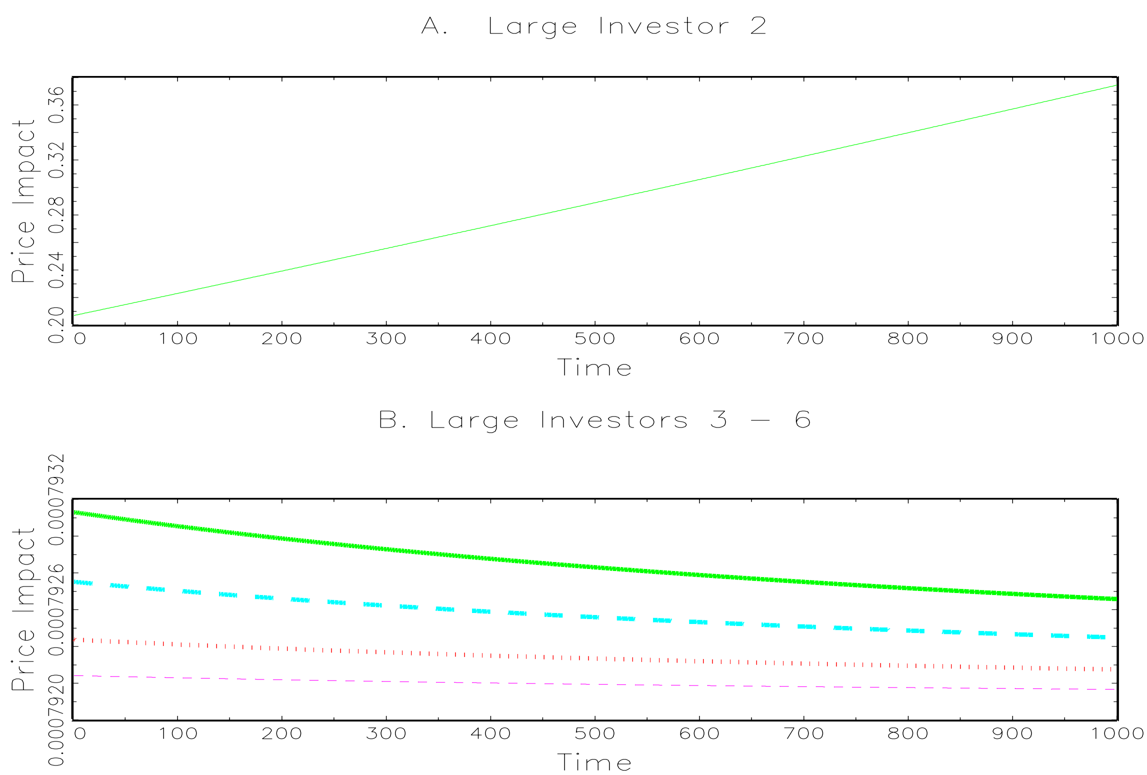
Notes: The figure presents the path of optimal asset holdings for a large investor when all investors learn at time 0 that he must liquidate his risky asset position by time 400 and then exit the market. The optimal asset holdings for the other investors are presented in figure 12. The equilibrium price path is presented in figure 10.

Figure 12: **Investors Asset Holdings During an Optimal Liquidation**



Notes: The figure presents other investors paths of equilibrium asset holdings when one large investor learns at time 0 that he must liquidate his risky asset holdings by time 400 and then exit the market. Asset holdings are presented for a scenario in which the liquidating investor can follow an optimal liquidation strategy (solid lines), and for a scenario in which his asset sales are concentrated at time 400 (dashed lines). The optimal trades for the liquidating investor are presented in figure 11. The price paths for the optimal and concentrated sales scenarios are presented in figure 10.

Figure 13: **Slope of Price Impact Functions by Large Investor**



Notes: The figure presents the price impact of large investors trades through time. Price impact is the slope of the demand curve that large investors face when deciding to purchase risky assets. The slope measures the per unit change in asset prices if the large investor buys 1 additional share when the positions of the other large investors are held fixed. When there is more than one large investor, the price impact per share sold varies by large investors risk aversion. More risk tolerant investors have a larger slope, i.e. their trades have a larger price impact.