Trading Strategies at Optimal Frequencies: Theory and Evidence

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Abstract. This paper studies a continuous-time stochastic game of trading activity in financial markets under asymmetric information. The model has the following features. First, informed and liquidity traders optimally control the timing of their order submissions. Second, they continuously choose whether to take or provide liquidity, issuing market or limit orders. Third, uninformed traders learn from the order flow optimally exploiting the information in the limit-order book. I construct an equilibrium in this setting and characterize (i) price formation and (ii) how optimal submission intensities and liquidity supply-demand behavior depend on private information and market conditions. I show that informed traders actively use both market and limit orders in equilibrium resulting in time-varying informed liquidity supply. After an information event, inter-arrival times are positively autocorrelated, the price impact of all order types increases and the limit-order book shows depth unbalances. The model nests the cases in which liquidity provision is decentralized (limit-order markets) and centralized (dealer markets). I find that the speed of information transmission into prices is lowered in the decentralized version. Since submission intensities are strategic, the time spacing of the data can be used to learn about traders’ optimal policies. I use a unique order-level dataset from the NYSE to test structural restrictions on optimal liquidity provision and find support for the model’s outcomes. The findings shed light on empirical and experimental results in the literature and have implications for inference methods with high frequency data.

Keywords: Market microstructure, asymmetric information, limit order book, insider trading, liquidity, market makers, limit orders, market orders, interarrival times, bid-ask spread.

JEL Codes: G12, G14, C63, C73, D82

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1 Introduction

In modern asset markets, participants interact with each other through asynchronous orders specifying conditions under which a trader is willing to buy or sell any given security\(^1\). The dynamic interaction of trading interests contained in orders shapes the evolution of liquidity, allocations and price formation. Given this central place in the exchange process, it is natural to consider the problem of optimal order submissions and its relation to the market outcomes we observe. This paper addresses these issues and contains the following contributions. First, I introduce a new framework to study the interrelated dynamic order decisions problem of heterogeneously informed traders in a market for a risky asset. The framework reproduces some of the salient features of important world exchanges where liquidity provision is decentralized (open limit-order books) or centralized (dealer markets\(^2\)). In the decentralized version, traders not only have control over the times of their submissions but they can choose between limit and market orders. Second, I show how decisions are affected in equilibrium by market conditions, private information and elements of market design. Optimal liquidity demand-supply schemes and price formation are analyzed. Third, I use simulations to characterize how the interaction of traders with different information shapes the joint evolution of market observables and derive empirical implications. Fourth, I conduct a comparative market design experiment to study the speed of information aggregation into prices. I find that this speed is lowered in markets where liquidity provision is decentralized. Fifth, using a unique order-level dataset from the NYSE, I test structural restrictions on optimal liquidity provision and provide new evidence that supports the model’s outcomes.

The specific setting considered is a continuous time market for a single risky asset with risk-neutral agents in a common value environment. I focus on the interactions that arise during a single information epoch: a new piece of information arrives at the market at time zero, it is observed a by single trader and is revealed to the market at large at a random time in the future. The informed agent seeks to maximize her profits by designing an optimal order placement strategy. Uninformed liquidity traders have certain trade pre-commitments but act strategically to minimize their expected costs. These agents observe the order flow and the state of the limit-order book and update their beliefs about the asset value using Bayes rule. I analytically derive the conditions that characterize a stationary equilibrium in Markovian strategies and provide an existence result. I use numerical methods to find agents’ optimal submission policies in equilibrium.

A notable feature of equilibrium behavior is that, independent of trading motives (information or liquidity), agents optimally play a double-role in a decentralized market, demanding and supplying liquidity simultaneously. That is, they simultaneously submit market orders and limit orders. The optimal frequency of submission of each order type, and the optimal demand-supply attitude for each type of market participant, depend on the value and expected duration of private information, and the state of

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\(^1\) Although other order types exist, the canonical types are market orders and limit orders. A market order is a trading instruction to buy or sell a given number of securities at the best available price at the market immediately. A limit order is a commitment to buy or sell a given number of securities only if the price reaches a certain value. Limit orders that are not yet executed are posted in a central file called a limit-order book. Orders remain in the book until they are canceled or find a matching trading counterpart (a market order with the opposite trading sign). We call a limit-order book ‘open’ when its status is visible to traders.

\(^2\) The canonical representation of financial markets with liquidity intermediaries or market makers at the center of the exchange process is found in the landmark papers of Kyle (1985) and Glosten and Milgrom (1985).
the market as summarized in the limit-order book. Traders’ behavior is intuitive and relates to a key trade-off between order types: execution uncertainty versus price improvement. Liquidity traders, at any given point in time, find a better price when they buy or sell the asset using limit orders. However, when order flow on the opposite side of the market is weak, they anticipate not having enough orders executed so as to meet their quantitative targets on timely basis. In such situations, they demand immediacy using market orders. Similarly, the informed trader finds it optimal to seek a better execution price using limit orders. The fear of losing her information advantage, in turn, stimulates the use of market orders to secure profits. Moreover, she seeks to optimally hide her trades by actively managing the relative use of each order type at any moment.

Order decisions are affected by market conditions and the elements of each trader’s information set. For an informed trader (also referred as the speculator), the single most important market factor is the discrepancy between the fundamental value of the asset, known to her, and the marketwise perception of that value. For any given state of market variables, the submission frequency of informed market and limit orders are inversely related to the degree of mispricing. Market orders are most sensitive to this factor.

One of the most powerful aspects of the model is that we can explicitly study traders’ optimal behavior as a function of the state of the limit order book (LOB, hereafter). Trading decisions depend on the number of orders posted on each side of the LOB (generally referred as depth) and the proportion of orders maintained by the speculator on any side of the LOB (referred here as the information content of the LOB). I find the following patterns for the order submissions of the informed trader. First, the frequency of limit and market orders decreases with the depth of the LOB on the same side of the order. Second, the submission frequency of market and limit order decreases and increases, respectively, with the depth of the LOB on the opposite side of the order. Third, market and limit order submissions are relatively less frequent whenever the information content of the LOB on the same side of the order is higher, or when the depth of both sides of the book increase evenly. I find the following patterns for the order submissions of uninformed orders. First, the frequency of uninformed limit and market orders decrease with the depth of the LOB on the same side of the order. Second, market order submissions are less frequent whenever the depth of the book increase evenly. Third, limit orders are more frequent, on average, in states of the market in which bid-ask spreads are higher\(^3\).

Since the informed trader actively uses limit orders in her optimal placement strategy, a market-making role for her arises endogenously in the model. In fact, for any given state of the LOB, we observe that when the degree of mispricing is sufficiently low, she submits limit orders more frequently than market orders. Since prices adjust dynamically to reflect information on the equilibrium path, this gives a time-varying characterization of speculator’s liquidity provision service. At early stages of the trading process, when the value of private information is high, the speculator trades aggressively with market orders. At advanced stages, when the value of information is low, she finds relatively more attractive to increase the relative frequency of limit orders. This behavior contrasts sharply with numerous examples of informed traders in the literature as strict consumers of liquidity. On the other hand, it is consistent

\(^3\)Since Biais et al. (1995), many empirical studies have documented that limit orders are less likely when the book is thick and more likely when the bid-ask spread is wide.
with the experimental evidence in a recent study by Bloomfield, O’Hara and Saar (Bloomfield et al., 2005).

Beyond price adjustments to information and the evolution of the limit order book, what determines speculator’s relative willingness to supply liquidity? I define a simple measure of equilibrium behavior that weights the relative use of limit orders to explore this question, and find the following. First, the informed trader is more willing to provide liquidity when her specific information is of longer expected duration. Insider knowledge about two companies merging that is likely to be announced next month, for example, will find the informed trader acting mainly as a market maker issuing limit orders. Short-lived information, like a quarterly earnings report that is likely to be released around 10:30am today, will find the informed trader executing outstanding limit orders aggressively. Second, an increase in the tick size increases (weakly) the informed trader’s relative liquidity provision for any given state of the market. Naturally, this effect is more important when bid-ask spreads are tight.

Uninformed liquidity providers (limit-order traders and dealers) monitor the market evolution and revise their beliefs about the asset value sequentially, extracting the information content of the order flow. This is a familiar feature in the class of sequential screening models of trade pioneered by Glosten and Milgrom (1985). However, in a decentralized market the task of uninformed liquidity providers is more complex. Given that informed traders may well post limit orders, uninformed liquidity providers need specific updating rules for each order type. Moreover, they may find it optimal to incorporate in their belief revisions the information contained in the limit order book as well. In doing so, they understand that different book configurations affect the dynamic incentives that each type of trader faces. They also understand that the unobservable proportion of informed orders in the book will shape informed order arrival rates. To exploit this last factor, they construct probability distributions over conceivable information contents of the LOB as an equilibrium product. I show the form of the pricing expressions that solve this problem in Section 4, and I characterize numerically how they change with the evolution of market states in Section 5. In equilibrium, the price impact of market (limit) orders is higher (lower) on average when the book is more populated. Since the price impact of orders can be measured empirically, the analysis provides a rich set of testable implications for periods in which asymmetric information is very likely (e.g. around earning announcements).

What can be expected in a continuous market out of the interaction of the different sort of agents? How do information asymmetries shape the expected evolution of observables? To explore these questions I use model’s equilibrium outcomes to simulate trading sessions and characterize the joint dynamics of market prices, quantities, orders and arrival times. In the advent of an information event, market and limit orders arrive more frequently and the time between arrivals show positive autocorrelation. This effect is driven by the speculators order placement, which decreases as trade progresses and prices approach the true asset value. The price impact of all orders type increases. Market orders have relatively more price impact in early stages, and limit orders in late stages. The bid-ask spread widens initially and decays progressively until it reaches the tick size. The order book, in turn, shows depth unbalances. In particular, when good news arrive the depth of the bid side increases faster than the one on the offer side and viceversa.

There is an empirical literature that studies the relationship between these variables. Consistent with
the results here, Engle and Dufour (2000) document that the price impact of orders is higher when the 
frequency of arrivals of marketable orders (those that induce immediate trades) is higher. The analysis 
in this paper further predicts that a similar pattern should be observed when non-marketable orders are 
considered. Ellul et al. (2007) find that when the depth of the limit-order book is unbalanced, the 
probability of limit-orders arriving on the thick side is higher. This paper suggests that the presence of 
long-lived informed traders may drive that empirical regularity. Even when traders’ marginal incentive 
to submit a limit orders does not increase with the depth of the book on the same side of the order, the 
profit-maximizing submission plan of the speculator generates such a dynamic pattern in our model.

In this framework, order arrival times are controlled stochastically by market participants\(^4\). In addition 
to offering new insights about strategic behavior, this feature brings a powerful econometric tool: observed 
order arrival times can be used to learn about traders’ optimal policies. I exploit this insight to test 
the following equilibrium implication: for any given set of conditioning states, limit order submission 
intensities form a monotone decreasing sequence with respect to the LOB’s depth on the same side of the 
order. The structural restriction has a natural economic intuition: traders find liquidity provision more 
attractive when the competition they face is less intense.

To conduct the test, I use a proprietary dataset from the NYSE’s System Order Data files which 
contains the complete record of orders (by the second) for twenty stocks (stratified by volume of orders) 
submitted during two weeks of May 2002. I design a suitable test statistic and use bootstrap methods 
for Markovian processes to approximate its distribution. I cannot reject the monotonicity restriction 
on buy orders for nineteen of the symbols and cannot reject it for any symbol on sell orders. I cannot 
reject the restriction for nineteen of the symbols when buy and sell orders submissions are tested jointly. 
Results also show that p-values are higher, on average, for stocks that are relatively more active. This 
fact indicates that, precisely for those stocks where competition in liquidity provision is more likely 
 to be a concern for participants, the model’s implied behavior on liquidity provision provides a better 
fit. Individual orders submitted by uninformed and informed traders cannot be differentiated by simply 
looking at arrivals in the dataset. However, to gain further insight I also test the conditions separately 
for situations in which asymmetric information is more and less likely. The structural restrictions cannot 
be rejected in either case, suggesting the following. First, liquidity traders, when we believe there are no 
asymmetries, submit limit orders conforming to the model’s implications. Second, when asymmetries are 
likely, uninformed and informed traders jointly do so as well. In addition to offering new evidence about 
traders’ behavior, the empirical exercise outlines ways to use the model for further econometric work, 
profiting from non-parametric estimation techniques that are fairly simple to implement.

The model is flexible enough to capture important elements of different microstructures. This feature 
allows us to compare alternative market designs within a single framework. Although comparative market 
design is not the primary goal of this paper, I exploit this feature to study one of the central questions 
in the micro-structure literature: how fast does information incorporate into prices? I conduct a simple 
experiment in which I calculate the distribution of the elapsed time between the arrival of new information 

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\(^4\)The presence of long-lived agents choosing optimal times for submitting orders can be seen as a formalization of the 
conjecture about ‘potential’ liquidity providers in Biats et al. (1995). Their empirical study suggests the existence of traders 
monitoring the market continuously and quickly responding to changing conditions by placing new limit orders or seizing 
trading opportunities.
and the time when the market price reflects it. I find that, controlling for trading volume, the speed of information transmission into prices is lower, on average, in a hybrid market (with dealers and a limit-order book) than in a pure dealers’ market. This suggests that a limit order market, by expanding traders’ set of possible actions, permits speculators to design trading strategies that delay information revelation further, a task which is in their own interest. This finding, may add more elements to the understanding of the trade-offs market regulators and exchange directors face in selecting trading venues. A limit-order market may provide liquidity better in situations of high distress, since informed traders find a market-making role. A dealer market, on the other hand, may aggregate information faster.

The motivation for endogenizing the time of order arrivals is largely empirical. Beginning in the nineties, financial markets’ data sets at the transaction level have been made available to scholars, spurring interest in empirical market microstructure. One of the most important features of this type of data is that virtually all transactions are irregularly spaced in time. This single feature fostered an important econometric literature pioneered by Engle and Russell (1998). These authors argue that financial data can naturally be seen as a marked point process. The arrival times form the points and the characteristics of the orders form the marks. A model of high-frequency financial data is then a set of restrictions on the distribution of both points and the marks. Numerous statistical models of these type have been proposed. A fundamental insight of this literature is that the time spacing of the data carries information (Engle and Russell, 2004).

However, the research that analyzes this key strategic dimension of trading activity in decentralized markets remains essentially statistical. There seems to be no structural framework in which market and limit order arrival times are linked to rational economic decisions. The model I propose heads in that direction. Instead of modeling points and marks directly using a statistical framework, I build from agents’ trading objectives and information. The distributional restrictions arise thus endogenously from the equilibrium of a dynamic game of incomplete information. The distribution of the points depend on the collection of traders’ optimal submission intensities at any moment. The distribution of the marks (order types) depend on the relative willingness of participants to take or provide liquidity. The equilibrium learning rule of liquidity providers, in turn, drives the price mark. Such structural approach may prove useful in empirical applications where the identification target is a hidden quantity linked to asymmetric information. Examples include the identification of the probability of an information event and properties of micro-structure noise in the estimation of integrated volatility.

Even when the interrelated dynamic problem that this paper deals with is fairly complex, there is no pretension of modeling every aspect of trading activity. Interesting strategic dimensions like order exposure are not analyzed. For tractability, we also restrict order size and submission price choices. Further, there is no pretension of fully modeling a real limit-order book. The standpoint taken in this paper emphasizes parsimony. The model considers a very stylized setting that seeks to captures three key aspects of such markets. First, the decentralized character of liquidity provision. Second, the ability

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5 Prominent examples are NYSE’s TAQ, TORQ and OpenBook, ECNs, Nastraq, London QMG, etc.
7 The model provides a simple consistent framework to incorporate order-level data in the development of these empirical applications. In doing so, one can exploit a much richer source of information. In the NYSE data set in this paper, for example, about 85% of all arriving orders are limit orders. Most of them are non-marketable and thus very hard to identify from a transaction-level dataset like NYSE’s TAQ.
of participants to choose the times of their order submissions during a continuous trading session. Third, the capacity of agents to observe the pending trading intentions of other market participants. I consider the least number of state variables that describe the accumulation of orders and that are able to drive agents’ time-varying incentives.

The remainder of the paper is organized as follows. Section 2, reviews the related literature. Section 3 introduces the general setup and notation. In section 4 the strategic problems of the different agents in the model is developed. The main results of this section are the conditions that characterize the equilibrium. Section 5 contains the main results about equilibrium behavior. Section 6 discusses implications of equilibrium behavior for market observables. Section 7 investigates the speed of information transmission in different market designs. Section 8 contains the description of the empirical exercise and results. Finally, section 9 concludes. All proofs are relegated to the appendix.

## 2 Related Literature

The literature that deals with order choices has been very active in recent years. The objective of this section is not to offer an exhaustive revision but to place the present paper in context, in light of the closest studies. Excellent recent surveys include Bias, Chester and Spatt (2005) and Parlour and Seppi (2008).

Order decisions in limit-order markets have been studied recently by Parlour (1998), Foucault (1999), Goetler Parlour and Rajan (2005) Foucault, Kandal and Kandel (2005), Hollified, Miller and Sandas (2004) and Rosu (2007). Since the trading process in a limit-order market is too complex to be fully modeled, each of these articles sheds light on different aspects. Their analysis differ from the framework presented here in one or more of the following respects. First, these papers deal with symmetric information environments. Second, some of these papers consider the dynamics of prices and quantities, but in a context where agents take a single order decision (or single trade) and disappear from the marketplace afterwards. While useful as a first approach to a fairly complex problem, this modeling standpoint cannot take on the type of time interrelated order placement problems that traders face in real markets. A well known instance of the type of behavior that cannot be fitted by a single action setting is the "working" of orders, a strategy by which a given trader manages a certain order by carefully placing many smaller orders over time. Third, these papers assume that traders arrive at the market at exogenous times. In compensation, working with such decision settings permits in some cases the incorporation of additional (more realistic) elements of the market microstructure not considered by this paper. Fifth, traders’ optimal decisions in this paper are not related to the introduction of special preferences, like heterogeneity of patience or random valuations for the asset.

This paper is closer in spirit to Back and Baruch (2007). These authors developed a model in which a long-lived informed trader solves a dynamic order submission problem, endogenizing the arrival times of her market orders. Their model is fundamentally different from this one in that traders with different information sets have different action spaces. In particular, a trader with superior information will only

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use market orders by construction. Moreover, uninformed order submissions are pure noise in their analysis. Here, the trading preferences and constraints of liquidity traders are modeled explicitly.

In research developed independently from mine, Goetler Parlour and Rajan (2008) propose a dynamic model in which informed traders can opt to place a single market or limit order (which could be revised later). Action times remain restricted though, whereas here they are optimal. They also observe that informed traders provide liquidity, and provide relatively more liquidity when the value of information is low.

The empirical literature on order submissions has almost entirely relied on reduced-form regressions. An exception is Hollifield et al. (2004) which tests a structural restriction between order execution probabilities and traders’ asset valuations using data from a single symbol in the Stockholm Stock Exchange.

3 The Model

3.1 The Market

Let the triple \((\Omega, \mathcal{F}, \mathbb{P})\) be a probability space. We consider a continuous time market for a single risky asset with Bernoulli distributed common liquidation value \(v \in \{v_L, v_H\}\). It is convenient to normalize \(\{v_L, v_H\} = \{0, 1\}\). The asset yields no dividends and all trades are anonymous.

There are two main classes of risk-neutral market participants in the market: dealers and traders.

**Dealers.** There is a large number of competitive liquidity providers (also called market makers or specialists in the paper) which post quotes for one unit of the asset at any moment of time. There is no stock accounting.

**Traders.** These market participants trade for information or liquidity reasons. To meet their objectives, they can issue both market and limit orders at any moment of time. All orders are of size one. They can trade with dealers or with other limit order traders.

3.2 Trading Venues

3.2.1 Limit Order Book

Ordinary traders can provide liquidity by issuing limit orders. Limit orders accumulate in a file called the limit-order book (LOB) until they are executed or cancelled. The LOB we consider is open: all the variables that describe its state at any moment (not including traders’ identities) are observable to any market participant.

**Depth:** Since all orders are of unitary size, the depth of the LOB at any price level equals the number of outstanding order at that price level.

**Execution Rules:** There is a strict price priority rule in the market: orders posted at more competitive prices execute first. There is no time priority. There is instead a proportional allocation rule: any two outstanding limit order posted at the same price have the same probability of being executed when a market order of the opposite sign arrives\(^9\).

\(^9\)Some exchange markets that operate with a limit-order book incorporate proportional allocation rules, usually with some form of time priority as well. A prominent example is that of the Chicago Mercantile Exchange’s Globex platform.
**Limit order prices:** There are $M \in \mathbb{N}_0$ acceptable prices on any side of the LOB. Limit order prices are normalized to be relative to the market-wide perception about the asset liquidation value, in a way we make specific in section 4. The case $M = 0$ indicates that the microstructure of the market does not allow limit orders, as in a pure dealer market. Whenever $M \geq 1$, traders are able to choose between taking liquidity issuing a market order or making it issuing a limit order. The case $M = 1$ represents a situation in which limit orders are price unconditional: when submitting a limit order only one limit price is available. Whenever $M > 1$, traders has greater freedom in choosing the level of aggressiveness of their orders. We focus hereafter on the case in which there is a single limit order price ($M = 1$). Having only one submission price at hand, traders clearly face a simpler decision problem. The essential trade-off between market and limit orders, execution uncertainty vs execution quality, is nonetheless preserved. Most of the important order-type-choice results we are after can thus be derived in the simpler setting.

Let $p_1^{+}$ and $p_1^{-}$ be the buy and sell limit order prices, respectively. These prices, so-called best bid and best ask prices, can take on any number on the real line but differ by a fixed quantity $\Delta > 0$. That is, it is required that $p_1^{-} \geq p_1^{+} + \Delta$. The quantity $\Delta$ then operates as a sort of "tick size" when the bid-ask spread approaches its value from above. This fact makes liquidity traders’ order choice problem non trivial even when no informed traders are present.

3.2.2 Dealers or specialists

We use the indicator variable $\chi$ to denote the presence of dealers in the market. When $\chi = 1$, dealers act in the market by continuously quoting bid and offer prices for one unit of the asset. Dealers’ prices incorporate the risk of trading with informed agents. In addition, they charge a liquidity provision service fee $\Lambda \geq 0$ that compensates them for order handling costs, inventory costs, etc. The quantity $\Lambda$ leaves dealers indifferent between providing liquidity and not participating.

When $\chi = 0$, traders' only source of liquidity is the limit-order book. This case corresponds to a so-called "pure limit order market" (PLOM).

3.2.3 Hybrid Market

We consider a hybrid market in which traders have access to dealers and an open limit order book as our benchmark case of analysis. The market we analyze is thus partially quote-driven and partially order-driven. Ordinary traders submitting market orders choose the trading venue that provides liquidity at the most convenient price for them. In the case where both trading venues offer equal prices, limit order traders have execution priority over dealers. Since both dealers and limit order traders are allowed to

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10The order size constraint is somewhat less restrictive than the submission price constraint. In this continuous time setting, any trader can issue any finite number of orders within any given time interval.

11If $M > 1$, $p_1^{+}$ represents the most competitive buy limit order price. The second most competitive buy LO price would equal $p_2^{+} = p_1^{+} - \Delta$, and the least competitive buy LO price $p_M^{+} = p_1^{+} - M\Delta$. Analogously, sell limit order submission prices would range between $p_1^{-}$ and $p_M^{-} = p_1^{-} + M\Delta$. Once the best buy and sell prices are determined in equilibrium, $\Delta$ fully structures the relationship between limit-order prices.

12Alternatively, dealers’ intermediation can be made stochastic by interpreting $\chi \in [0, 1]$ as the probability of finding a dealer to trade with. We believe our main results would not be affected by such a possibility, and thus we focus on the case in which $\chi = \{0, 1\}$.

13This is consistent with trading rules in important asset markets. The NYSE, for example, gives priority to limit order traders over the specialist’s proprietary trading.
provide liquidity to other traders, we refer to them generically as "liquidity providers".

Table 1 summarizes the different cases that can be analyzed within the framework. The reference case in the following sections correspond to \((\chi = 1, M = 1)\). We investigate market microstructure differences regarding the price process in section 7.

<table>
<thead>
<tr>
<th>(M)</th>
<th>(\chi = 0)</th>
<th>(\chi = 1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>Dealers’ Market: Market orders only</td>
<td>Dealers’ Market: Market orders only</td>
</tr>
<tr>
<td>1</td>
<td>PLOM: Order type choice only</td>
<td>Hybrid: Order type choice only</td>
</tr>
<tr>
<td>&gt; 1</td>
<td>PLOM: Order type and price choice</td>
<td>Hybrid: Order type and price choice</td>
</tr>
</tbody>
</table>

Table 1: Cases of Analysis

3.3 Information Structure and Trader Types

All market participants observe the order flow, the state of the limit order book and dealers’ quotes continuously. Traders belong to one of two groups depending on their participation motives: liquidity traders and informed traders.

A single informed trader (or speculator) learns the realization of \(v\) at time zero. A public release of information about the asset value takes place at a random time \(\tau\) distributed exponentially with parameter \(r\). The revelation time is independent of any other variable in the model. These facts are common knowledge. This agent trades to maximize her profits dynamically. When the true value is revealed to the market his information becomes worthless. Since there are two possible liquidation values, there are thus two types of informed traders: "high type" corresponding to \(v_H\) and "low type" corresponding to \(v_L\).

Liquidity traders act for exogenous reasons that may or may not be related to the asset liquidation value, like portfolio rebalancing, index trading, hedging, etc. We consider two types of liquidity traders: strategic and non-strategic (noise traders)\(^{14}\).

There is a single strategic buyer and a single strategic seller. They adopt a dynamic placement strategy that seeks to minimize the expected cost of their trades, subject to a certain quantitative target constraint defined in section 4.

Non-strategic liquidity traders simply introduce noise. Their orders submissions occur following a Poisson process with constant intensities. The arrival intensities of buy and sell noise orders are equal, but may differ between different order types. Noise orders best represent the aggregate order submissions of a number of small market participants that trade rarely, or without specific quantitative targets.

3.4 Traders’ Actions

Market orders. By a market order (MO) we mean the following trading instruction: buy (or sell) one unit of the asset immediately at the most favorable price available in the market. Provided \(\chi = 1\), this trading instrument is always available to traders. In contrast, in a pure limit order market, this trading alternative is available only when the order book is not empty.

\(^{14}\)Most of the microstructure sequential trade models assume that liquidity traders are passive to changes in market conditions, and trade following an ad-hoc probabilistic pattern. An important exception is Admati-Pleiderer (1988).
**Limit Orders.** By a limit order (LO) we mean the following trading instruction: buy (or sell) one unit of the asset only if the execution price is not worse for the trader than a prespecified value. Such value is, at any moment, the most competitive price that compensate traders for adverse selection risks, as explained in section 4.

**Revisions.** Uninformed traders monitor the market continuously. Any new information it is incorporated on the price of their outstanding limit orders as soon as it is available. That is, anytime that an arriving order induces a change in asset value beliefs, the price of posted limit orders will reflect it immediately.

**Cancellations.** Limit orders are cancelled with an exogenous probability at each moment. Traders’ available actions at any moment of time are then: to issue a buy or sell market order of size one, to issue a buy or sell limit order of size one, or to do nothing.

### 3.5 Market Variables and Notation

#### 3.5.1 Order Processes

We use $X$ for the counting process of the informed trader’s orders, $Z$ for the counting process of liquidity traders’ orders, and $Y$ for the counting aggregate order process ($Y = X + Z$). Subscripts $S$ and $N$ label, respectively, the counting processes of strategic and non-strategic liquidity traders ($Z = Z_N + Z_S$). Superscripts denote a particular order type. We index order types generically by $o$ in the set $O = \{m, \ell, c, +, -\}$, where: (i) $m$, $\ell$, $c$ denote market order, limit order and cancellation, respectively, and (ii) $+$ and $-$ denote buy and sell orders, respectively. For example, $X_t^{m+}$, $X_t^{\ell-}$, $Z_t^c$ denote the total number of informed buy market orders, informed sell limit orders and liquidity buy limit order cancellations up to time $t$, respectively. In equilibrium traders submit market and limit orders with a certain (optimal) intensity. We use the small letters to denote point intensities of the above counting processes. For example, $x_t^{m+}$ and $z_t^{\ell-}$ correspond to the time $t$ point intensities of informed buy market orders and liquidity sell limit orders, respectively. Since noise liquidity traders submit buy and sell orders with the same intensity at any moment, we drop the sign superscript hereafter for notation economy ($z_t^{\ell-} = z_t^{\ell-} = z_t^{+}$ and $z_t^{m+} = z_t^{m+} = z_t^{m-}$).

Let $\mathcal{F}_t^U = \sigma(\{Y_u : u \leq t\})$, and $\mathcal{F}_t^L = \sigma(\{Y_u, X_u : u \leq t\})$, that is, the $\sigma-$fields generated by $Y$ and $(Y, X)$ respectively up to time $t$ ($\{\mathcal{F}_t^U : t \in [0, \infty)\}, \{\mathcal{F}_t^L : t \in [0, \infty)\}$ are thus filtrations of $\mathcal{F}$). In subsequent sections we work with two filtered probability spaces, $(\Omega, \mathcal{F}, \{\mathcal{F}_t^U\}, \mathbb{P})$ and $(\Omega, \mathcal{F}, \{\mathcal{F}_t^L\}, \mathbb{P})$, corresponding to the resolution of information over time available to uninformed and informed traders, respectively. The filtered spaces satisfy the usual conditions, as defined by Protter (2005).

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15 We solved a version of the model in which the informed trader is able to strategically cancel orders at any time. Within a single information event, we found that this cancellations take place only off the equilibrium path. The informed trader does not find optimal to cancel orders on the side of his private informations. She has an incentive to optimally cancel orders on the opposite side, but such limit orders are not submitted on the equilibrium path. Similarly, it is not optimal for strategic liquidity traders, with fixed trading instructions, to cancel outstanding limit orders. The exogenous cancellation rate decreases the frequency of the states in which the LOB is full in the solution algorithm.
3.5.2 State Variables

Let $A_t$ and $B_t$ denote, respectively, the time $t$ depth of the bid and ask sides of the book (the total number of outstanding orders on each side). Also, let $A^I_t$ and $B^I_t$ stand for the number of outstanding orders submitted by the informed trader up to time $t$ (hereafter, "informed orders"). To describe the depth of the limit-order book on both sides we use $D = (B, A)$.

An important strategic quantity is the informational composition of the LOB, that is the fraction of informed orders on the ask and bid sides of the order book. The informed fraction process $F = \{(F^A_t, F^B_t) : 0 \leq t < \tau\}$ is defined by

$$F^B_t \overset{\text{def}}{=} \frac{B^I_t}{B_t + 1_{\{B_t = 0\}}} \quad F^A_t \overset{\text{def}}{=} \frac{A^I_t}{A_t + 1_{\{A_t = 0\}}} \quad (1)$$

and where $1$ is an indicator function. The indicator functions in $1$ simply avoid indeterminacies and will be omitted hereafter for notation economy.

Let $p_t$ be the conditional expectation of the asset value given liquidity providers’ information up to time $t$, that is $p_t = \mathbb{E}[v|\mathcal{F}^U_{t-}]$. The quantity $p_t$ represents the market’s consensus value at time $t$. We refer to the sequence $\{p_t : t \geq 0\}$ as the "price process". Uninformed traders update beliefs at any time $t$ at when new information arrives. Note that since time is continuous, their prior belief for that matter is $p_{t-} = \mathbb{E}[v|\mathcal{F}^U_{t-} = \bigcup_{s < t} \mathcal{F}^U_s]$.

Let $s_t = (p_t, D_t) \in \mathcal{S}$ collect uninformed traders’ time $t$ state variables. Let $s^I_t = (p_t, D_t, F_t) \in \mathcal{S}^I$ collect speculators’ time $t$ state variables. The essential difference between $s$ and $s^I$ is that uninformed traders cannot observe whether outstanding limit orders are informed or not. However, whenever the book is empty, $s_t$ is a sufficient statistic for $s^I_t$.

Finally, for any given $s^I = (p, D, F)$ we use the subscripts $(x,o), (z,o), o \in \mathcal{O}$ to denote the change in the state variables induced by a arriving order. For example, $s^I_{zt+}$ denote the change in $s^I$ induced by the arrival of an informed buy LO. For the case of market orders, this change might be random due to the random allocation rule. More detail of these expressions can be found in the appendix.

3.6 Submission costs

There are no monitoring costs. However, order submissions are not free of costs. Flow costs are directly related to submission intensities of any order type as follows: $C^o(x^o) = \kappa(x^o)^\phi$, where $\kappa > 0$, $\phi > 1$.

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16 Given the made assumptions, $F^A_t$ and $F^B_t$ are $\mathcal{F}^I$-measurable stochastic processes that change over time (whenever the corresponding side of the book is non-empty) according to

$$F^A_t = \left\{ \frac{A^I_t + dX^+_t - dY^+_{t+1}I_{hit}}{A_t + dY^-_t - dY^+_{t+1} - dY^-_{t+1}} : 0 \leq t < \tau \right\}$$

$$F^B_t = \left\{ \frac{B^I_t + dX^+_t - dY^+_{t+1}I_{hit}}{B_t + dY^-_t - dY^+_{t+1} - dY^-_{t+1}} : 0 \leq t < \tau \right\}$$

where $I_{hit}$ is an indicator function that equals one only if an informed order is executed. Conforming with standard notation, $W_t = \lim_{u \downarrow t} W_u$.

17 Note that $p_t$ is not the actual price of the asset due to adverse selection and the tick size. It can be viewed as the price that, in a frictionless market, a certain trader would obtain were she able to signal herself as uninformed.
and \( o \in \mathcal{O} \). That is, all orders are equally costly. Submission costs are additive, that is trader \( i \)'s total submission flow cost is given by

\[
C(x_i) = \kappa \sum_{o \in \mathcal{O}} (x_i^o) \delta
\]  

The separable structure of 2 simplifies the subsequent analysis\(^{18}\). These costs may have a counterpart in reality. For example traders may employ agents like a broker. In this case, more frequent submissions would require more frequent contacts with the broker and higher administrative costs. Some exchanges impose costs directly. Intrade, an important prediction market, charges commissions for every executed order. The NYSE charges traders for limit orders that stay in the LOB for more than fifteen minutes.

4 Strategic Analysis

The market game outline above has (i) strategic liquidity traders placing orders to minimize expected costs, (ii) a single trader seeking to maximize profits by optimally exploiting his superior information (iii) noise traders submitting orders randomly and (iii) liquidity providers continuously monitoring the market and using newly available information to update their quotes and limit order prices using Bayes rule. We argue that there is no equilibrium in which the speculator’s submission times are predictable to uninformed market participants.

Proposition 4.1: There is no equilibrium of the model in which the informed trader submits orders at \( F^\mathcal{U} \)-times.

The intuition is that, if the informed trader were to place an order in equilibrium at a time that other agents can anticipate, once that time is reached the speculator finds optimal to deviate. Then, if there is an equilibrium it must be one in which informed orders are submitted at random times\(^{19}\).

We will then consider an equilibrium in which an informed trader of type \( i \in \{H, L\} \) submits orders with a certain probability. The concept of submission intensity is central to our analysis. Let \( x_i^o, o \in \mathcal{O} \) be defined by

\[
x_i^o(s_t) = \lim_{\delta \to 0} \Pr \left( \frac{X_i^{o+\delta} - X_i^o > 0}{\delta} \middle| s_t \right)
\]

Similarly for uninformed traders,

\[
z_i^o(s_t) = \lim_{\delta \to 0} \Pr \left( \frac{Z_i^{o+\delta} - Z_i^o > 0}{\delta} \middle| s_t \right)
\]

Within any interval of time \( \Delta t \) the speculator of type \( i \) will submit, for example, buy market orders with probability \( x_i^{o+\Delta t} + O(\Delta t^2) \) and sell limit orders with probability \( x_i^{o-\Delta t} + O(\Delta t^2) \). At most one order arrives at any time \( t \) \( \mathbb{P} - \text{a.s.} \) and thus, \( \mathbb{P}(dX_i^o = 1|s_{t-}) = \mathbb{E}(dX_i^o|s_{t-}) = x_H^o(s_{t-})dt \). We formalize this idea and its relation to the processes contained in \( X \).

Definition: A strategy for the informed trader is a set of functions \( x = \{x_i^o : i \in \{H, L\}, o \in \mathcal{O}\} \)

\(^{18}\)The assumed cost function make traders’ best responses unique for each set of state variables. From a computational point of view, costs make the solution algorithm well defined. Given that the computational state space is necessarily finite, otherwise any limit order submission level would be optimal when the maximum level of depth in the LOB has been reached.

\(^{19}\)An analogous argument was used by Bach and Baruch (2004) in a pure dealers’ market.
within the class of mappings \( S^I \rightarrow \mathbb{R}_+ \) such that the following stochastic processes

\[
\begin{align*}
\mathcal{M}^+ &= \left( X_t^{m+} - v \int_0^t x_H^+(s_u^-)du - (1 - v) \int_0^t x_L^+(s_u^-)du : t < \tau \right) \\
\mathcal{M}^- &= \left( X_t^{m-} - v \int_0^t x_H^-(s_u^-)du - (1 - v) \int_0^t x_L^-(s_u^-)du : t < \tau \right) \\
\mathcal{L}^+ &= \left( X_t^{l+} - v \int_0^t x_H^+(s_u^-)du - (1 - v) \int_0^t x_L^+(s_u^-)du : t < \tau \right) \\
\mathcal{L}^- &= \left( X_t^{l-} - v \int_0^t x_H^-(s_u^-)du - (1 - v) \int_0^t x_L^-(s_u^-)du : t < \tau \right)
\end{align*}
\]

are \((\mathcal{F}_t \cup \mathcal{V})\)-martingales (i.e. martingales with respect to the speculator’s information set).

A strategy for uninformed traders can be defined similarly.

A direct consequence of Proposition 4.1 is that equilibrium submission intensities must be finite for each order type\(^{20}\). That is, there exists a number \( \overline{\pi} < \infty \) such that

\[
\max_{i \in \{H, L\}, o \in \mathcal{O}, s^I \in S^I} \left\{ x_i^o(s^I) \right\} < \overline{\pi}
\]

and similarly for uninformed traders. To clarify these ideas, Figure 1 shows the evolution of the LOB. The book is empty at the center of the figure and we trace its possible evolutions up to two orders on each side. Each dotted arrow in Figure 1 corresponds to a possible transition and the corresponding transition intensities are displayed (showing only the LOB state as argument).

One of the strategic aspects of the informed trader behavior that we investigate is whether in equilibrium the high type submits sell orders and the low type submits buy orders. This behavior is sometimes called blufing. We refer to M-blufing to denote a behavior in which a trader motivated by superior information submits both buy and sell market orders. That is, for any \( v \in \{v_H, v_L\} \), both \( \mathcal{M}^+ \) and \( \mathcal{M}^- \) are non-degenerate processes. Similarly, L-blufing denotes a behavior such that both \( \mathcal{L}^+ \) and \( \mathcal{L}^- \) are non-degenerate processes.

In the remainder of this section we study the problem that uninformed and informed traders face and derive the conditions that characterize an equilibrium.

### 4.1 Liquidity Providers and Price Formation

Competitive dealers and limit order traders use the information contained in the order flow to update their beliefs about the asset liquidation value anytime a market event takes place (an order arrives or is cancelled). Since an informed trader may use market and limit orders in equilibrium, each order type will carry specific information that agents seek to distill optimally. We now formally define the concept of price impact function.

**Definition:** (price impact functions). The price impact function associated with order type \( o \in \mathcal{O} \) is

\(^{20}\)Since orders will only arrive at separate times in equilibrium, the stochastic differential game I analyze is piecewise deterministic (see, for example, Dockner et al. 2000).
denoted $q^o : S \rightarrow \mathbb{R}$ and represents the mapping

$$s \mapsto \mathbb{E}[v|s, dX_t^o = 1] = \mathbb{P}(v_H|s, dY_t^o = 1)$$

For example, suppose the $\mathcal{F}^U$-state is $s$, then $(q^m(s) - p)$ and $(q^m(s) - p)$ represent the change (jump) in the price process caused by the arrival of buy and sell market orders, respectively. The set of updating functions is $Q$.

Beliefs then evolve accordingly to the following stochastic differential equation

$$dp_t = \sum_{o \in \mathcal{O}} [q^o(s_{t-}) - p_{t-}] (dX_t^o + dZ_t^o) \quad (7)$$

The fact that the term $(dX_t^o + dZ_t^o)$ appears on the RHS of 7 reflects that liquidity providers cannot distinguish traders’ motivations.

The tick size, in the particular sense of section 3, and liquidity providers’ pricing behavior, motivate the following definition.

**Definition:** (bid, ask, dealers’ quotes) The bid and ask prices, denoted $a$ and $b$ respectively, are at any time given by

$$a(s) = \max\{q^+(s), p + \frac{1}{2}\Delta\}, \quad b(s) = \min\{q^-(s), p - \frac{1}{2}\Delta\}$$
Dealers’ quotes, denoted $a^D$ and $b^D$, are at any time given by

$$a^D(s) = a(s) + \Lambda, \quad b^D(s) = b(s) - \Lambda$$

The bid and ask prices defined above are effectively the limit order prices available in the LOB. Given the price impact of incoming orders, any buy and sell limit orders submitted at time $t$ will be posted, respectively, at prices equal to

$$p_{1,t}^{f+} = a(q_t^{f+}(s_{t-}), B_{t-} + 1, A_{t-})$$
$$p_{1,t}^{f-} = b(q_t^{f-}(s_{t-}), B_{t-}, A_{t-} + 1)$$

Further, as uninformed limit-order traders update their beliefs about the asset value, outstanding orders prices will reflect those updates (limit order prices are relative to the market-wide perception of the asset value). Note that when $M = 1$, dealers’ quoted prices are no better than limit order prices. Thus, dealers intermediate when a market order finds no liquidity on the corresponding side of the book.

We now find the exact expressions in $Q$ that will arise in equilibrium. The main complication uninformed agents face is imperfect knowledge about the informational composition of the LOB, which affect in turn speculators’ actions. At any time $t$, they need to use information contained in $\mathcal{F}_t^U$ to form beliefs about such hidden states. With that aim, denote by $\mathcal{F}(\mathcal{D})$ the set of conceivable informed order fractions for any observable book $\mathcal{D}$. Further, we denote with a tilde the $\mathcal{F}_t^U$—intensities of the informed trader for any order type $o \in \mathcal{O}$. That is,

$$\tilde{x}^o(s) = p \sum_{F \in \mathcal{F}(\mathcal{D})} x^o_H(s^I) \mathbb{P}(F|s, v_H) + (1 - p) \sum_{F \in \mathcal{F}(\mathcal{D})} x^o_L(s^I) \mathbb{P}(F|s, v_L)$$

Consequently, for any order type $o \in \mathcal{O}$ we have $\tilde{y}^o(s) = \tilde{x}^o(s) + z^o(s)$. The following proposition, which is simply a Bayes rule derivation, specifies the way in which liquidity providers update their beliefs following the arrival of a new order.

**Proposition 4.2:** Consider a non-random time $t < \tau$, with $s_{t-}$ equal to $s \in \mathcal{S}$ and $s_t^I$ equal to $s^I \in \mathcal{S}^I$ and. Consider a given speculator’s strategy $x$ and liquidity traders’ strategy $z$. Then, the expected value of the asset conditional on a buy order arriving at time $t$ is

$$q^{m+}(s) = \frac{p}{\tilde{y}^{m+}(s)} \left[ \sum_{F \in \mathcal{F}(\mathcal{D})} x^{m+}_H(s^I) \mathbb{P}(F|s, v_H) + z^{m+}(s) \right]$$
$$q^{f+}(s) = \frac{p}{\tilde{y}^{f+}(s)} \left[ \sum_{F \in \mathcal{F}(\mathcal{D})} x^{f+}_H(s^I) \mathbb{P}(F|s, v_H) + z^{f+}(s) \right]$$

for market and limit orders, respectively. Analogous expressions hold for sell buy and limit orders.

The updating expressions like 8 represent value-weighted conditional probabilities. The weights correspond to $p, v_H = 1$ and $v_L = 0$, each for a possible source: an uninformed trader, a high type and low

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Formally, $\mathcal{F}(\mathcal{B}) = \{ F : F \in \mathbb{Q}^2_+, A^I \leq A, B^I \leq B, (A^I, B^I, A, B) \in \mathbb{N}^4_0 \}$.

For instance if $\mathcal{B} = (1, 0)$ then there are only two possibilities: either the single buy order in the book was sent by a noise trader (in which case $F = (0, 0)$) or by an informed trader (so $F = (1, 0)$). Thus, $\mathcal{F}(\mathcal{B}) = \{(0, 0), (1, 0)\}$. 

16
type informed trader, respectively. Note that updating functions in Proposition 4.2 are defined in terms of model’s parameters, the informed trader strategy, and the quantities \( P(F|s,v) \). We refer to the set of beliefs

\[
\{P(F|s,v) : F \in \mathcal{F}(D), s \in S, v \in \{v_H, v_L\}\}
\]  

(10)

generically as \( \hat{F} \). We explain in the appendix the algorithm with which this set is calculated and give further details about the price impact of cancellations.

Example 4.1: Suppose there are no limit orders available \((M = 0)\) and that market orders’ submission frequencies are constant. That is for buy orders \( x_i^{m+}(s^t) = x_i^{m+} \in R, i \in \{L, H\} \), \( z^{m+}(s) = z^{m+} \in R \). Then

\[
q^{m+}(s) = \frac{p(x_i^{m+} + z^{m+})}{px_H^{m+} + (1-p)x_L^{m+} + z^{m+}}
\]  

(11)

Expression 11, a particular case of 8, can be seen as a Glosten-Milgrom updating rule in a continuous time framework (similar to Easley et al., 1996).

For every \( t < \tau \) and \( s \in S \), we assume that in equilibrium traders strategies are such that buy orders are more likely when the asset is undervalued and sell orders are more likely when the asset is overvalued\(^{22}\). Under these mild assumptions, the price updating functions of Proposition 2 satisfy \( q^{m+}(s) > p > q^{m-}(s) \) and \( q^{l+}(s) > p > q^{l-}(s) \). Both conditions are verified in the computed equilibrium of Section 5.

### 4.2 The Informed Trader

We seek to derive in this section the optimal submission intensities of the speculator. This trader needs to decide at any moment (i) whether to submit orders or not (ii) which is the optimal frequency of submissions and (iii) what order types are more convenient to her. In the subsequent analysis we normalize the interest rate to zero.

Start considering the speculator submitting a buy market order at time \( t \) when the \( \mathcal{F}^U \)-state of the market is \( s \in S \). Three possible types of execution can take place. Provided the ask side of the LOB is non empty (i) she will get the security from a limit order trader with probability \((1 - F^{A})\) paying the amount \( a(s) \), and (ii) she will hit one of his outstanding sell limit orders realizing zero profit (an event we call "self trade") with probability \( F^{A} \). Third, if the ask side of the LOB is empty she will get the unit from a dealer at the price \( a(s) + \Lambda \). The expected instantaneous profit from this action, conditional on his information set \( \mathcal{F}^L \cup v \), is

\[
\pi_i^{m+}(s^t) = 1_A[v - a(s)](1 - F^{A}) + (1 - 1_A)[v - a(s) - \Lambda]
\]

where \( 1_A \) is an indicator function that takes value one when \( A > 0 \). Similarly, in the case of a sell order

\[\begin{align*}
P(dY_t^{o+}) &= 1[s, v_H] > P(dY_t^{o+} = 1[s, v_L]) \quad \text{and} \quad \{o+\} \in \{m+, \ell+\}, \quad \{o-\} \in \{m-, \ell-\}.\end{align*}\]
market order, the expected instantaneous profit is

\[ \pi_i^m(s^I) = 1_B [b(s) - v] (1 - F^B) + (1 - 1_B) [b(s) - \Lambda - v] \]

where \( 1_B \) equals one when \( B > 0 \).

Suppose now that the informed trader keeps \((A^I, B^I) \geq (0, 0)\) limit orders in the book at time \( t \). Recall that the probabilities of a buy and sell market orders arriving within \( dt \) are \([x_i^{m+}(s^I) + z^m] dt\) and \([x_i^{m-}(s^I) + z^m] dt\) respectively. Then, given the proportional allocation rule, execution probabilities of sell and buy informed orders within any infinitesimal time interval \( dt \) are respectively:

\[
F^A [x_i^{m+}(s^I) + z^m] dt \\
F^B [x_i^{m-}(s^I) + z^m] dt
\]

where \( i \in \{H, L\} \). An execution of a limit order carries profits for the speculator only if the counterpart of the transaction is a liquidity trader. Hence, within any interval \( dt \), the \( F^I_t \)-expected profits of outstanding limit orders on the bid and ask sides are respectively

\[
\pi_i^{\ell+}(s^I) = F^B [v - b(s)] (z^m dt) \\
\pi_i^{\ell-}(s^I) = F^A [a(s) - v] (z^m dt)
\]

Based on the discussion above we can write the informed trader objective functional as

\[
\mathbb{E}^I \left[ \int_0^\tau \pi_i^{m+}(s^I_{t^-}) dX_t^m + \int_0^\tau \pi_i^{m-}(s^I_{t^-}) dX_t^m + \int_0^\tau \pi_i^{\ell+}(s^I_{t^-}) dZ_t^m + \int_0^\tau \pi_i^{\ell-}(s^I_{t^-}) dZ_t^m - \int_0^\tau C_i(t) dt \right]
\]

where the expectation is conditional on the informed trader information\(^{23}\). Note that each of the first four terms in 12 accounts for the profits obtained through a different trading instrument: the first term corresponds to buy MOs, the second to sell MOs, the third and fourth to buy and sell LOs, respectively.

The informed trader will pick a submission strategy to maximize the time zero expectation of his profits as given by 12. In doing so, she will take as given the expressions like 8 and the submission strategies of the uninformed traders.

Let \( V_i : S^I \to \mathbb{R}, i \in \{H, L\} \) denote the value function of a speculator of type \( i \). Since the objective functional 12 is stationary, we can define the informed trader value at any moment as a function of her state variables \( s^I \).

**Lemma 4.1:** The Bellman equation of an informed trader of type \( i \in \{H, L\} \) is given by

\[^{23}\text{Note that the integrators in 12 are stochastic processes. For this object to be well defined it is usually required that (i) the integrand is a previsible process and (ii) the integrators are semimartingales. The first condition is satisfied since all the integrands are adapted L-processes and the second is satisfied since the integrators are point processes of finite variation (see for example Rogers and Williams, 2000).}\]
\[ rV_i(s') = \max_{x \in \mathcal{X}} \left\{ x_i^{m+} \pi_i^{m+}(s') + x_i^{m-} \pi_i^{m-}(s') + z^{m-} \pi_i^{m-}(s') + x_i^{m+} \pi_i^{m+}(s') \right\} \] (13)

\[-\kappa \sum_{o \in \mathcal{O}} (x'_o)^\phi + \sum_{o \in \mathcal{O}} x'_o(s') \left[ \tilde{V}_i(s'_{zo}) - V_i(s') \right] + \sum_{o \in \mathcal{O}} z^o(s) \left[ \tilde{V}_i(s'_{zo}) - V_i(s') \right] \]

where \( \mathcal{O} = \{ m+, m-, \ell+, \ell-, c+, c- \} \) and \( \tilde{V}_i(s'_o) \) stand for the expectation of type's \( i \) value function conditional on a given market event \( o \in \mathcal{O} \) taking place.

The four terms of the RHS of 13 reflect the instantaneous expected profits corresponding to buy/sell market and limit orders respectively. The fifth term groups trader \( i \)'s submission costs. Finally, the last two terms collect the expected jump in \( i \)'s value function caused by her actions and liquidity traders’ actions, respectively. Note that informed trader’s limit order submissions or cancellations change continuation values, but do not bring any instantaneous monetary profit (they do not induce an execution).

The maximization problem on the RHS of the Bellman equation determines the optimal submission strategy for the informed trader. The following Proposition expresses informed trader’s optimal intensities as a function of contemporary market states, the induced jumps in the value function and the model parameters.

**Proposition 4.3.** Suppose the state of the market is \( s' \in S \). Then the optimal submission intensities of the informed trader of type \( i \in \{ H, L \} \) are given by

\[ x_i^{m+}(s') = \max \left\{ 0, \left( \frac{1}{\kappa \phi} \left[ \tilde{V}_i(s'_{zm+}) - V_i(s') + \pi_i^{m+}(s') \right] \right)^{\frac{1}{\phi-1}} \right\} \] (14)

\[ x_i^{m-}(s') = \max \left\{ 0, \left( \frac{1}{\kappa \phi} \left[ \tilde{V}_i(s'_{zm-}) - V_i(s') + \pi_i^{m-}(s') \right] \right)^{\frac{1}{\phi-1}} \right\} \] (15)

\[ x_i^{l+}(s') = \max \left\{ 0, \left( \frac{1}{\kappa \phi} \left[ \tilde{V}_i(s'_{zl+}) - V_i(s') \right] \right)^{\frac{1}{\phi-1}} \right\} \] (16)

\[ x_i^{l-}(s') = \max \left\{ 0, \left( \frac{1}{\kappa \phi} \left[ \tilde{V}_i(s'_{zl-}) - V_i(s') \right] \right)^{\frac{1}{\phi-1}} \right\} \] (17)

Note that the optimal frequency of any order type decreases with cost parameters \( \kappa \) and \( \phi \) as expected. However, it is not as obvious how intensities are affected by the remaining parameters that define a market environment. In section 5 we calculate numerically the effect of meaningful parameter changes on optimal submission policies.

### 4.3 Strategic Liquidity Traders

There is a single strategic buyer and a single strategic seller. Each strategic liquidity trader (hereafter, SLT) has a quantitative target to meet, and seeks to place orders dynamically to reduce expected costs.\(^{24}\)

\(^{24}\)It is straightforward to consider many buyers and sellers provided they share the same target functions in a symmetric equilibrium. To simplify the presentation only one agent of each type is considered.
They observe the order flow, the depth of the limit order book and bid and offer prices at any moment of time.

To reduce the number of state variables in the system, we make strategic liquidity traders to take decisions based on their average position in the depth of the LOB. That is, when the observable state of the market is \((p, B, A)\), a strategic buyer forms a belief about the how many of the \(B\) outstanding orders he holds on the bid side of the book. Likewise for the strategic seller\(^{25}\). On any side of the LOB, a given limit order could have been placed by a SLT, a non-strategic liquidity trader or a speculator. We let \(\tilde{F}_S^B, \tilde{F}_N^B, \tilde{F}_I^B\) and \(\tilde{F}_S^A, \tilde{F}_N^A, \tilde{F}_I^A\) denote the respective \(\mathcal{F}^U\)-expected positions in the LOB on the bid side and on the ask side, respectively\(^{26}\).

### 4.3.1 Quantitative Target

SLTs need to either buy or sell, in expectation, a certain number of units of the asset over the next instant. The expected execution rate target is denoted \(\epsilon\), and is the same for the buyer and the seller. The trading instruction that they need to fulfill could be informally expressed as "buy (sell) at least \(\epsilon dt\) orders on average at any moment".

Given other traders’ strategies, the constraints that the buyer and seller must satisfy are

\[
\begin{align*}
\epsilon & \leq z^{m+}_S(s) + (\tilde{z}^{m-}_S(s) + z^{m-}_S(s) + z^{0}_N(s)) \left(\tilde{F}_S^B(s)\right) \quad (18) \\
\epsilon & \leq z^{m-}_S(s) + (\tilde{z}^{m+}_S(s) + z^{m+}_S(s) + z^{0}_N(s)) \left(\tilde{F}_S^A(s)\right) \quad (19)
\end{align*}
\]

Note that each trader has direct control over the expected execution rate through the submission intensity of market orders. Note also the interrelations between 18 and 19: if the buyers decides to submit market orders more frequently, he will induce more executions of sellers’ limit orders and vice versa. Limit order submission intensities, in turn, provide only indirect control over the expected execution rate: the probability of submitting a limit order that is instantly executed is of order \(dt^2\).

### 4.3.2 Trading Costs

Strategic liquidity traders seek to minimize total costs, given by the sum of submission costs and transaction costs. Submission costs are as given in 2 and labeled \(C^B\) and \(C^S\) for buyers and sellers, respectively. Transaction costs are at any time computed as the difference between what traders pay (receive) for buying (selling) the asset and the price they perceive as fair (the security \(\mathcal{F}^U\)-expected value). Let the mapping \(\psi^o : S \to \mathbb{R}\) denote the transaction cost associated with order type \(o \in \mathcal{O}\). Suppose the

\(^{25}\)This simplification can be easily lifted, but at the cost of increasing significantly the computational time required to achieve an equilibrium.

\(^{26}\)Whenever there are orders in the LOB, these beliefs must satisfy

\[
\begin{align*}
\sum_i \tilde{F}_S^B(s) + \tilde{F}_N^B(s) + \tilde{F}_I^B(s) &= 1 \\
\sum_i \tilde{F}_S^A(s) + \tilde{F}_N^A(s) + \tilde{F}_I^A(s) &= 1
\end{align*}
\]
\(\mathcal{F}^U\)-state of the market is \(s\), then

\[
\psi^m(s) = [(a(s) - p)1_A + (a(s) + \Lambda - p)(1 - 1_A)]
\]

\[
\psi^m(s) = [(p - b(s))1_B + (p - b(s) - \Lambda)(1 - 1_B)]
\]

for market orders and

\[
\psi^f_+(s) = b(s) - p
\]

\[
\psi^f_-(s) = p - a(s)
\]

for limit orders. Note that transaction costs can be positive or negative since \(\psi^m_+(s) > 0, \psi^m_-(s) > 0\) and \(\psi^f_+(s) < 0, \psi^f_-(s) < 0\).

### 4.3.3 Optimal Strategies

Consider the strategic buyer. This trader takes the behavior of other traders as given and chooses an order placement strategy to minimize

\[
\mathbb{E}^U \left\{ \int_0^T \psi^m_+(s) dZ^m_+ + \int_0^T \psi^f_+(s) \left( \tilde{F}^B_S(s) \right) dY^m_+ + \int_0^T C^B_t dt + \Psi(s+) \right\}
\]

where \(\Psi(s+)\) represents the continuation value of his program at the moment in which the asset fundamental value is revealed (and the informed trader leaves the market). The constraint this trader faces is given by the quantitative target 18. The problem that sellers face is analogous, once 18 is replaced by 19. Let \(W^B\) and \(W^S\) denote the value function of buyers and sellers respectively. As before, let \(t_s^+\) denote the change in state variables in \(s\) when order \(\ell^+\) arrives, etc.

**Proposition 4.4:** Given the informed trader behavior \(x\) and beliefs \((Q, \tilde{F})\), the \(\mathcal{F}^U\)-cost minimizing submission strategies are (i) for buyers

\[
z^m_+(s) = \max \left\{ 0, \epsilon_S - \gamma^m_-(s) \left( \tilde{F}^B_S(s) \right) \right\}
\]

\[
z^f_+(s) = \max \left\{ 0, \frac{1}{\kappa^B} [W^B(s) - W^B(s^+)] \frac{\sigma_1}{\sigma_1} \right\}
\]

and (ii) for sellers

\[
z^m_-(s) = \max \left\{ 0, \epsilon_S - \gamma^m_+(s) \left( \tilde{F}^A_S(s) \right) \right\}
\]

\[
z^f_-(s) = \max \left\{ 0, \frac{1}{\kappa^B} [W^S(s) - W^S(s^-)] \frac{\sigma_1}{\sigma_1} \right\}
\]

Intuitively, strategic liquidity traders use market orders only when the expected number of their limit orders executions is not enough to meet the target, as in 21 and 23.
5 Equilibrium

5.1 Existence

We search for a stationary equilibrium in Markovian strategies. Since players use Markov strategies, the evolution of the game’s states is Markov. The equilibrium has the following components.

**Definition** : (Equilibrium). The set of functions \((x, z, Q, \tilde{F})\) constitute an equilibrium if the following conditions are satisfied:

(i) Given belief functions \(Q\) and \(\tilde{F}\), and liquidity traders submission strategy \(z\), the informed trader strategy \(x\) maximize his profits.

(ii) Given the informed trader equilibrium strategy \(x\) and belief functions \(Q\) and \(\tilde{F}\), submission strategy \(z\) achieves buyer and sellers’ quantitative targets at minimum cost.

(iii) Given traders’ equilibrium strategy \(x\) and \(z\), belief functions \(Q\) and \(\tilde{F}\) are determined by Bayes rule.

The equilibrium is stationary since time is not a state variable. The Markov specification requires agents to condition only on the current state of the game (the ones that are observable to them). An agent behaves optimally in every state- that is measurable with respect to his information set- irrespective of whether such state is on or off the equilibrium path. Since the objective functions are stationary, \(\tau\) is independently distributed and the influence of past play is captured in the current states, there is a one-to-one correspondence between subgames and states (see e.g. Doraszelski and Satterthwaite, 2007). Accordingly, agents’ beliefs in states that are not visited must be reasonable (given by Bayes rule).

**Proposition 5.1** : (Existence) Let \(x\) be given by 14-17. Let \(z\) be given by 21-24 and \(Q\) as in Proposition 4.2. Let beliefs \(\tilde{F}\) be determined by Bayes rule. Let \(V_{i}, i \in \{L, H\}\) be as in Lemma 1. Then \((x, z, Q, \tilde{F})\) constitute an equilibrium.

We do not provide a uniqueness result. However, the main qualitative features of the computed equilibrium were robust to both parameter and initial conditions changes. In other words, when a computational equilibrium was achieved, the main patterns of such an equilibrium were the same.

The informed trader will gradually transmit information through her orders’ placements. Prices will adjust responding to those orders. Will prices reflect information completely over time? This of course depends on equilibrium behavior. If the informed trader does not bluff ‘significantly’, prices will reflect the information asymptotically. The following proposition displays sufficient conditions for convergence.

**Proposition 5.2** : Suppose that for every possible state of the game \(s^t \in S^t\), bluffing intensities are bounded in equilibrium as follows

\[
x_{m}^{n-} < \rho_m(s)(x_{m}^{n+} + z_{m}) - z_{m}, \quad x_{H}^{f-} < \rho_t(s)(x_{H}^{f+} + z_{t} + z_{c}) - (z_{t} + z_{c})
\]

for the high type, and

\[
x_{L}^{m+} < \rho_{m}^{-1}(s)(x_{L}^{m-} + z_{m}) - z_{m}, \quad x_{L}^{f+} < \rho_{t}^{-1}(s)(x_{L}^{f-} + z_{t} + z_{c}) - (z_{t} + z_{c})
\]

(25) (26)

for the low type, where \(\rho_m(s) = (q^{n+}(s) - p) / (p - q^{m-}(s))\) and \(\rho_t(s) = (q^{f+}(s) - p) / (p - q^{f-}(s))\).

Then, (i) conditional on \(v = v_{H}\) the price process is a \(F^t\)-R submartingale (ii) conditional on \(v = v_{L}\) the
price process is a \( \mathcal{F}^t \)-\( \mathcal{R} \) supermartingale (iii) asset value beliefs converge to the realization of \( v \mathbb{P} - a.s. \)

Since optimal policies are not available in close form, a rapid checking of conditions 25 and 26 is not at hand. However, we do not observe bluffing in the computed equilibrium for any parametrization.

5.2 Parametrization

We compute equilibrium strategies numerically. We define next a baseline parametrization against which we will compare a number of alternatives. The tick size and dealers’ liquidity premium equal 1/20. The revelation time parameter \( r \) equals 0.25. Noise traders’ Poisson intensities \( z_N \) and \( z_{\Delta} \) are 1/2 (equal for buys and sells). With this parametrization, the information asymmetry is expected to last for four units of time and eight noise orders of each type are expected to arrive before revelation. Strategic liquidity traders’ execution target \( \epsilon \) is 0.6. The submission cost parameters are \( \kappa = 0.2 \) and \( \phi = 2 \).

The algorithm is iterative. Since we work with an interrelated optimization problem, we cannot appeal to standard value iteration convergence theorems. However, the algorithm converged with an error tolerance of \( 10^{-8} \). The time to convergence was relatively low given the state space size and, as expected, most sensitive to the choice of the revelation time parameter \( r \). The continuous time formulation of the game reduces its computation time significantly\(^{27}\).

5.3 Model Outcomes

5.3.1 Traders’ Order Decisions

We observe a number of interesting features in the computed equilibrium. First and foremost, the informed trader actively provides liquidity with limit orders. Moreover, her behavior is sensitive to both the remaining value of information and the state of the order book.

**Result 5.1** *(Informed Trader Equilibrium Behavior)* (i) The informed trader actively uses limit orders in his optimal order placement strategy (ii) The optimal submission frequencies of both market and limit orders decrease when private information becomes less valuable, limit order intensities decrease at a lower rate. (iii) Limit orders submissions decrease with the depth of the LOB on the same side of the order and with the depth of the opposite side (iv) Market order submissions decrease with the depth of the LOB on the same side of the order and increase with depth on the opposite side (v) Market and limit order submissions are relatively less frequent whenever the speculator participation in the book liquidity is higher or, for a given number of informed orders, when the depth of both sides of the LOB increase evenly.

[FIGURE 3 ABOUT HERE]

Figure 3 depicts the main equilibrium features of the high type speculator’s behavior. Panels a and b display the equilibrium submission intensities of market and limit orders respectively. We observe that an increase in the depth of the bid side \( B \) decreases both MO and LO frequencies. Buy LO intensities decrease since the execution of an extra over any given time interval is more uncertain. The effect on buy market orders is consistent with the increase in the price impact function of that order type, as we discuss next.

\(^{27}\)See Doraszelski and Judd (2007) for an extensive discussion of this point.
The effect of a change in the depth of the LOB on the opposite side of the order is different for market and limit orders. An increase in $A$ in panels b and a decreases buy LO frequencies but increases buy MO frequencies. Note that $A$ does not change the probability of execution of buy orders directly through the state variable $F^B$, but reduces uninformed liquidity sellers incentives to submit sell market orders. This ultimately induce less buy limit orders’ executions. These observations suggest that, as reported in Bloomfield et al. (2005), changes in the depth of one side of the book affect order submissions on both sides. The model’s equilibrium suggest that on average, LO submissions are more sensitive to changes in depth on the same side of the order whereas MO change more with depth on the opposite side of the order.

The dotted lines of panels a and b in figure 4 display the effect of a simultaneous increase in the depth of the ask and bid sides, keeping the LOB’s depth balanced. This change reduces limit order intensities significantly, and it has a negative effect on MO intensities as well28.

The rightmost panel of figure 3 shows the effect of changes in the informational composition of the LOB on the speculators’ strategy (captured by an increase in the state variable $F^B$). The impact on buy limit and market orders submissions is negative. This downward shift reflects that whenever informed orders are posted in the book, it is relatively more attractive for the high type to wait for their execution instead of clustering additional LO or issuing new MO (with the corresponding information transmission cost). The increase in $F^B$ also affect sell orders (not displayed in Figure 3). In particular, it decreases the submission intensities of sell market orders and increases slightly the intensities of sell limit orders. This effect is intuitive since, after an increase in $F^B$, the chances of a low type self-trading are greater. This makes the use of limit orders relatively more attractive for the low type. This effect has not been explored empirically, arguably because the information content of posted orders is not observable.

Panel c of Figure 3 also provide hints about changes in relative liquidity supply-demand behavior as prices adjust to information. Indeed, we can observe that when the price discovery process is sufficiently advanced (price beliefs are close to one in the figure) the informed trader submits limit orders more frequently than market orders.

**Result 5.2 (Relative Informed Liquidity Provision)**

(i) The behavior of the informed trader change in response to the dynamic adjustment of prices to information: for any given state of the book she takes (provides) relative more liquidity when the value of their information is high (low). The relative willingness to provide liquidity (ii) decreases with book depth of the book on the same side of the order when the value of information is low and increases when the value of information is high (iii) decreases with depth on the opposite side of the order and (iv) decreases with the participation in book liquidity on the same side of the order.

[FIGURE 4 ABOUT HERE]

Result 5.2 is best illustrated in Figure 4, which display the computed ‘relative liquidity provision

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28 Similar effects were reported in empirical investigations: Biais et al (1995), Griffiths (2000) and Ranaldo (2004) all found that limit orders are more likely to arrive in thin limit order books.
index’ \( R^+_H : S^I \rightarrow [0,1] \) defined as

\[
R^+_H(s^I) \overset{\text{def}}{=} \frac{x^+_H(s^I)}{x^+_H(s^I) + x^-_H(s^I)}
\]

For the high type speculator, this ratio increases for any configuration of the LOB as prices approach one. Interestingly, this change in trading attitude of the informed trader from an aggressive market order trader to a dealer-like liquidity provider was observed in the experimental environment of Bloomfield et al. (2005)\(^{29}\).

Overall, we observe a general pattern: even when both market and limit order intensities depend on the whole set of state variables, market orders are relatively more sensitive to changes in the value of information and limit orders are relatively more sensitive to changes in the state and information content of the limit order book.

We now discuss the equilibrium implications about strategic liquidity traders behavior.

**Result 5.3** (Uninformed Traders’ Equilibrium Behavior) (i) Submissions of uninformed limit and market orders decrease with the depth of the book on the same side of the order and (ii) uninformed limit order submissions are less frequent whenever the depth of the book increases on both sides of the book. (ii) Limit order submissions tend to be higher in states of the market in which bid-ask spreads are higher.

An increase in the depth of the bid side drives the downward shift in buyer’s LO and MO frequencies in panels a and b of figure 5. These effects are intuitive. Whenever the number of posted orders on a given side of the LOB is higher, for a given level of MO arrivals, the expected execution of liquidity orders will be higher. Since liquidity traders try to minimize the cost of reaching a given expected execution rate, they will thus submit relatively fewer orders.

Finally, figure 6 displays the value functions of both informed and strategic liquidity traders for a given LOB configuration. Panels a and b indicate that changes in the number of informed orders in the book have a significant impact on the informed trader expected profits. Changes in the number of uninformed orders have a second order of magnitude effect. In particular, note that when the high (low) type speculator holds an order on the offer (buy) side of the book, her expected value drops dramatically. Such decrease in her expected profits, for any given price prior, rationalizes why the informed trader does not bluff with limit orders in the computed equilibrium.

Interestingly, strategic liquidity traders submit orders even when the execution target is zero. In this case, they use limit orders only and participate to make small profits in expectation as pure market makers. Their profits stem from the liquidity provision service to non-strategic liquidity traders. A market-making role arise thus endogenously also for strategic uninformed traders, even when no trade pre-commitments are set.

\(^{29}\)The results in Bloomfield et al. (2005) support parts (i)-(ii) of Result xx.2. One noticeable difference is found in (iii): increases in depth on the opposite side of the order tend to increase relative liquidity provision in the lab. Part (iv) of Result xx.2 is not investigated by the authors.
5.3.2 Price Impact

In an equilibrium setting, liquidity providers’ quoting and pricing behavior is affected by the behavior of every market participant. Both liquidity traders and the speculator flow closely market events and change their beliefs about the asset value and the composition of the limit order book. The following result summarizes liquidity providers’ equilibrium use of order flow and LOB’s information.

**Result 5.4** *(Equilibrium Price Impact Functions)* (i) The price impact of market orders increases (in absolute value) with the LOB’s depth on the same side of the order and, to a lesser extent, with depth on the opposite side (ii) The price impact of limit order decreases in absolute value with LOB’s depth on the same side and, to a lesser extent, with depth on the opposite side (iii) A balanced increase in depth of the limit order book is associated with higher price impact of market orders and lower price impact of limit orders.

Result 5.4 is illustrated in Figure 7. Note the asymmetric shape of the different functions in the figure. Price impacts are higher on average for relative low values of \( p \) in panels a and c (buy market and limit orders, respectively), and they are relative higher for high values of \( p \) in panels b and d (sell market and limit orders, respectively). This is consistent with optimal submissions discussed above. In the case of buy market orders, for example, we found that the speculator submits more orders when the value of information is high (panel a of Figure 3) whereas liquidity traders tend to submit relatively more orders of this type when the value of information is low. This makes buy market orders more (less) informative for lower (higher) \( p \) values.

Note also that, as we approach extreme values of \( p \), the updating impact power of market and limit order on the same side differ. In particular, for values close to one, buy market orders are less informative than buy limit orders (note in figure 3 that \( x_{H}^{m+} < x_{H}^{l+} \) in that range), whereas sell market orders are less informative than sell limit orders for values of \( p \) close to zero. This interesting fact is yet another consequence of speculators’ endogenous willingness to provide liquidity at advanced phases of the price discovery process.

**[FIGURE 7 ABOUT HERE]**

5.4 Strategies and Market Environment

We perform a number of parameter change exercises to evaluate the behavior of equilibrium objects under different market environments.

**Result 5.5.** *(Information horizon and informed trader’s strategies)* The expected duration of the information advantage is positively related with the submission intensities of market orders and negatively related with the submission intensities of limit orders. An increase in the expected information horizon increases (weakly) the informed trader relative liquidity provision for any given state of the market.

The top panels of Figure 8 show the effect of a decrease in the expected duration of the information advantage (an increase in \( r \)) on the high type informed trader. We notice the informed trader increases the intensity of MO submissions and decreases the intensity of LO submissions. This behavior is very intuitive: with a reduction in the expected time to exploit her informational advantage, the speculator
finds more urgency in realizing profits, a goal that can be achieved increasing the relative use of market orders. Further, a shorter expected horizon makes the execution of her limit orders less likely, making this instrument relatively less attractive. Consequently, we observe in the top rightmost panel of figure 8 that \( R^+ \) decreases for any given market state.

**Result 5.6.** *(Tick size and informed trader’s strategies)* The tick size is negatively related with the submission intensities of market orders and positively related with the submission intensities of limit orders. An increase in the tick size increases (weakly) the informed trader relative liquidity provision for any given state of the market.

The middle row of panels in Figure 8 depict the effect of a decrease in the "tick size" parameter \( \Delta \) on the high type speculators’ strategies. This parameter directly influence the price improvement achieved by LO, and consequently a decrease in its value is expected to make LOs relatively less attractive. This is what we indeed observe in the new equilibrium. Note that the change on market order intensities is negligible except for beliefs values close to one, precisely when the bid-ask spread reduces to a value equal to its minimum \( \Delta \). Consequently, \( R^+ \) increases tangibly only at advance stages of traders’ interaction.

**Result 5.7.** *(Exogenous shifts in liquidity supply-demand)* An exogenous simultaneous increase in uninformed liquidity demand and supply increases informed market and limit order submissions.

The bottom panels of Figure 8 show the effect of an increase in exogenous liquidity demand-supply given by parameters \( z^N_m, z^N_f \). This change makes noise market orders (both buy and sell) to arrive more frequently and thus induce more executions of orders in the limit order book. Such an increase in execution rates make LO relative more attractive to the speculator since it reduces times to executions. At the same time, an increase in \( z^N_m \) makes easier for the speculator to hide his information trading with market orders. A visual inspection of equation 8 makes this point clearer. As a result, the speculator submits market orders more often, as the leftmost bottom panel of Figure 8 shows. Consequently, an exogenous increase of exogenous liquidity demand in the market creates more profit opportunities for the speculator, who increases his trading activities both as liquidity a consumer and as a liquidity provider. The effect on relative liquidity provision though is ambiguous (for the particular parametrization considered it is positive, as displayed in figure 8).

**[FIGURE 8 ABOUT HERE]**

### 6 The Dynamics Market Variables

We use the computed equilibrium objects to simulate the evolution of a number of interesting market variables, that are jointly determined on the equilibrium path. Letting \( w \) be the vector of such variables, we obtain \( w \)'s conditional distribution for any initial state of the game \( s^0_0 \cup v \)

\[
f(w|\gamma, x, z_S, Q, s^0_0, v)
\]

where \( \gamma = [z^N_m, z^N_f, z^N_c, r, \Delta, \phi, c] \) is the market environment, \( Q \) is the set of equilibrium price impact functions and \( x \) and \( z_S \) represent the equilibrium order placement strategy of the informed trader and

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\(^{30}\)A similar effect is found in Kaniel and Liu (2006).
strategic liquidity traders, respectively. To compute this distribution we use a thinning algorithm similar to the proposed by Ogata (1981), adapting it to the case of a multivariate doubly stochastic process with a random horizon (the details of this algorithm can be found in the appendix). We use the output of this algorithm in the following subsections. We focus on the dynamic impact of an information event at time $t = 0$ on market variables. We set $D_0 = (0, 0)$, $p_0 = 1/2$ and $v = 1$, and we simulate 3000 trading sessions. During each market session, orders arrive at the market and trades take place until the actual asset value is publicly announced. A natural time scaling to consider is event time, denoted ‘pooled order arrivals time’ in this context. One unit of time correspond to a market event (an order arrival or cancellation). We adopt this time measure in figures 9-11.

6.1 Evolution of Liquidity

Figure 9 displays the evolution of traders’ expected submission intensities and dealers’ differential probability of intermediation, after the arrival of good news. The informed traders’ submissions are displayed on the top panel and suggest the following. First, we observe that both buy market and limit order intensities are initially high and decay progressively as the price discovery process takes place. Limit orders submission intensities decay at a slower rate. Consequently, the relative use of limit orders increases over time consistently with the analysis in section 5. Second, note that this agent does not submit sell orders with positive probability. In other words, blufing is not observed on the equilibrium path\(^{31}\). A direct consequence is that cancellation of informed order is not observed on the equilibrium path either\(^{32}\).

These results also suggest that in the presence of asymmetric information, interarrival times of both market and limit orders are serially correlated.

**Empirical Implication 6.1.** Following the arrival of news to the market (i) the mean time interval between orders decrease on the side of information for all order types. On the side of information, market and limit orders show positive autocorrelation. (ii) On the side of information market orders arrive relatively more frequently on average when spreads are high (information value is high), and limit order arrive relatively more frequently when spreads are low (information value is low).

Patterns of serial correlations in order submissions are studied in Biais et al. (1995), Engle and Russell (1998), Engle and Dufour (2000), among others. These authors suggest that serially correlated arrivals may be due to the presence of informed traders in the markets. The explicit role of asymmetric information in this exercise provide support to these conjectures.

Uninformed order submissions in the middle panel of Figure 9 highlight the interrelated nature of buyers and sellers. At early stages of the trading session, strategic liquidity sellers and buyers generate a similar flow of market orders. As buy limit orders accumulate in the order book, less market order

---

\(^{31}\) Back and Baruch (2004) showed in a dealer market sufficient conditions for bluffing with market order possible. Bluffing arise when price beliefs approach the true value asymptotically and order size approaches zero. Note that order size remains fixed in our analysis.

\(^{32}\) The cancellations of buy limit orders would reduce her expected profits, as suggested by figure 6, and sell limit orders are not submitted.
executions will be needed to secure the target rate. This generates asymmetries in the sign of liquidity order flow as displayed.

Finally, the bottom panel shows the excess probability that dealers will intermediate on the buy side. As the informed trader progressively consumes liquidity on the sell side of the book, this excess probability increases.

6.2 Prices, Spreads and Price Variability

The behavior of prices and price impact functions is depicted in Figure 10. Panel a shows an important fact, the pattern of order placements that arise in the market pushes the approximation of liquidity providers’ beliefs to the true asset value. As price discovery progresses. Second, we observe that the price impact of all arriving orders is significant in early stages, precisely when the speculator places orders more aggressively. Over time, the rate of information transmission through trades diminishes. Note that cancellations do not have impact on uninformed agents’ belief at the beginning of the trading session simply because the LOB is empty at time zero. Naturally, in the absence of informed cancellations, the cancellation of a buy order induces a downward revision in beliefs and vice versa.

The evolution of the price updating functions, in turn, shape the evolution of the bid-ask spread, displayed in panel b. At early stages, when the informed trader places order aggressively, the value of the spread peaks. It decays over time as a greater number of orders is processed.

Finally, note that early in the trading session the instantaneous price variability is high. Given that every type of arriving order has a price impact in equilibrium, the volatility of asset value belief is greater at all times than transaction price volatility (the latter is only affected by market orders arrivals). The following empirical implication summarizes the discussion.

**Empirical Implication 6.2.** Following the arrival of news to the market (i) Transaction prices drift on average toward the fundamental value of the asset (ii) The price impact of all orders increase, and decay progressively. The bid-ask spread follows the same pattern until it equals the tick size (iii) The price impact of cancellations have the opposite sign of the price impact of limit orders (iv) Transactions price’ instantaneous volatility jumps up.

In accordance with our results, Engle and Dufour (2000) observed that the price impact of orders is higher when trading intensity is high.

6.3 The Limit Order Book

The top left panel of Figure 10 shows the dynamic evolution of the order book variables. Starting from a time zero situation in which the order book is empty, the depth of both the bid and ask sides of the book grow initially as traders post orders. However, the influence of the high type speculator makes the bid side of the book to grow faster. In the advent of a positive information event the relative depth of the book \( B - A \) shows then an inverted u-shape over time. Intuitively, an inverted u-shape evolution where \( B > A \) (u-shape when \( A > B \)) indicates that the asset is overvalued (undervalued) by the market.
Empirical Implication 6.3. Following the arrival of news to the market, the limit-order book’s depth shows unbalances. Higher bid depth is an indication of the asset being undervalued. Higher ask depth is an indication of the asset being overvalued.

The dynamic pattern on LOB’s variables induce a bias the distribution of depth during an information event. In the case of good news arriving, the distribution of relative depth is biased to the right as panel d of figure 10 shows.

7 Market Design and Price Discovery

A fundamental theme in the microstructure literature is the following question: how fast does information incorporate into prices?. To shed light on the topic, this section studies the distribution of the following first passage time

\[ T = \inf_{t>0} \{ p(t) \geq 1 - k\Delta | v = 1, s_0^i \} \]  

in different market designs. Intuitively, \( T \) represent the first time at which, for any initial belief \( p_0 \in (0, 1 - k\Delta) \), the asset price places \( k\Delta \) units away from the true state or closer. It provides a simple way of measuring the speed of revelations. Performing the analysis for different integers \( k \) permits to draw more robust conclusions.

In the absence of informed traders, order arrivals would not shift prices in the model towards the true fundamental value. In such situation, public announcements would take place at a random time \( \tau^* \exp(r) \), and consequently \( ET = r^{-1} \). We want to study how the actions of the speculator impact the distribution of \( T \) in the case of a pure dealers’ market and in the case with dealers and a LOB (hybrid market). A pure quote-driven dealers’ market, like the one analyzed in Glosten and Milgrom (1985), can be viewed as a particular case of the standard model, once the limit order book is eliminated.

We then simulate the distribution of random time \( T \) in each market type\(^{33}\). To compare the effects of the informed trader actions with different make types, a standpoint about the level of non-informed trading activity is in order. Note first that strategic liquidity traders have a trivial problem to solve, since limit orders are eliminated from their choice set. Consequently, their market orders submission rate needs to match the expected execution target exactly at any time. We fix that level in the dealers’ market at \( z_D^m \) on both sides. To make the comparison consistent, the criteria we adopt is to set \( z_D^m \) at a level that achieves in equilibrium the same volume of trade for the uninformed trade as \( z^m \) and \( z^f \) do jointly in the model with a LOB (see the appendix for more details about the calculation of \( z_D^m \)).

Figure 12 displays the results for the case with initial conditions \( p_0 = 1/2, D_0 = (0, 0) \), and \( r = 0.1, k = 2 \). The left panel corresponds to the a hybrid market . We observe that the action of the informed trader impacts the distribution of \( T \) non trivially. In particular, its expected value decreases to from 10 to 8.23 units of time. The core of the revelation process, however, does not take place in a short period

\(^{33}\)A model without a limit order book is significantly easier to solve and simulate numerically since both the action and state spaces are reduced dramatically wrto the the market with a LOB.
of time. The speculator optimally uses her information advantage to realize profits carefully delaying information transmission.

The right panel of figure 12 displays the distribution of \( T \) in the dealers’ market. The informed also delays information leakage, but the expected value of \( T \) is reduced to 7.28\(^{34} \). The speed of information transmission into prices is lowered in the market that includes a limit order book. This suggests that a limit order market, by expanding traders’ set of possible actions, permits speculators to design trading strategies that delay information revelation further.

[FIGURE 12 ABOUT HERE]

8 Empirical Evidence

8.1 Structural Restrictions

The equilibrium analysis in section 5 implies that both informed and uninformed traders are on average more willing to provide liquidity to other traders when the depth of the limit-order book on the same side of the order is lower. For buy limit orders, given any set of conditioning variables, if \( B' > B \)

\[
\begin{align*}
  z_i^{f+}(B,\cdot) & > z_i^{f+}(B',\cdot) \\
  x_i^{f+}(B,\cdot) & > x_i^{f+}(B',\cdot)
\end{align*}
\]

This condition on optimal intensities holds both for the informed trader and liquidity traders, and under any informational regime \( i \in I = \{-1, 0, 1\} \). In other words, it is satisfied in cases of asymmetric information when bad news arrive (indexed \( i = -1 \)), when good news arrive (indexed \( i = 1 \)) and when no informed trader is present\(^ {35} \) (indexed \( i = 0 \)). As a consequence, it will hold for aggregate intensities of the form \( y = x + z \) as well.

**Condition 1 (Book Depth Monotonicity +).** For any information regime \( i \in I \), and any given information value \( \|p - v\| \in R_+ \), and ask depth of the LOB \( A \), we have, whenever \( B' > B \)

\[
y_i^{f+}(\|p - v\|, A, B) > y_i^{f+}(\|p - v\|, A, B')
\]

**Condition 2 (Book Depth Monotonicity -).** For any information regime \( i \in I \), and any given information value \( \|p - v\| \in R_+ \), and bid depth of the LOB \( B \), we have, whenever \( A' > A \).

\[
y_i^{f-}(\|p - v\|, B, A) > y_i^{f-}(\|p - v\|, B, A')
\]

\(^{34}\)The same pattern is found for other choice integers \( k \).

\(^{35}\)To conclude that the condition holds for situations of symmetric information as well, I designed an additional equilibrium algorithm in which only uninformed traders and dealers are present, although they incorporate that new information may arrive in the future. Under such conditions, new orders have no price impact and the bid-ask spread equals \( \Delta \). Strategic liquidity traders still deal with an optimal execution problem. Their equilibrium strategy in this case, not reported for brevity in the main body of the paper, still satisfies the book depth monotonicity condition.
The test will be based on 28 and 29. The economic intuition behind the test is strong: independently of trading motives, it is relatively more attractive to "make" liquidity when liquidity supply is scarce and to "take" liquidity when supply competition is intense.

Note that the conditioning variable in 28 and 29 is \( \|p - v\| \) and not simply \( p \). From a strategic point of view, it is the distance between market and full information prices what drives the actions of the informed trader. In the formulation of the model we kept the realization of \( v \) fixed in the analysis and hence \( p \) was sufficient. In a real market, many information events may have happened for any given symbol and thus \( v \) is better seen as an unobservable stochastic process. One limit on the direct implementation of 28 and 29 is that the value of information \( \|p - v\| \) is not observable either. I will use the bid-ask spread to proxy for this variable. This choice relies on the fact that in equilibrium bid-ask spreads tend to be higher on average in states where the value of information is high. The rationale of this choice is further discussed in the econometric appendix.

8.2 Data Description

Data Sources. We used two different sources of data in the analysis. Both samples span two weeks of trading between May 6-17 2002. In that period, decimalization was in place as well the OpenBook information system\(^{36}\). First, we used Trades and Quotes (TAQ) database, distributed by the NYSE, as the source of market’s observable variables traders use to condition their strategies upon. Second, we used NYSE’s proprietary System Order Data (SOD) for order specific information. The SOD database includes detailed information (by the second) on all orders that arrive at the exchange through NYSE’s SuperDot system or that are entered by the specialist. The files contain about 99% of the exchange’s orders, which represented three quarters of total NYSE traded volume in 2002.

Symbol selection. To select the sample symbols were sorted according to order-arrival activity during the first day in the sample. Two symbols of each decile were randomly selected\(^{37}\). Our sample then contains twenty symbols representing different type of assets as regards market activity.

Variables construction. Let \( W_N = \{w_1, ..., w_N\} \) represent the sample for a given symbol. Observations in \( W_N \) are indexed chronologically according to the order of arrival to the NYSE’s electronic platform. An observation \( n \) in \( W_N \) contains the following fields. First, order-specific information \( \{t_n, o_n\} \), where \( t_n \) stands for order \( n \)'s calendar arrival time and \( o_n \) stands for order \( n \)'s type. Second, it contains observable market aggregates \( \{A_n, B_n, \sigma_n\} \), where \( A_n \) and \( B_n \) represent total depth at the best available prices on the offer and bid side of the LOB, respectively, and \( \sigma_n \) stands for the value of the bid-ask spread at time \( t_n \). Hereafter we refer to traders’ best-available prices on any side of the LOB as ‘prices at the market’. From order specific information, we construct the sequence of order interarrival times: for each order type \( o \), \( T_n = t_n^o - t_{n-1}^o \). Consequently, an observation \( n \) in \( W_N \) represents the vector \( w_n = \{A_n, B_n, \sigma_n, T_n, o_n\}\(^{38}\) .

To test 28 and 29 I consider only non-marketable\(^{39}\) limit orders that are submitted at the market,\(^{32}\)

---

36 OpenBook service was introduced on January 24, 2002, for all NYSE securities simultaneously. OpenBook operates between 7:30 A.M. and 4:30 P.M and showed at the moment of introduction the aggregate limit-order volume available in the NYSE Display Book system at each price point.

37 Only symbols with "normal trading activity" were considered. In particular, stocks that traded continuously, did not experiment splits or delisting during the sample period, etc.

38 Additional detail about the sample and variable construction is provided in the econometric appendix.

39 Limit orders are non-marketable if their submission price is strictly below the best available price on the other side of
during normal trading hours. Table 2 in appendix F shows the number of limit orders that arrived during our sample period for each of the selected symbols, which are sorted by trading activity. It is worth noting that limit orders at the market represent a very important part of total limit orders in the sample. Note that the relative importance of limit orders at the market is higher on average for stocks that are relatively less traded (86.6% of total non-marketable limit orders on average for the top ten symbols in Table 2), but still represents a very important proportion for relatively more active stocks (66.7% for the bottom ten symbols).

8.3 Test Design

8.3.1 Estimation of Conditional Intensities

A necessary step in testing the structural restrictions is to estimate conditional submission intensities of the form

\[ \gamma^\alpha(s_t) = \lim_\delta \to 0 \frac{\Pr(Y_{t+\delta}^\alpha - Y_t^\alpha > 0 | s_t)}{\delta} \]  (30)

Since we need to condition on multiple market states, it is convenient to work with categories. I will group (i) depth observations in quintiles of its empirical distribution and (ii) I will use two categories for the bid-ask spread, those in the first quartile of its empirical distribution and up. A given market variables triple \( s_i = (A_i, B_i, \sigma_i) \) at time \( t_i \) will then belong to the set \( S = \{(A_k, B_{k'}, \sigma) : k, k' = 1, \ldots, K = 5; \sigma \in \{\sigma_1, \sigma_2\}\} \)

of size fifty. This selection proved reasonable to avoid having too many 'holes' in the state space, that is states with no observations on it. The criteria for the spread categories has an economic angle. Individual orders submitted by uninformed and informed traders cannot be differentiated by simply looking at arrivals in the dataset. However, to gain further insight I can test the conditions separately for situations in which asymmetric information is more and less likely. The first bid-ask spread category will collect the situations in which this variable is either at its minimum (one cent) or very close to it. Following the model’s insights, we believe these are situations of likely symmetric information. The second category collects the cases in which the spread is relatively high. We believe this are situations of likely asymmetric information.

We will estimate 30 using the following simple non-parametric estimator

\[ \tilde{\gamma}^\alpha(s) = \left( \frac{1}{n(o,s)} \sum_{i=1}^{n(o,s)} \left( t_i^\alpha - t_{i-1}^\alpha \right) \right)^{-1} \]
where \( n(o, s) \) is the number of times that order type \( o \) arrived in state \( s \in S \), and \( t_{00} = 0 \). Consistency results for this class of estimators are provided in Karr (1991). Figures 13 and 14 display the estimated buy and sell limit order intensities, respectively, for the most actively traded stock in our sample.

8.3.2 Test Construction

To suitable construct the test statistic, let

\[
D_k^+(\sigma, A) \overset{\text{def}}{=} \min\{0, y^{\ell^+}(\sigma, A, B_k) - y^{\ell^+}(\sigma, A, B_{k+1})\}
\]

and define \( D_k^-(\sigma, B) \) analogously. Note that if 28 and 29 hold, both \( D_k^+(\cdot) \) and \( D_k^-(\cdot) \) will be zero for any set of conditioning states. Further, let

\[
T^+(\sigma, A) \overset{\text{def}}{=} \sum_{k=1}^{K-1} (D_k^+(\sigma, A))^2
\]

and define \( T^-(\sigma, B) \) and \( T^- \) analogously. Finally, let

\[
T^+ \overset{\text{def}}{=} \sum_{\{\sigma, A\}} \sum_{k=1}^{K-1} (D_k^+(\sigma, A))^2
\]

and

\[
T^- \overset{\text{def}}{=} \sum_{\{\sigma, B\}} \sum_{k=1}^{K-1} (D_k^-(\sigma, A))^2
\]

Naturally, if traders’ behavior satisfies the stated monotonicity restriction on book depth for every depth level, and for every set of conditioning states, then 32 and 33 will equal zero exactly. Consequently, our null and alternative hypothesis will be given by\(^{43}\).

\[
H_0^+ : T^+ = 0, \quad H_1^+ : T^+ > 0
\]

\[
H_0^- : T^- = 0, \quad H_1^- : T^- > 0
\]

We will use the estimation set \( \{\tilde{y}_N^a(s) : s \in S\} \) to construct estimates of \( T^+, T^- \) for each symbol in the sample. To approximate the distribution of \( \hat{T}_N^+, \hat{T}_N^- \), we implement an instance of Horowitz’s Markov Conditional Bootstrap (MCB) (Horowitz, 2003). Further details about the implementation can be found in the Econometric Appendix.

8.3.3 Empirical Results

Table 3 shows the results of the test for buy limit orders. The null hypothesis \( T^+ \) cannot be rejected at a 5% significance level for 19 of 20 stocks.

Further, two interesting insights can be drawn from Table 3. First, p-values are higher on average for relatively more active stocks. This is intuitive since optimal management of liquidity competition is

\(^{43}\)Similar one-sided test statistics have been used recently by Chernozukov et al. (2007) among others.
likely to less of a concern for stocks that are traded only rarely. It may also reflect that the assumption of continuous monitoring of market variables (and timely reactions to information) better represents trading activity for stocks that are relatively more active. Second, conditioning on low spread levels, \( H_0 \) cannot be rejected either for 19 of the stocks and, conditioning on high spreads, cannot be rejected for any stock. Further, p-values are higher on average for high-spread intensities. This results for specific spread values suggest the following. First, when we believe there are no asymmetries, liquidity traders submit limit orders conforming to the model’s implications. Second, when asymmetries are likely, uninformed and informed traders jointly do so as well.

Table 4 show the results for of the test for sell limit orders. As we can observe, \( H_0^- \) cannot be rejected for any of the symbols in the sample at a 5% significance level. It cannot be rejected either when conditioning of specific spread values. We observe that p-value share the same patterns reported above for buy orders. Finally, Table 5 reports the results for buy and sell orders jointly. In this case, we cannot reject the hypothesis for 19 of the symbols in the sample. Further, for the joint test p-values are higher on average than the corresponding to the separate buy-sell tests. Overall, the data seems to support conditions 28 and 29 quite strongly.

9 Concluding Remarks

Despite the economic importance of electronic trading platforms in world’s asset markets, little is known about how traders make decisions there. Key strategic dimensions like order type choice and optimal submissions’ timing failed to be integrated in a dynamic equilibrium model. Unsurprisingly, the task of distilling the economics in order-level data has remained elusive. I introduced in this paper a new framework that incorporate these elements and reproduces salient features of the working operation of important asset exchanges. I characterize the equilibrium behavior of traders who have an information advantage and those who don’t, and I analyze the consequences of their interaction on market variables’ dynamics.

This paper also explores using the framework in comparative market design. A quote-driven dealers’ market and an order-driven limit-order market were considered to study the speed of aggregation of information into prices. This area of research is still green. To the best of my knowledge, the only antecedent of using a dynamic model to compare market designs is given in Back and Baruch (2007) who compare the equilibriums of a limit-order market and a floor exchange. Additional questions, that are beyond the scope of this paper, can be explored. How do elements of the design affect volume and price volatility? How do they affect the welfare of different participants? Which are the specific effects of changes in regulations or trading protocols on each market design? The answers to these types of questions are of primary importance for academic economists, investors and market regulators alike.

Further, the model also provides basis for structural empirical work with high-frequency data. The empirical exercise in this paper exemplifies the use of the model in that respect. The example highlights an important fact: enriching the strategic analysis by incorporating strategic submission times provides a bridge between the theory and the data. Since arrivals times convey strategic information in the model, observed arrival times can be used to learn about traders’ policies.
Beyond the empirical analysis in this paper, more sophisticated inference procedures could be designed in interesting applications. One promising area is the empirical identification of information events. The identification strategy of periods with and without asymmetric information adopted in section 8, which relies on bid-ask spreads only, does not fully exploit the model’s structure. The fact that multiple market observables - prices, volumes, arrival times and order types - are jointly determined within the model during such events, suggests that a much broader set of variables could be employed in identification schemes. State of the art techniques in hidden Markov models\textsuperscript{44} can be combined with models' relationships to identify such events in real time, an avenue that the author is currently exploring.

More generally, the structure developed in this study provides a tool for using order-level data (e.g. NYSE’s SOD) in a diverse range of applications in which transaction-level data (e.g. NYSE’s TAQ) has only been considered.

One important example of such research areas is the non-parametric realized volatility literature. The recent focus of this literature has been on designing methods that are robust to the microstructure noise components of price observations. However, very little is known about the distribution of those components. Giving the current dissatisfaction with simple (ad-hoc) noise schemes, authors like Hansen and Lunde (Hansen and Lunde, 2006) have advocated the use of information in the limit order book. But, in the absence of a structural framework, no systematic way of making sense of order sequences or the information in the limit-order book is available. This paper may offer new directions to implement such proposals.

\textsuperscript{44}See, for example, in Cappe’ et al. (2005).
REFERENCES


(with )


Victor Chernozhukov"


JAIN 2005, COMPLETE


Appendix A: More about the Trading Game

A1. Expressions for changes in Value functions

The expected change in $V(s^t)$, for any given $s^t = (D,F,p)$, caused by the arrival of a market order is denoted $\tilde{V}(s^t_o)$ where $o \in \{m+,m-\}$. We then have \(^{45}\):

\[
\tilde{V}_i(s^t_{m+}) = \mathbb{E}(V_i(s^t_{m+})|dY^m_{t+} = 1)
\]

\[
= (1 - 1_A)V_i(D,F,a) + F^A V_i \left( B, (A - 1)^+, F^B, \left( \frac{A^t - 1}{\max\{A - 1,1\}} \right)^+, a \right) \tag{36}
\]

\[
+ (1 - F^A)V_i \left( B, (A - 1)^+, F^B, \left( \frac{A^t - 1}{\max\{A - 1,1\}} \right)^+, a \right) \tag{37}
\]

\[
\tilde{V}_i(s^t_{m-}) = \mathbb{E}(V_i(s^t_{m-})|dY^m_{t-} = 1)
\]

\[
= (1 - 1_B)V_i(D,F,b) + F^B V_i \left( (B - 1)^+, A, \left( \frac{B^t - 1}{\max\{B - 1,1\}} \right)^+, F^A, b \right) \tag{38}
\]

\[
+ (1 - F^B)V_i \left( (B - 1)^+, A, \left( \frac{B^t - 1}{\max\{B - 1,1\}} \right)^+, F^A, b \right) \tag{39}
\]

A2. Cancellations

Whenever a liquidity trader cancels a limit order, uninformed liquidity providers may want to revise backwards previous belief updates that order induced. If for example an uninformed buy limit order is canceled, they may want to correct their beliefs downwards. To fully revise their previous updates, traders would need to keep track of the sequence or orders and prices instead of simply contemporary market variables. To approximate that revision process in the numerical solution we use

\[
q^o(s) = \frac{1}{y^o(s)} \left[ p \sum_{F \in F(D)} x^o_{H}(s^t)P(F|s,v_H) + z^o \left( q^{o d} \right)^{-1}(p) \right] \tag{40}
\]

where $d = \{+, -\}$ for $o \in \{c+, c-\}$ respectively.

The difference with say 8 lies in the value with which a liquidity cancellations is weighted. The term $(q^{o d})^{-1}(p)$ represents the mapping $q^{o d}(p) \to p$, for any given $D$.

\(^{45}\)The role of $\max$ operators in 36 and 38 is simply to avoid $\frac{0}{0}$ indeterminancies.
Appendix B: Proofs

Proof of Proposition 4.1

Suppose the contrary. Consider without loss of generality a high type informed trader (IT). Her strategy would then be a sequence of $\mathcal{F}^U$-measurable stopping times $\tau_1, \tau_2, \ldots$ at which she submits a specific order type. Consider any time $t$ at which the order book is empty. Suppose that the current belief is $p$. Suppose further that the equilibrium strategy of the IT prescribes submitting an order (either limit or market) at a $\mathcal{F}^U$-time $\tau_1 \geq t$. If in equilibrium the informed trader were to follow such strategy, and there were no order at time $\tau_1$, liquidity providers would conclude that they are facing a low type speculator and would set their asset value beliefs equal to 0 thereafter. Hence, when $\tau_1$ is reached, the optimal reaction would be for the IT to refrain from trading and buy the asset afterwards, once the absorbing state $p = 0$ has been reached. Hence, there is no equilibrium strategy in which the IT submits an order at $\mathcal{F}^U$-times when the book is empty.

Thus, the only remaining possibility is that the first limit order arriving, say at time $t_1$, comes from a noise trader. In this case the IT state set would be $s^U = (p', D', F') = 0$, and it will be known to liquidity providers. Suppose now that under $s^U$ the equilibrium strategy of the IT prescribes submitting an order (either limit or market) at $\mathcal{F}^U$-time $\tau_2 \geq t_1$. If in equilibrium the IT were to follow such strategy, and there were no order at time $\tau_2$, market makers would believe they are facing a low type IT and would set their asset value beliefs equal to 0 thereafter. By iterating this argument forward in the same fashion we conclude that there is no strategy in equilibrium in which informed orders are submitted at $\mathcal{F}^U$-times. $\square$

Proof of Proposition 4.2

Consider time $t < \tau$, and let $s_{t-}$ be equal to $s = (p, D)$. Note that using Bayes rule and the Law of total probability we can decompose $q^o(s) = P(v_H|s, dY^o_t = 1), o \in O$, as follows

$$P(v_H|s, dY^o_t) = \frac{P(v_H|s)P(dY^o_t = 1|s, v_H)}{P(dY^o_t = 1|s)} = \frac{pP(dY^o_t = 1|s, v_H)}{P(dY^o_t = 1|s, v_H) + P(dY^o_t = 1|s, v_L)}$$

Now, reapplying the law of total probability we find that $P(v_H|s, dY^o_t = 1)$ equals

$$P(dY^o_t = 1|s, F, v_H) P(F|s, v_H) + (1 - p) \sum_{F \in F(D)} P(dY^o_t = 1|s, F, v_L) P(F|s, v_L)$$

Finally, since $P(dY^o_t = 1|s, F, v_H) = x_H^o(s^f) + z^o(s)$ and $P(dY^o_t = 1|s, F, v_H) = x_H^o(s^f) + z^o(s)$ we obtain 8 and 9. $\square$

Proof of Lemma 4.1
Consider objective function 12 of type $i \in \{L, H\}$. The associated Bellman can be written as

$$rV_i(s^I) = AV_i(s^I) + h_i(s^I, x)$$

where $A$ is the infinitesimal generator of the jump Markov process we deal with and $h$ is a running cost function (see for example Fleming and Soner, 2006). The generator $A$ is given by

$$AV_i(s^I) = \sum_{\tilde{s}^I \in S^I} \left[ V_i(\tilde{s}^I) - V_i(s^I) \right] \Pi_i(s^I, \tilde{s}^I)$$

where $\Phi_i(s^I)$ is the intensity with which a jump will occur under state $s^I$ and where $\Pi_i(s^I, \tilde{s}^I)$ is the probability distribution of the post jump location $\tilde{s}^I$ (see for example Ethier and Kurtz, 2005). The function $h_i(\cdot)$, in turn, is given by

$$h_i(s^I, x) = -c\left( (x^m_i)^\phi + (x^m_i)^\phi + (x^{\ell+}_i)^\phi + (x^{\ell-}_i)^\phi \right) + x^{m+}_i \pi^{m+}_i + z^m_i F^A \pi^{m-}_i + x^{m-}_i \pi^{m-}_i + z^m_i F^B \pi^{m+}_i$$

Finally, note that the jump intensity of each market event determine the form of $\Pi_i$, and the aggregate jump intensity $\Phi_i$ is given by

$$\Phi_i(s^I) = \sum_{o \in \mathcal{O}} (x^o_i(s^I) + z^o_i), i \in \{L, H\}$$

Rearranging we obtain 13

**Proof of Proposition 4.3**

To obtain expressions 14-17, take first order conditions of 13 with respect to every order type $o \in \{m+, m-\}$. An interior solution requires

$$x^o_i(s^I) = \frac{1}{\kappa o} \left[ V_i(s^I) - V_i(s^I) + \pi^o_i(s^I) \right]$$

Hence, the optimal order submission intensity for type $i$ is given by 14 and 15 for $o = m+$ and $o = m-$ respectively. Similarly, for order type $o \in \{\ell+, \ell-\}$, the optimization problem of type $i$ requires for an interior solution

$$x^o_i(s^I) = \frac{1}{\kappa o} \left[ V_i(s^I) - V_i(s^I) \right]$$

Hence, the optimal order submission intensity for type $i$ is given by 16 and 17 for $o = m+$ and $o = m-$ respectively. □

**Proof of Proposition 4.4**

We will develop the solution for strategic liquidity buyers. For sellers the analysis is similar and thus omitted here. Following the same logic used in Lemma 1, the Bellman equation of strategic liquidity
buyers is given by

\[ 0 = \min_{z^m, z^i} \kappa ((z^m + z^i)^\phi + (z^m + z^i)^\phi) + z^m \psi^m(s) + z^i \psi^i(s) \left( \tilde{F}_U^B(s) \right) \] (43)

\[-(z^m + (s) + q_N)[W_1^B(s_{m+}) - W_1^B(s)] - z^m(s)\left[ W_1^B(s_{i+}) - W_1^B(s) \right] \]

\[-q_N[W_1^B(s_{m-}) - W_1^B(s)] - q_N\left[ W_1^B(s_{i-}) - W_1^B(s) \right] \]

where \( \mathcal{O} = \{ m+, m-, i+, i-, c+, c- \} \) and

\[ \tilde{\Psi}(s_i) = pW_i^B(1, \bar{B}, \bar{A}) + (1 - p)W_i^B(0, \bar{B}, \bar{A}) \]

\[ \bar{B} = \text{round}(\tilde{F}_U^B(s)B), \bar{A} = \text{round}(\tilde{F}_U^A(s)A) \]

Note that the MO submission intensity \( z^m \) is pinned down directly by the constraint 18. Thus,

\[ z^m(s) = \max \left\{ 0, \epsilon - \bar{g}^m(s) \left( \frac{1}{n_B} \tilde{F}_S^B(s) \right) \right\} \]

Now, taking first order conditions of 43 w.r.t limit order intensity, in an interior solution

\[ z^i(s) = \frac{1}{k^\phi} [W_1^B(s) - W_1^B(s_{i+})]^{\frac{1}{\phi - 1}} \]

justifying 22.\( \square \)

**Proof of Proposition 5.1**

We follow the strategy outlined in Doraszelski and Judd (2008) for dynamic games in continuous time. Note though that their proof does not apply directly since the state space considered in this game is not finite. We consider here Schauder-Tychonoﬀ’s ﬁxed point theorem (see e.g. Rudin, 1991) with respect to a mapping from the set of values and policies to itself.

As discussed in section 3, intensities belong in equilibrium to the bounded set \([0, \bar{F}]\) where

\[ \bar{F} \geq \max_{i \in \{L, H\}, o \in \mathcal{O}, s' \in S^i} x_1^o(s') \]

and \( \{x_1^o(s')\}_{o \in \mathcal{O}} \) is given by 14-17. Hence, policies for the informed trader of type \( i \) belong to the set \([0, \bar{F}]\). This fact also makes submission costs bounded at any time by

\[ C = 4\kappa \bar{F}^\phi \]

Since intensities are bounded in equilibrium, only a ﬁnite number of orders will be submitted by the speculator during any time ﬁnite interval \( \mathbb{P} - a.s. \) Since for any given \( r > 0 \) information revelation will take place in a \( \mathbb{P} - a.s. \) ﬁnite time, and given that a maximum proﬁt of \( |v_H - v_L| \) can be achieved by any order execution, we conclude that proﬁts will be ﬁnite \( \mathbb{P} - a.s. \). Hence \( V_L, V_H \) will be ﬁnite \( \mathbb{P} - a.s. \). Let
Let \( V(.) \) denote the informed trader value, where \( V_i(.) \) is a mapping from \( S^I \) to \( [\underline{V}, \overline{V}] \). Let also \( x(.) \) denote the policies of the informed, where \( x_i : S^I \rightarrow [0, \infty]^4 \). Also, recall that \( Q \) is a set of mappings from \( S \subset S^I \) to real values in \([v_L, v_H]\).

Pointwise define the mapping \( f(\cdot) = (f^V(\cdot), f^x(\cdot), f^Q(\cdot))' \) from the set of values, policies and beliefs by

\[
\begin{align*}
  f^{V,i,s'}(V(\cdot), x(\cdot), Q(\cdot)) &= \left\{ \max \left\{ \underline{V}, \min \left\{ \overline{V}, \frac{1}{r} \max_{x_i \in [0,\infty]^4} G^i(x_i, z(s), Q(s), s^I, V_i(\cdot)) \right\} \right\} \right\}
  \quad f^{x,i,s'}(V(\cdot), x(\cdot), Q(\cdot)) = \left\{ \arg \max_{x_i \in [0,\infty]^4} G^i(x_i, z(s), Q(s), s^I, V_i(\cdot)) \right\}
  \quad f^{Q,s'}(x(\cdot)) = \{Q(x(s^I), s)\}
\end{align*}
\]

for each type \( i \in \{L, H\} \) and state \( s^I \in S^I \), where \( G_i \) is the maximand in the Bellman equation 13 for player \( i \), and \( Q(x(s^I), s) \) is given by the expressions in Proposition 4.2. The mapping \( f \) then maps \((V(\cdot), x(\cdot), Q(\cdot))\) into itself.

Fix \( i \in \{L, H\} \) and \( s^I \in S^I \). Note that \( G^i \) is a continuous function of values, beliefs and policies. By Berge’s theorem of the maximum then

\[
\max_{x_i \in [0,\infty]^4} G^i(x_i, z(s), Q(s), s^I, V_i(\cdot))
\]

is a continuous function of \( z(s), Q(s), \) and \( V_i(\cdot) \). Thus, \( f^{V,i,s'}(\cdot) \) is a continuous function of \( V(\cdot), x(\cdot), Q(\cdot) \) that maps into \([\underline{V}, \overline{V}]\) by construction.

Consider now the mapping \( f^x(.) \) fixing \( i \in \{L, H\} \) and \( s^I \in S^I \). According to the theorem of the maximum, \( f^{x,i,s'}(\cdot) \) is compact valued and upper hemicontinuous. Given equations 14-17 it is clear that it is also single valued. Consequently, \( f^{x,i,s'}(\cdot) \) is a continuous function of yields that is a continuous function of \( V(\cdot), x(\cdot), Q(\cdot) \) that maps into \([0,\infty]^4 \).

Finally, it is clear that \( f^{Q,s'}(\cdot) \) is a continuous function of \( x(\cdot) \) that maps onto \([v_L, v_H]^4 \). We conclude that \( f \) is a continuous mapping from a compact and convex subset of a locally convex topological vector space into itself. By the Schauder-Tychonoff theorem there exist \((V(\cdot), x(\cdot), Q(\cdot))\) such that \((V(\cdot), x(\cdot), Q(\cdot)) \in f((V(\cdot), x(\cdot), Q(\cdot))) \).

**Proof of Proposition 5.2**

We will show that in the case \( v = v_H \) the price process is a \( \mathcal{F}^I - R \) submartingale. The case for \( v = v_L \) is similar.

From 7 we know that, given \( s^I \in S^I \), the expected change in \( p \) per unit of time equals

\[
\delta_H(s^I) = [q^m^+(s) - p](x^m_+(s^I) + z^m) + [q^m^-(s) - p](x^m^-(s^I) + z^m) + [q^l^+(s) - p](x^l_+(s^I) + z^l + z^c) + [q^l^-(s) - p](x^l_-(s^I) + z^l + z^c) \tag{44}
\]

\[
\delta_H(s^I) = [q^m^+(s) - p](x^m_+(s^I) + z^m) + [q^m^-(s) - p](x^m^-(s^I) + z^m) + [q^l^+(s) - p](x^l_+(s^I) + z^l + z^c) + [q^l^-(s) - p](x^l_-(s^I) + z^l + z^c) \tag{45}
\]

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Consider the first two terms or the RHS of 44. Note that their sum will be positive whenever

\[
\begin{align*}
[q^{m^+}(s) - p](q^{m^+}_H + z^m) &> [p - q^{m^-}(s)](x^{m^-}_H + z^m) \\
\rho_m(s)(x^{m^+}_H - z^m + z^m) &> x^{m^-}_H
\end{align*}
\]

Similarly, the sum of the last two terms of the RHS of 44 will be positive anytime

\[
\rho_m(s)(x^{m^+}_H + z^m) - z^m - z^m > x^{m^-}_H
\]

Thus, under the made assumptions \( \delta_H(s^I) > 0 \ \forall s^I \in S^I \).

Consider now the process \( \{p'_t : t \geq 0\} \) satisfying 7 and \( p'_0 = p_0 \) so that \( p'_t = p_t \ \mathbb{P}-a.s. \) over \([0, \tau)\). Let \( T = \inf\{t : p'_t \in \{v_L, v_H\}\} \). Then

\[
\mathbb{E} \left[ p'_{t\wedge T} - p'_{\xi\wedge T} \mid \mathcal{F}^I_{\xi} \right] = \mathbb{E} \left[ \int_{\xi\wedge T}^{t\wedge T} \delta_H(s^I_{u^-}) du \mid \mathcal{F}^I_{\xi} \right] \geq 0
\]

where \( \xi < t \). This shows that when \( v = v_H, \ p' \) is a R-submartinale w.r.t. the informed trader’s information set. Since \( \{p'_{t\wedge T}\} \) is right continuous and integrable the Submartingale Convergence Theorem guarantees that \( p' \) converges to an integrable limit \( p'_\infty(\omega) \overset{\text{def}}{=} \lim_{t \to \infty} p'_t(\omega) \) for a.e. \( \omega \in \Omega \) (Karatzas and Shreve (1991) p.17). In particular, taking \( \xi = 0 \) and letting \( t \to \infty \)

\[
\mathbb{E} \int_0^T \delta_H(s^I_{u^-}) du = \mathbb{E} [p'_\infty | v_H] - p_0 < \infty
\]

and we conclude that \( \int_{[0,T]} \delta(s^I_{u^-}) du \) is finite \( \mathbb{P}-a.s. \). Since \( \hat{p}_t = p_t \) for any \( t < \tau \) this proves the statement. \( \square \)
Appendix C: Numerical Methods

Computation of Equilibrium

To compute equilibrium strategies, we work on a price grid on $[0, 1]$ of size $n$ and we define $K$ to be the maximum number of orders on each side of the book at any time. That is, if there are $K$ orders on the ask side at time $t$, and $dY_i^{K-} = 1$ at time $t' > t$, then $A_t$ is still equal to $K$. The discretized state space is then of size $(K+1)^4 \times n$ (although we don’t need to consider states in which $B^I > B$ or $A^I > A$).

We set in the displayed results $n = 200$ and $K = 5$. For robustness, we experimented with different grid sizes $n = \{50, 100, 200, 300\}$, different values of $K$ and different initial guesses.

The algorithm is iterative and works in the following manner.

Step 0: We make an initial guess of the value function and the optimal policies of each type of speculator and each order type. We also make an initial guess of the updating behavior of liquidity providers and about their beliefs on informed fractions under each possible state.

Step 1: Stop if the convergence criteria is met. Else, based on current guesses and the system of equations 14-17 we compute new guesses $\hat{V}_i(s^I)$ and $\hat{x}_i(s^I)$ for each element of the (discretized) state space $S^I$ according to

$$\hat{x}_i(s^I) \leftarrow \max \left\{0, \left(\frac{1}{\kappa^I} \left[ V_i(s^n_o) - V_i(s^I) + \pi_i^o\right]\right)\right\}$$

where $i \in \{L, H\}$, $o \in O$, and $\pi_i^o$ equals $\pi_i^{m+}$ if $o = m+$, $\pi_i^m$ if $o = m-$, and 0 in any other case.

With these updated policies, we update the value function according to

$$\hat{V}_i(s^I) = \delta V_i \left\{ \frac{1}{r + \Phi_i(s^I)} \Psi_i(s^I) \right\} + (1 - \delta)V_i(s^I)$$

where $\Phi_i(s^I)$ is the (controlled) jump intensity of the system given by 42, and where

$$\Psi_i(s^I) = \{ (\hat{x}_i^{m+} + z^m) V_i(s^I_{m+}) + \hat{x}_i^{m+} \pi_i^{m+} + z^m F A \pi_i^{m-} + \hat{x}_i^{m-} + z^m V_i(s^I_{m-}) + \hat{x}_i^{m-} \pi_i^{m-} + z^m F B \pi_i^{m+} + \hat{x}_i^t V_i(s^I_{t+}) + z^t 1_{(K)} B V_i(s^I_{t+}) + z^t 1_{(K)} A V_i(s^I_{t-}) \}$$

The indicator functions in 46 are defined by $1_{(K)}^B = 1\{B \leq K\}$ and $1_{(K)}^A = 1\{B \leq K\}$. Finally, the constant $\delta \in (0, 1)$ is introduced for numerical stability purposes only. Note that we update the value functions by using the recently updated policy guesses, so we indeed use a Gauss-Siedel scheme. This approach has the advantage of using new information as soon as it becomes available (Doraszelski et al, 2007).

Step 2: Based on the updated guesses of policies and the current beliefs about informed fractions, we recalculate all the price impact functions following the formulas in Proposition 4.2.

We stop and return to Step 1 with a certain probability $p_s$ that is increasing in the degree of convergence of the algorithm in each successive iteration. With probability $(1 - p_s)$ instead we move to Step 3.
Step 3: Based on the current guess for the value functions, policies and price updating functions we simulate $N = 200,000$ trading sessions. Each trading session evolves as follows

- Draw an initial belief $p_0 \sim U[0, 1]$
- Following Algorithm B2 simulate all the point processes describing order flow and information revelation. Update the states of the system and the intensities of every process after any event.
- Proceed in this fashion until information is revealed or the price gets closer than $1/2n$ units away from the absorbing states $\{0, 1\}$.

For each simulated market we account the number of times a given state is visited. If a given state is not visited, we give equal probability to feasible informed fraction under such state. We approximate $P(F|p, D)$ by using relative frequencies. Return to Step 1. □

**Stopping rule.** We use an adaptive stopping rule like the one proposed in Doraszelski and Judd (2007). Define $Y_j = \| \hat{V}_L^{(j)} + \hat{V}_H^{(j)} \|$ to be the Euclidean norm of the $j$-th guess of the sum of the value functions. The iteration is stopped when

$$\|Y_{l+1} - Y_l\| \leq (1 - \rho)\varepsilon$$

where $\varepsilon$ is a pre-specified tolerance and $\rho$ is given by

$$\rho = \left( \frac{\|Y_l - Y_{l-1}\|}{\|Y_k - Y_{k-1}\|} \right)^{-\frac{1}{k}} \quad (47)$$

In 47 $l$ corresponds to the first iteration such that $\|Y_l - Y_{l-1}\| \leq \varepsilon$ and $k$ is the first iteration in which $\|Y_k - Y_{k-1}\| \leq 10\varepsilon$. By construction, this stopping rule estimates the convergence factor from past iterates. In all the equilibrium computations we chose $\varepsilon = 1 \times 10^{-8}$. This adaptive rule is expected to avoid a premature termination of the iteration procedure vis a vis the ad hoc rule $\|Y_l - Y_{l-1}\| \leq \varepsilon$.

**SIMULATION TECHNIQUES**

For simplicity of notation let $\omega_o(s^I)$ stand for the equilibrium intensity of submission of order type $o \in O$ in state $s^I \in S^I$. The algorithm goes as follows

**Step 0:** Fix the value of $v \in \{v_L, v_H\}$ and draw an information horizon $\tau$ from an exponential distribution with parameter $r$.

**Step 1:** Define a piecewise constant function $\omega^*(t)$ such that $\omega^*(t) \geq \omega(t) \equiv \sum_{o \in O} \omega_o(s^I_t)$ for every $t \in [0, \tau]$ and $s^I_t \in S^I$.

**Step 2:** Simulate a non-homogeneous Poisson process with intensity $\omega^*(t)$ over $(0, \tau]$. Denote the arrival times by $t^*_{1, t^*_{2, ..., t^*_{N^*_I}}}$.

**Step 3:** Set $k = 1$, and the vector $i \in \mathbb{N}^{\#O}$ counting order arrivals equal to 0.

**Step 4:** Keep arrival time $t^*_j$ with probability with probability $[\omega^*(t) - \omega(t)]/\omega^*(t)$. If it is kept, attach a mark $o \in O$ with probability $\omega_o(t^*_j)/\omega(t^*_j)$. If the mark is $o'$ is selected, record the arrival time set $t^*_{i(o')} = t^*_j$.

**Step 5:** Set $k$ equal to $k + 1$ and update the state of the game $s^I_{k+1}$ accordingly.
Step 6: If $k > N_T^*$ stop, otherwise go to step 4. □

Algorithm B2 is justified broadly by Proposition 1 in Ogata (1981) since the class of point processes that he contemplates include multivariate doubly stochastic Markov point processes. Step 0 is added since the original version contemplates a fixed time interval.

We choose function $\omega^*(\cdot)$ to be constant. In particular

$$\omega^* = \sum_{o \in O} \left( \max_{s' \in S_i} x_i^o(s') + z^o \right)$$

Submission Rates in Dealers Markets

For any initial state $s_0 \in S_0$ and parameter set $\gamma \in \Gamma$, let $\overline{Z} = \overline{Z}(s_0, \gamma)$ be the expected number of executions that liquidity traders achieve through the use of their trading instruments, both market and limit orders. We estimate $\overline{Z}$ using simulation algorithm B2. Call $\hat{Z}$ the output of such estimation procedure. The objective is to calculate a constant rate of market order submissions that leads to the same expected number of executions $\overline{Z}$ in a pure dealer market (where traders cannot submit limit orders). For a given revelation parameter $r$, and expected number of executions $\hat{Z}$ the new market order intensity $z^m_D$ is defined by the following equation

$$\hat{Z} = \mathbb{E} \int_0^r 2z^m_D dt$$

$$= 2z^m_D \mathbb{E} r = \frac{2z^m_D}{r}$$

Then, the new rate is given by $z^m_D = \frac{1}{2} \hat{Z} r$. 

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Appendix D: Econometrics

More about Variable Construction

Trades and Quotes (TAQ)

$t$: Bid-ask spread. Best offer price - best Bid price at $t$ (number of cents)

$A_t$: Offer depth (100 shares lots) at the best offer price (specialist’s quoted depth + LOB’s depth)

$B_t$: Bid depth (100 shares lots) at the best bid price (specialist’s quoted depth + LOB’s depth)

The values of this variables are averaged between any two order arrivals.

System Order Data (SOD)

$t_i$: $i$th Interarrival time corresponding to order type $o$

Interarrival times of the first order of any given day were computed as the number of seconds from the opening of the same day. If two orders have the same time stamp, one of them is randomly sent forward one second.

$o_i$: Order type indication.

We used the following filters: (1) only straight Limit Orders were considered (no stop limit, etc.). Marketable Limit orders were considered Market orders. Only orders submitted at the market, with respect to TAQ best available prices were considered. We only use orders that arrived during normal trading hours.

Proxying for the Value of Information

The spread \( \sigma_i = b_i - a_i \) gives an indication of the distance \( ||p_t - v_i|| \). It is not a perfect statistic since \( \sigma_i(||p_t - v_i||) \) is not biyective. Moreover, we know from the model’s equilibrium that conditional on \( v=1 \) (\( v=0 \)) beliefs are a submartingale (supermartingale). Hence, starting at \( \frac{v_H-v_L}{2} \) (at info arrival), prices tend to be in \([\frac{v_H-v_L}{2}, v_H]\) when \( v_H \) and the other half when \( v_L \).

Over those ranges, \( \sigma_i(\cdot) \) is decreasing. On average then, \( \sigma \) will give relevant information about the degree of price convergence and will be used as a proxy of \( ||p_t - v_i|| \).

Markov Conditional Bootstrap Implementation

Preliminary Steps

Estimate the marginal distribution of the triple of states \( s = \{\sigma, B, A\} \), \( p_s \)

Estimate the conditional transition kernel \( Q(s_{j+1}|s_j) \)

Estimate the intensity set \( \{y^o(s) : s \in \mathcal{S}\} \).

Bootstrap Steps

Step 1: Draw \( s_0^* \) from \( p_s \).

Step 2: Draw \( s_{j+1}^* \) from \( Q(\cdot|s_j^*) \) simultaneously with \( t_k^{l,-}, t_k^{l,+} \) from \( \{y^o(s_j^*) : s_j^* \in \mathcal{S}, o \in \mathcal{O}\} \)

Step 3: Repeat step 2 till \( S^* = \{s_j^* : j = 1, ..., J\} \) and \( t^* = \{t_o^* : o \in \ell_+, \ell_-, k = 1, ..., K\} \) have been obtained.

Step 4: From \( S^* \) and \( t^* \) estimate \( \{\hat{y}^o(s) : s \in \mathcal{S}\} \), then construct and store the bootstrap test statistics \( \hat{T}^+, \hat{T}^- \).

Step 5: Repeat steps 1-4 many times. Calculate \( \hat{P}(\hat{T}^+ \leq m), \hat{P}(\hat{T}^- \leq m) \).
Appendix E: Figures

Figure 3: High type informed trader’s buy order submissions. Panels (a) and (b) displays market and limit orders submissions, respectively, as a function of beliefs of the asset liquidation value and the state of the limit order book (b) Limit orders optimal submission intensities. (c) displays market and limit order optimal submission intensities for the case of zero informed orders (solid lines) and positive number of informed orders on the bid side of the limit order book.
Figure 4: High type informed trader’s relative liquidity provision index (for buy orders). Each line corresponds to the ratio of limit order intensities to total submission intensities on the same order sign. (a) Displays the effect of increases in bid depth of the LOB (b) displays the effect of increases in the ask depth of the LOB (c) displays the effect of increases in the proportion of posted orders on the bid side held by the informed trader.

Figure 5: Strategic Liquidity Traders. (a) Displays the effect of increases in bid depth on buy limit order intensities (b) displays the effect of increases in the bid depth of the LOB on buy market orders intensities (c) displays the equilibrium bid ask spread as a function for different levels of depth on the bid side of the LOB (d) Displays the effect of increases in depth of the LOB, keeping the book balanced, on sell limit orders intensities.
Figure 6: Value functions. (a) High type informed trader expected profits (b) Low type informed trader expected profits (c) Strategic liquidity buyer expected minimum costs (d) strategic liquidity seller expected minimum cost.

Figure 7: Belief Update Functions. This figure shows the price impact of different order types as a function of current market beliefs. Each line correspond to a particular configuration of the limit order book. In the case of sell orders, negative values are shown for visual clarity (a) Buy market orders (b) Sell market orders (c) Buy limit orders (d) Sell limit orders.
Figure 8: Market Environment and high type speculator equilibrium behavior. This figure shows the strategic consequences of changes in the distribution of the duration of information advantage (top panels), and increase in the minimum distance between bid and ask prices (central panels), and an increase in both market and limit order submission intensities of noise liquidity traders. The first column of panels correspond to the change in buy market order submission intensities, the second column to buy limit orders, and the third column to the relative liquidity provision index. The intensities shown correspond to a situation in which the limit order book is fixed (depth is balanced) and it is representative of the qualitative effects on strategies across different book configurations.
Figure 9. Expected Equilibrium Behavior of Traders and Dealers’ differential probability of intermediation. (a) Expected order submission intensities for the high type informed trader. This trader submits buy market and limit orders with the expected intensity displayed. (b) Expected order submission intensities of the uninformed strategic liquidity buyer and seller. (c) Difference between the probability of dealers’ intermediating with buy and sell market orders. Time scale corresponds to the pooled counting order arrival-cancellations process.
Figure 10. Expected Evolution of Prices, Price volatility and Price Impact Functions. (a) Expected evolution of market asset value beliefs (b) Expected value of the Bid-Ask spread. The lower bound corresponds to the tick size. (c) Instant price variability of asset value beliefs and transaction price process. Price impact functions characterize the change in markets’ consensus value of the asset following the arrival of a specific order. (d) Impact Functions of Market orders (e) Limit orders (f) Cancellations of limit orders. Time scale corresponds to the pooled counting order arrival-cancellations process.
Figure 11. Expected Evolution and Distribution of the LOB’s depth during an information event (good news). (a) Expected evolution of the bid and ask depths of the LOB, and relative depth. Relative depth equal the bid depth minus the ask depth. (b) Distribution of the bid side of the book (c) Distribution of the ask side of the book (d) Distribution of the relative depth. The actions of the informed trader induce an asymmetric shape.
Figure 12. Speed of information transmission into prices. Left: Hybrid, limit-order market and dealers. Right: pure dealers’ market.
Figure 13. Estimated buy limit order intensities for Procter Gamble CO (PG). Top: Intensities conditioning different levels of ask depth of the LOB as a function of bid depth. Low spread case. Bottom: Intensities conditioning different levels of ask depth of the LOB as a function of bid depth. High spread case. The structural restriction tested predicts that the slope of these curves, for each conditioning set of variables, is not positive.
Figure 14. Estimated sell limit order Intensities for Procter Gamble CO (PG). Top: Intensities conditioning different levels of bid depth of the LOB as a function of ask depth. Low spread case. Bottom: Intensities conditioning different levels of bid depth of the LOB as a function of ask depth. High spread case. The structural restriction tested predicts that the slope of these curves, for each set of conditioning variables, is not positive.
## Appendix F: Tables

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**Table 2:** Number of limit orders (buys and sells) in the NYSE’s SOD sample May 6-17, 2002 for selected symbols. Column (a) total number of limit orders (b) Limit orders submitted at a price strictly below the best available on the opposite side of the market (c) Non-marketable limit orders submitted at the best price available on the market or better.
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<th>(a) T+ spread=1 p-value</th>
<th>(b) T+ spread=2 p-value</th>
<th>(c) T+ p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>DVI</td>
<td>0.001 0.06</td>
<td>0.000 0.30</td>
<td>0.001 0.29</td>
</tr>
<tr>
<td>MTN</td>
<td>0.000 0.15</td>
<td>0.000 0.00</td>
<td>0.000 0.29</td>
</tr>
<tr>
<td>RUS</td>
<td>0.001 0.16</td>
<td>0.000 0.47</td>
<td>0.001 0.44</td>
</tr>
<tr>
<td>IO</td>
<td>0.978 0.01</td>
<td>0.000 0.81</td>
<td>0.978 0.01</td>
</tr>
<tr>
<td>LNR</td>
<td>0.000 0.47</td>
<td>0.001 0.58</td>
<td>0.000 0.67</td>
</tr>
<tr>
<td>ABM</td>
<td>0.002 0.33</td>
<td>0.000 0.94</td>
<td>0.002 0.79</td>
</tr>
<tr>
<td>GGG</td>
<td>0.001 0.51</td>
<td>0.001 0.70</td>
<td>0.001 0.77</td>
</tr>
<tr>
<td>EME</td>
<td>0.000 0.40</td>
<td>0.001 0.92</td>
<td>0.001 0.91</td>
</tr>
<tr>
<td>BGP</td>
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<td>0.000 0.92</td>
<td>0.002 0.70</td>
</tr>
<tr>
<td>ORI</td>
<td>0.001 0.74</td>
<td>0.002 0.55</td>
<td>0.002 0.78</td>
</tr>
<tr>
<td>PCP</td>
<td>0.000 0.86</td>
<td>0.001 0.78</td>
<td>0.001 0.93</td>
</tr>
<tr>
<td>ATI</td>
<td>0.000 0.79</td>
<td>0.002 0.85</td>
<td>0.002 0.93</td>
</tr>
<tr>
<td>ANN</td>
<td>0.001 0.93</td>
<td>0.001 0.97</td>
<td>0.002 0.95</td>
</tr>
<tr>
<td>SNA</td>
<td>0.000 0.93</td>
<td>0.001 0.98</td>
<td>0.001 0.97</td>
</tr>
<tr>
<td>EXC</td>
<td>0.001 0.99</td>
<td>0.005 0.76</td>
<td>0.006 0.93</td>
</tr>
<tr>
<td>FD</td>
<td>0.000 0.99</td>
<td>0.004 0.97</td>
<td>0.004 0.98</td>
</tr>
<tr>
<td>WB</td>
<td>0.004 0.76</td>
<td>0.002 0.91</td>
<td>0.005 0.94</td>
</tr>
<tr>
<td>SO</td>
<td>0.002 0.96</td>
<td>0.005 0.93</td>
<td>0.007 0.99</td>
</tr>
<tr>
<td>ADI</td>
<td>0.006 0.99</td>
<td>0.004 0.94</td>
<td>0.010 0.98</td>
</tr>
<tr>
<td>PG</td>
<td>0.002 0.99</td>
<td>0.002 0.87</td>
<td>0.005 0.92</td>
</tr>
</tbody>
</table>

Table 3: Buy limit orders. Results of the structural test on optimal liquidity provision behavior. Column (a) Test statistic value for intensities conditioned on low bid-ask spread (b) Test statistic value for intensities conditioned on high bid-ask spread (c) Test statistic value not conditioning on the spread. All p-values correspond to Horowitz’s MCB Bootstrap values.
<table>
<thead>
<tr>
<th>Symbol</th>
<th>(a) T-spread=1 p-value</th>
<th>(b) T-spread=2 p-value</th>
<th>(c) T-p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>DVI</td>
<td>0.001 0.08 0.000 0.48</td>
<td>0.001 0.000 0.82 0.44</td>
<td>0.001 0.22</td>
</tr>
<tr>
<td>MTN</td>
<td>0.001 0.12 0.000 0.39</td>
<td>0.001 0.000 0.84 0.44</td>
<td>0.001 0.26</td>
</tr>
<tr>
<td>RUS</td>
<td>0.001 0.23 0.000 0.82</td>
<td>0.001 0.000 0.84 0.44</td>
<td>0.001 0.44</td>
</tr>
<tr>
<td>IO</td>
<td>0.000 0.53 0.000 0.84</td>
<td>0.000 0.000 0.84 0.44</td>
<td>0.000 0.88</td>
</tr>
<tr>
<td>LNR</td>
<td>0.003 0.11 0.000 0.67</td>
<td>0.003 0.000 0.84 0.44</td>
<td>0.003 0.26</td>
</tr>
<tr>
<td>ABM</td>
<td>0.001 0.56 0.001 0.85</td>
<td>0.001 0.000 0.84 0.44</td>
<td>0.001 0.89</td>
</tr>
<tr>
<td>GGG</td>
<td>0.000 0.65 0.001 0.84</td>
<td>0.000 0.000 0.84 0.44</td>
<td>0.000 0.91</td>
</tr>
<tr>
<td>EME</td>
<td>0.000 0.37 0.000 0.94</td>
<td>0.001 0.000 0.84 0.44</td>
<td>0.001 0.86</td>
</tr>
<tr>
<td>BGP</td>
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<td>0.004 0.000 0.84 0.44</td>
<td>0.004 0.42</td>
</tr>
<tr>
<td>ORI</td>
<td>0.000 0.89 0.000 0.86</td>
<td>0.001 0.000 0.84 0.44</td>
<td>0.001 0.97</td>
</tr>
<tr>
<td>PCP</td>
<td>0.000 0.74 0.000 0.76</td>
<td>0.001 0.000 0.84 0.44</td>
<td>0.001 0.90</td>
</tr>
<tr>
<td>ATI</td>
<td>0.000 0.82 0.001 0.99</td>
<td>0.001 0.000 0.84 0.44</td>
<td>0.001 0.99</td>
</tr>
<tr>
<td>ANN</td>
<td>0.002 0.84 0.004 0.92</td>
<td>0.006 0.000 0.84 0.44</td>
<td>0.006 0.97</td>
</tr>
<tr>
<td>SNA</td>
<td>0.000 0.99 0.001 0.99</td>
<td>0.001 0.000 0.84 0.44</td>
<td>0.001 0.99</td>
</tr>
<tr>
<td>EXC</td>
<td>0.001 0.98 0.003 0.94</td>
<td>0.004 0.000 0.84 0.44</td>
<td>0.004 0.99</td>
</tr>
<tr>
<td>FD</td>
<td>0.002 0.97 0.001 0.99</td>
<td>0.003 0.000 0.84 0.44</td>
<td>0.003 0.99</td>
</tr>
<tr>
<td>WB</td>
<td>0.007 0.77 0.002 0.87</td>
<td>0.009 0.000 0.84 0.44</td>
<td>0.009 0.90</td>
</tr>
<tr>
<td>SO</td>
<td>0.002 0.92 0.001 0.97</td>
<td>0.003 0.000 0.84 0.44</td>
<td>0.003 0.98</td>
</tr>
<tr>
<td>ADI</td>
<td>0.007 0.99 0.000 0.99</td>
<td>0.007 0.000 0.84 0.44</td>
<td>0.007 0.99</td>
</tr>
<tr>
<td>PG</td>
<td>0.001 0.99 0.004 0.82</td>
<td>0.004 0.000 0.84 0.44</td>
<td>0.004 0.96</td>
</tr>
</tbody>
</table>

**Table 4**: Sell limit orders. Results of the structural test on optimal liquidity provision behavior. Column (a) Test statistic value for intensities conditioned on low bid-ask spread (b) Test statistic value for intensities conditioned on high bid-ask spread (c) Test statistic value not conditioning on the spread. All p-values correspond to Horowitz’s MCB Bootstrap values.
<table>
<thead>
<tr>
<th>Symbol</th>
<th>(a) Spread=1 T</th>
<th>p-value</th>
<th>(b) Spread=2 T</th>
<th>p-value</th>
<th>(c) T</th>
<th>p-value</th>
</tr>
</thead>
<tbody>
<tr>
<td>DVI</td>
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<td>0.09</td>
<td>0.001</td>
<td>0.59</td>
<td>0.002</td>
<td>0.31</td>
</tr>
<tr>
<td>MTN</td>
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<td>0.16</td>
<td>0.000</td>
<td>0.45</td>
<td>0.001</td>
<td>0.35</td>
</tr>
<tr>
<td>RUS</td>
<td>0.001</td>
<td>0.27</td>
<td>0.001</td>
<td>0.64</td>
<td>0.002</td>
<td>0.56</td>
</tr>
<tr>
<td>IO</td>
<td>0.978</td>
<td>0.01</td>
<td>0.000</td>
<td>0.94</td>
<td>0.978</td>
<td>0.02</td>
</tr>
<tr>
<td>LNR</td>
<td>0.003</td>
<td>0.28</td>
<td>0.001</td>
<td>0.76</td>
<td>0.004</td>
<td>0.55</td>
</tr>
<tr>
<td>ABM</td>
<td>0.003</td>
<td>0.52</td>
<td>0.001</td>
<td>0.98</td>
<td>0.004</td>
<td>0.95</td>
</tr>
<tr>
<td>GGG</td>
<td>0.001</td>
<td>0.71</td>
<td>0.002</td>
<td>0.88</td>
<td>0.002</td>
<td>0.94</td>
</tr>
<tr>
<td>EME</td>
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<td>0.51</td>
<td>0.001</td>
<td>0.99</td>
<td>0.002</td>
<td>0.98</td>
</tr>
<tr>
<td>BGP</td>
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<td>0.001</td>
<td>0.99</td>
<td>0.006</td>
<td>0.73</td>
</tr>
<tr>
<td>ORI</td>
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<td>0.93</td>
<td>0.002</td>
<td>0.83</td>
<td>0.003</td>
<td>0.97</td>
</tr>
<tr>
<td>PCP</td>
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<td>0.94</td>
<td>0.001</td>
<td>0.89</td>
<td>0.002</td>
<td>0.98</td>
</tr>
<tr>
<td>ATI</td>
<td>0.001</td>
<td>0.93</td>
<td>0.002</td>
<td>0.99</td>
<td>0.003</td>
<td>0.94</td>
</tr>
<tr>
<td>ANN</td>
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<td>0.97</td>
<td>0.008</td>
<td>0.98</td>
</tr>
<tr>
<td>SNA</td>
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<td>0.002</td>
<td>0.99</td>
<td>0.002</td>
<td>0.95</td>
</tr>
<tr>
<td>EXC</td>
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<td>0.96</td>
<td>0.008</td>
<td>0.95</td>
<td>0.010</td>
<td>0.96</td>
</tr>
<tr>
<td>FD</td>
<td>0.002</td>
<td>0.97</td>
<td>0.005</td>
<td>0.98</td>
<td>0.007</td>
<td>0.97</td>
</tr>
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<td>WB</td>
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<td>0.96</td>
<td>0.015</td>
<td>0.98</td>
</tr>
<tr>
<td>SO</td>
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<td>0.006</td>
<td>0.99</td>
<td>0.009</td>
<td>0.99</td>
</tr>
<tr>
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<td>0.005</td>
<td>0.99</td>
<td>0.017</td>
<td>0.98</td>
</tr>
<tr>
<td>PG</td>
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<td>0.98</td>
<td>0.006</td>
<td>0.94</td>
<td>0.009</td>
<td>0.95</td>
</tr>
</tbody>
</table>

**Table 5**: Buy and sell limit orders. Results of the structural test on optimal liquidity provision behavior. Column (a) Test statistic value for intensities conditioned on low bid-ask spread (b) Test statistic value for intensities conditioned on high bid-ask spread (c) Test statistic value not conditioning on the spread. All p-values correspond to Horowitz’s MCB Bootstrap values.