What Is the Risk-Free Rate?

A Model and Empirical Tests in a Market with Frictions

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Job Market Paper

November 7, 2008

Abstract

I study the properties of implied interest rates from futures and put-call parity relations, and compare it to other market rates commonly used by academics and practitioners. I show that in a market with borrowing and short-selling costs, the price of futures and put-call parity relations is affected by demand pressure. I apply the model to the futures market and obtain a closed-form solution for futures prices that is a function of the risk-free rate and a latent demand factor. My model-implied risk-free rate estimated using S&P 500 index futures compares favorably with other commonly used candidates. I also estimate the model-implied demand factor and verify that is related to observable proxies for demand pressure in the futures market. I also show empirically that relative mispricing is positive (negative) when buying (selling) pressure is high and is difficult to borrow (short-sell the underlying). These results extend to futures and put-call parity relations in other indices as well.

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1 Introduction

Measuring and understanding the risk-free rate is a fundamental question in financial economics. The valuation of real and financial assets depends crucially on using the correct risk-free rate. It is a well-known fact that even if markets are incomplete and there is no risk-free asset, the absence of arbitrage opportunities implies the existence of a pricing kernel that generates an *implied risk-free rate*. In perfect markets, the risk-free rate measures the cost of borrowing *and* lending *unlimited* units of the riskless asset. In real markets, however, agents face borrowing constraints that prevent them from borrowing unlimited amounts. In that case, the risk-free rate is a latent variable that cannot be observed directly but that can nevertheless be estimated from traded assets (Black, 1972).

In this paper I study the properties of implied interest rates from derivative instruments, and compare it to other market rates commonly used by academics and practitioners. In doing so I take into account the fact that there are costs for borrwing both the riskless and the risky asset.

In the first part of the paper I provide a set of stylized facts about implied risk-free rates from futures and put-call parity relations written on major indexes: S&P 500, Nasdaq and Dow Jones Industrial Average (DJIA). If market are frictionless and complete, by inverting the cost-of-carry formula for futures and the put-call parity relation for options, it is possible to obtain the implied risk-free rate that economic agents use to price those instruments. I find that on average implied interest rates from both futures and options lie between Treasury and LIBOR rates. From January 1998 to December 2007, three-month rates implied from futures prices are on average 48 bp above Treasury and 5 bp below LIBOR, whereas implied interest rates from options are on average 50 bp above Treasury and 3 bp below LIBOR. Thus, these simple implied rates are very similar regardless of whether they are extracted from futures or options prices, and are on average much closer to borrowing (LIBOR) rather than lending (Treasury) rates. This result is consistent with the common industry practice of using LIBOR rates as a proxy for the risk-free rate when valuing derivative contracts, and also with previous findings in the literature (Brenner and Galai, 1986).

If borrowing rates differ from lending rates, or if agents are constrained of borrowing infinite amounts, an arbitrageur that provides liquidity in a particular derivatives market will be unable to hedge the derivative perfectly if she is exposed to exogenous demand shocks. In this case demand pressure for the derivative will affect prices, in a similar manner as in the model originally studied by Garleanu, Pedersen, and Poteshman (2007) for option markets. If this is the case, implied interest rates will in general be biased estimates of the true risk-free rate at a particular point in time.

In the second part of the paper I provide a solution for this problem. I derive a continuous-time equilibrium model that takes into account the fact that borrowing rates are higher than lending rates, and that agents are constrained of borrowing infinite amounts of *both* the riskless and the risky asset. In order to derive a close form solution for the model, I make the simplifying assumption that borrowing costs increase linearly with demand. This assumption is consistent with the fact that borrowing rates may differ from lending rates, and that it is more expensive to borrow larger amounts of the riskless or risky asset . I obtain a tractable model that can be estimated directly from the data. In the paper, I specialize the model to price futures contracts. It is important to note, however, that the model can also be used to price synthetic forwards obtained from put-call parity relations, or extended to price other derivatives.

In the model there are two types of agents: *arbitrageurs* and *traders*. Arbitrageurs operate in a competitive market and cannot act strategically to minimize borrowing costs. Traders can be of two types: fundamental traders and hedgers. In this paper, I do not distinguish between these two types of traders and only assume that all traders have an exogenous demand for the derivative. If traders' demand is positive, then arbitrageurs short the derivative, buy the risky asset and borrow. In equilibrium, arbitrageurs borrow when traders' demand is positive, which is exactly when borrowing costs are higher. In a competitive market, arbitrageurs set the price of the derivative such that they are indifferent between taking the opposite side of the trade or do nothing. This equilibrium price will be higher than the price obtained in an otherwise equivalent frictionless economy. A similar logic applies if traders want to short the derivative. In that case, arbitrageurs buy the derivative, short the underlying asset and pay lending fees. In equilibrium, the price of the derivative should be set lower than in an otherwise equivalent frictionless economy in order to motivate arbitrageurs to take the long position.

The model implies that the equilibrium futures price is an exponential-affine function of the risk-free rate and a *demand factor*. From the model it is also possible to derive the equilibrium term-structure of interest rates at any given point in time. These results imply that in this economy futures prices can be computed as the expectation of the spot price under some equivalent martingale measure.

The model generates three main testable implications. First, the demand factor should, unless in very specific cases, be priced. Second, the implied spot rate should on average follow closely the overnight U.S. federal funds rate, although this need not necessarily be the case in periods of market stress. Third, the demand factor obtained from *prices* should actually be correlated with actual proxies for demand pressure.

In the third part of the paper, I test the model using S&P 500 index futures. Using state-space methods and the Kalman filter, I estimate model parameters and state-variables. I find three important results. First, the implied spot rate from the model follows closely the overnight fed funds rate, although it is not always the same. If the model is correct, The model implied spot rate is on average 15 bp above the fed funds rate. Also, implied short-term rates are on average similar to market rates computed in the first part of the paper. Second, I find that the demand factor is indeed priced, which is in line with previous findings in the literature (Figlewski, 1984). Hence, arbitrageurs are exposed to an additional source of risk when taking a position in the futures market, i.e. "index arbitrage" is risky.

In the last part of the paper, I verify the third testable implication from the model and show that the demand factor implied by the model is related to actual demand pressure in the S&P 500 futures market. I find that the demand factor is positively correlated with the net futures demand by large speculators, even after controlling for transaction costs, liquidity and index cash market volatility. Furthermore, I show that futures mispricing is also related to a measure of market sentiment: the net position in the S&P 500 futures market by large speculators. This relation is expected since it is known from the behavioral finance literature (see e.g. Han, 2008) that this demand by large speculators is related to sentiment proxies. Thus, the model opens a channel through which sentiment can affect the *relative* valuation of the derivative and its underlying asset. I also show empirically that relative mispricing is positive (negative) when buying (selling) pressure is high and is difficult to borrow (short-sell the underlying). Finally, I show that these results extend to futures and put-call parity relations in other indices as well.

In order to further highlight the importance of the topics of this paper, in the next subsection I present the evolution of interest rates during the 2007-08 liquidity and credit crisis, and compare it with the estimate obtained from the model.

1.1 A Case Study: The 2007-08 Liquidity and Credit Crisis

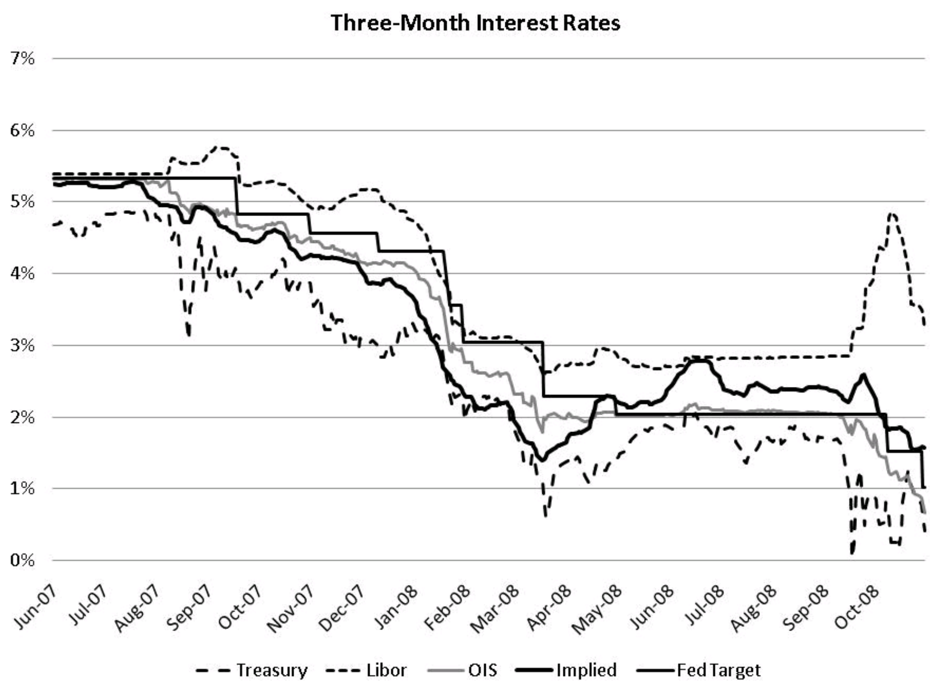

Figure 1: Interest Rates During the 2007-08 Credit Crisis

Defining the risk-free rate has become an increasingly difficult task as the 2007-08 liquidity and credit crisis unveils. Figure 1 shows the evolution of four different three-month interest rates from June 2007 until September 2008: the Treasury rate, proxied by the yield on a three month Constant Maturity Treasury (CMT), the London Interbank Offered Rate (LIBOR), the Overnight Index Swap (OIS) rate, and the implied risk-free rate obtained from S&P 500 prices.

The two traditional benchmarks that academics and practitioners generally use, namely Treasury and LIBOR rates, respectively, have increasingly being drifting apart during this period. The so-called TED spread, defined as the difference between LIBOR and Treasury rates was at the end of September 2008 at an all time high. In this scenario, some academics and practitioners have started to look for other alternatives as a proxy for the risk-free rate. In particular, Brunnermeier (2008) and Michaud and Upper (2008) propose to look at the OIS rate as a good benchmark for the risk free-rate. The OIS rate is the fixed leg on a swap contract written on the geometric average of the U.S. federal funds rate. The figure shows that the OIS rate have been between Treasury and LIBOR rates during the crisis, although during the second part of the crisis this rate has been closer to Treasury rather than LIBOR rates.

The figure also show the implied interest rate obtained from S&P 500 futures contracts, one of the most liquid exchange-traded derivatives. Since futures are continuously marked-to-market and traders are required to post a margin, futures can be considered default-free. As can be seen from the figure, both the OIS and the implied rate were moving closely together until the mid of March 2008. However, on March 17th 2008, the implied rate starts drifting upwards and away from the OIS rate, moving rapidly close to LIBOR. Coincidentally, March 16th 2008 marks the day on which Bear Stearns was sold for $2 a share to JP Morgan in order to avoid bankruptcy. From the figure, it looks that this event actually triggered the implied rate to diverge from the OIS rate. Also interesting from the figure is that implied and Treasury rates increase at the same time.

The previous case study highlights the importance of obtaining a better understanding of the risk free rate. The use of an incorrect risk-free rate can generate the illusion of arbitrage profits even when markets are perfect and fully efficient. This seems to be a particularly acute problem during crisis periods since during these times the TED spread increases the most.

1.2 Related Literature and Discussion

While there is a huge literature on dynamic term structure modelling[1] using Treasury bonds and LIBOR rates, the use of derivatives for extracting information about the risk-free rate has been mostly ignored by financial economists. A notable exception is Brenner and Galai (1986), who are probably the first to estimate implied risk-free rates from put-call parity relations on stock option prices. In related work, Brenner, Subrahmanyam, and Uno (1990) also look at implied risk-free ratesusing Nikkei index futures data. Liu, Longstaff, and Mandell (2006) obtain implied risk-free rates from plain-vanilla swap contracts, but they use the three-month General Collateral (GC) repo rate as a proxy for the three-month risk-free rate. I make no initial assumptions about what the risk-free rate should be. Feldhütter and Lando (2008) also use swap data to estimate the risk-free rate. They are closer in spirit to this paper in the sense that they extract the risk-free rate from

[1]This literature is too large to put all the references here. Two recent surveys of the field are Dai and Singleton (2003) and Piazzesi (2003).

swap and corporate bond data alone without making initial assumptions about the what the risk-free rate should be. However, I use a much larger time-series and I cross-validate my implied rate with different assets. Another main difference between my paper and previous literature is that I account for the fact that demand pressure can affect security prices, and hence distort implied risk-free rate estimates in significant ways.

On a more indirect way, this paper is also related to the literature that computes risk-neutral distributions, or pricing kernels, from option prices[2]. In theory, the conditional mean of the pricing kernel is inversely related to the risk-free rate. Thus, the implied interest rate estimated in this paper might help cross-validate the estimation of risk-neutral distributions.

This study also fits into a large literature that studies index futures arbitrage[3]. By futures mispricing it is usually understood that the observed price deviates from the usual cost-of-carry formula. The main focus of this literature has been to relate *absolute mispricings* to liquidity and transaction costs of the underlying cash index. In this paper I focus instead on understanding the *sign of the mispricing.* Transaction costs on the underlying cash index operate symmetrically on deviations from no-arbitrage values and thus cannot provide an explanation for positive or negative deviations from fair-value. This paper also relates to the literature that analyzes put-call parity violations in the options market[4].

On the theoretical side, the model I present in this paper draws heavily on Garleanu et al. (2007) and Vayanos and Vila (2007). The main extension from Garleanu et al. (2007) is that in this paper the channel through which demand affects derivative prices is the existence of constraints on borrowing, for both the riskless and the risky asset. In Garleanu et al. (2007), the channel that links demand and derivative prices is the inability of market makers to hedge their inventories because of frictions related to the options market, like stochastic volatility, jumps, discrete trading and transaction costs. Vayanos and Vila (2007) apply a similar idea to the fixed-income market, although in a very difference setting.

[2] See, for example, Breeden and Litzenberger (1978), Banz and Miller (1978), Jackwerth and Rubinstein (1996), Figlewski (2008).

[3] See, for example, Modest and Sundaresan (1983), Kawaller, Koch, and Koch (1987), Brenner, Subrahmanyam, and Uno (1989), Brenner et al. (1990), Yadav and Pope (1990), Chung (1991), Klemkosky and Lee (1991), Subrahmanyam (1991), Bessembinder and Seguin (1992), Chan (1992), Sofianos (1993), Miller, Muthuswamy, and Whaley (1994), Neal (1996), Roll, Schwartz, and Subrahmanyam (2007).

[4] See, for example, Lamont and Thaler (2003), Ofek, Richardson, and Whitelaw (2004), Cremers and Weinbaum (2008).

In the model I abstract from transaction costs that are generated by trading flows and instead concentrate on borrowing costs that are generated by holding portfolio balances. In the last section of the paper, however, I show that even after controlling for transaction costs there is still a positive relation between derivatives demand and prices.

Finally, this paper fits into a broader literature that looks at how market frictions prevent arbitrageurs in some circumstances from profiting of arbitrage opportunities[5]. Effectively, I show that borrowing costs open a channel through which derivatives demand can affect no-arbitrage prices, even when the payoff is linear in the underlying.

2 Interest Rates

2.1 Literature Review

2.1.1 Market Rates

It is common for empirical researchers in finance to use the yield on Treasury bills and notes as a proxy for the risk-free rate. The intuition for this is simple: Treasury bills and bonds are backed by the full faith and credit of the U.S. government. As such, they are the safest investment available for an investor whose consumption is denominated in U.S. dollars. However, there is now a large literature documenting discrepancies between treasury and other interest rates. Some authors suggest that these discrepancies might be due to transaction costs and taxes[6]. Other authors suggest that investors perceive a benefit from owning a treasury security[7].

A major implication of this literature is that using Treasury rates to price other assets might introduce pricing biases. As a consequence of using the wrong proxy for the risk-free rate, one might be inclined to believe that there is an apparent mispricing in a derivative contract when in fact its price is correct. This would explain why, as will be shown in a later section, the use of Treasury rates increases the mispricing of futures and option contracts with respect to other possible proxies for the risk-free rate.

On the other hand, It is common for practitioners to use LIBOR rates as a proxy for the risk-free

[5] See, for example, Shleifer and Vishny (1997), Xiong (2001), Gromb and Vayanos (2002), Duffie, Gârleanu, and Pedersen (2002), Liu and Longstaff (2004), Kondor (2007).

[6] See, for example, Amihud and Mendelson (1991), Kamara (1994), Elton and Green (1998), Strebulaev (2002)

[7] See, for example, (Jarrow and Turnbull, 1997, Longstaff, 2004, Krishnamurthy and Vissing-Jorgensen, 2008)

rate when valuing derivative contracts. Indeed, as noted by Hull (2002, p. 94),

> "[...] banks and other large financial institutions tend to use the LIBOR rate rather than the Treasury rate as the "risk-free rate" when they evaluate derivatives transactions. The reason is that financial institutions invest surplus funds in the LIBOR market and borrow to meet their short-term funding requirements in this market. They regard LIBOR as their opportunity cost of capital."

Even though LIBOR represents interest rates charged on uncollateralized loans between banks and hence they are subject to credit risk, there are several authors that argue that LIBOR rates carry very low credit risk. For example, Collin-Dufresne and Solnik (2001) and Grinblatt (2001) indicate that LIBOR rates are almost risk-free since they are refreshed top-credit-quality rates. Feldhütter and Lando (2008) find that default-risk embedded in LIBOR rates does not contribute significantly to the long-term swap spreads.

However, during the 2007-08 credit crisis LIBOR rates have increased above "normal" levels. Michaud and Upper (2008) and Brunnermeier (2008) suggest using the Overnight Index Swap (OIS) rate as a proxy of the risk-free rate. Overnight Index Swap contracts are swap contracts written on the overnight U.S. federal funds rate. As such, they roughly measure expected overnight funding costs for banks and financial institutions. In this particular swap agreement, the floating leg pays the geometric average of the overnight rate. Michaud and Upper (2008) indicate that credit risk of these contracts is small since there is no exchange of principal and parties are required to post collateral. Furthermore, these swap contracts do not involve any initial cash-flows, increasing the liquidity of these contracts. Historical data on OIS rates is available from Bloomberg starting in January 2002 for maturities ranging up to five years.

Another alternative is to use the three-month General Collateral (GC) repo rate as a proxy for the risk-free rate (Longstaff, 2000). The three-month GC rate is available since 1991 from Bloomberg through Garban, a large and well-know Treasury securities broker. However, an informal examination of the three-month GC rate shows that it stays constant during several days at a time, suggesting that it might lack liquidity.

2.1.2 Implied Rates

As shown in the previous sub-section, knowing exactly which interest rate to use as a proxy for the risk-free rate is a difficult task. An alternative solution for this problem is to compute the implied risk-free rate that market participants use to price derivative contracts.

Trying to estimate implied interest rates from certain derivative contracts is certainly not new. In an early paper, Brenner and Galai (1986) estimate implied risk-free rates from put-call parity relations on stock option prices. They find that implied risk-free rates are similar with respect to other short-term rates, and closer to lending rather than prevailing borrowing rates. Brenner et al. (1990) obtain implied interest rates from Nikkei index futures contracts. They also find that implied risk-free rates are similar to other short-term rates. Liu et al. (2006) and Feldhütter and Lando (2008) obtain implied risk-free rates from plain-vanilla swap contracts.

In this paper I propose to use index futures and put-call parity relations to compute implied risk-free rates. Although in theory any futures or option contract could be used to obtain implied risk-free rates, in practice there are several issues that need to be addressed when computing implied rates. First, it is desirable that the derivative used to obtain an implied rate is simple to price. Otherwise there could be a concern that the implied rate is driven by the model assumptions. In this respect, futures contracts, and put-call parity relations from European options, are perhaps the simplest derivatives available. Second, all dividends paid by the underlying asset must be quantifiable. It is well known that many commodities have associated convenience yields that are difficult to estimate (see e.g. Schwartz, 1997). Thus, using commodity futures to estimate an implied risk-free rate would be problematic. Third, the derivative used to estimate the risk-free rate has to be liquid. For example, Brenner, Eldor, and Hauser (2001) show that illiquidity can be a serious problem for options in some cases.

2.2 Data

To obtain implied risk-free rates, I use futures and options on three major indexes: S&P 500, Nasdaq and Dow Jones Industrial Average (DJIA). The period I study in this section is from January 1998 to December 2007. I decide to start in January 1998 because DJIA futures and options only start trading after this date. In a subsequent section, I present results spanning a longer time period

(January 1989 to December 2007) in which I only use S&P 500 futures contracts.

Ideally, it would be interesting to look at a wide array of underlying assets and different derivatives to obtain more precise estimates of implied risk-free rates. However, there are some restrictions on the array of instruments or underlying assets available for use, as indicated previously. First, if the underlying pays a dividend, it is necessary to estimate that dividend with precision. Second, in the case of options, the put-call parity relationship only holds exactly for European options. Third, it is important for the derivatives used in the analysis to be liquid and actively traded. For these reasons I study futures and options written on three major indexes: S&P 500, Nasdaq, and the Dow Jones Industrial Average (DJIA). First, dividends on these indexes are observed and can be easily forecasted. Second, options on these indexes are European and put-call parity relations should hold exactly. Third, futures and options on these indices, specially on the S&P 500, are very liquid and trade for a wide arrange of maturities and strike prices.

2.2.1 Dividends

Dividend data for each index is obtained from Bloomberg, and is available starting in January 1988 for the S&P 500, and January 1993 for Nasdaq and DJIA. The top panel of Figure 2 displays the average dividend yield per year and per index. The dividend yield has been on average increasing slowly for all indexes during this time period. Also, it is highest for the DJIA, followed very closely by the S&P 500. Nasdaq has very low dividend yields as would be expected from technology stocks, although the dividend yield has increased significantly in the last years. Nevertheless, year to year variations are small indicating that dividends are highly predictable, at least for up to 12 months.

Despite the year to year predictability, dividend payments are also highly non-stationary within the year. The bottom panel of Figure 2 shows that dividend payments peak in February, May, August and November for all indexes. The difference between a high dividend yield month and a low dividend yield month can be significant, specially for the DJIA and the S&P 500.

In order to take into account the seasonality of dividend payments I compute the dividend yield at time t for each contract expiring at time $t + \tau$ as:

$$\delta_t(\tau) = \frac{1}{\tau}\left(\sum_{n=t-1}^{t+\tau-1} \frac{d_n}{S_n}\right), \tag{1}$$

where d_n is the ex-dividend payment in dollars at day n and S_t is the S&P 500 spot price at time t. Since variation in dividend payments from year to year is small, this estimate of the dividend yield is likely to be accurate Roll et al. (2007), at least for up to one year. It is important to note, however, that market participants might have better information of future dividend yields at the time they trade index futures and option contracts. This is turn can have an impact on implied interest-rates.

2.2.2 Futures and Options

Data on futures for each index is obtained from Bloomberg. Data on S&P 500, Nasdaq and DJIA starts in January 1983, May 1996 and January 1998, respectively. There are several contracts available for trading on each index with maturities ranging from 3 to 24 months. All contracts expire the third Friday of March, June, September, and December of each year. Thus, every three months a new contract is issued. Despite the fact that there are eight maturities available for trading in S&P 500 futures, trading concentrates heavily in the first three contracts with the shortest maturity. Among all indexes S&P 500 futures are by far the most traded and hence the most liquid futures contract.

Table 1, Panel A presents summary statistics of the open interest in S&P 500, Nasdaq and DJIA futures contracts from January 1998 to December 2007. For each stock index the table summarizes the proportion of open interest for three, six, nine, and twelve-month contracts, respectively, with respect to the total open interest on that day. On average, a very large proportion of the total open interest concentrates in the closest-to-maturity contract. The twelve-month contract contributes on average to less than 1% of the total open interest for all indexes. For this reason I follow (Roll et al., 2007) and include in my analysis the three, six, and nine-month contracts only.

Data on options is obtained from OptionMetrics. Data on S&P 500 and Nasdaq options starts in January 1996, while for DJIA it starts in January 1998. Options on indexes also follow the issuance cycle of futures contracts. The main difference in terms of data availability with respect to futures contracts is that for each maturity there are several options with different strike prices. Table 1, Panel B presents summary statistics of the open interest in S&P 500, Nasdaq and DJIA options contracts from January 1998 to December 2007. Open interest is grouped by maturity. For each stock index the table summarizes the proportion of open interest for three, six, nine, and twelve-month options, respectively, with respect to the total open interest on that day. As with futures, on average a very large proportion of the total open interest concentrates in the closest-to-maturity option contracts. However, more open interest is concentrated in the longer maturity contracts than in futures. For options, the twelve-month contract contributes on average to approximately 5% of the total open interest for all indexes.

2.3 Stylized Facts about Implied Interest Rates

In this section, I compute implied interest rates from index futures and put-call parity relations on index options. For the case of futures contracts, I follow Brenner et al. (1990) and compute the implied risk-free rate from the usual cost-of-carry formula given by

$$R_t(\tau) = \frac{1}{\tau} \log\left(\frac{F_t(\tau)}{S_t}\right) + \delta_t(\tau), \tag{2}$$

where τ is the maturity of the contract, $F_t(\tau)$ is the closing futures price, S_t is the closing spot price, and $\delta_t(\tau)$ is the dividend yield from time t up to time $t+\tau$. If there are no frictions in the market, the absence of arbitrage opportunities requires that $R_t(\tau)$ equals the risk-free rate. Any deviation from equation (2) would represent an arbitrage opportunity, the existence of some friction, or a combination of both. Over large periods, however, any discrepancy should disappear since otherwise it would represent a systematic arbitrage opportunity.

To obtain implied risk-free rates from options, I use the put-call parity relationship and compute:

$$R_t(\tau) = -\frac{1}{\tau} \log\left(\frac{S_t e^{-\delta_t(\tau)} - C_t(K,\tau) + P_t(K,\tau)}{K}\right), \tag{3}$$

where $C_t(K,\tau)$ and $P_t(K,\tau)$ are the prices of a call and put option with maturity τ and strike K, respectively. It is important to note that this relation is exact only if the options are European, which is the case for options on the indexes that I include in the analysis.

Table 2 presents summary statistics for several types of short-term interest rates: Treasury, LIBOR and Implied rates obtained from index (S&P 500, Nasdaq, and DJIA) futures and put-call parity relations, for the period January 1998 to December 2007. All rates are continuously compounded. Treasury and LIBOR rates are presented for three, six and twelve-month maturities.

Treasury rates are Constant Maturity Treasuries (CMTs) rates obtained from the U.S. Department of the Treasury.

Implied rates from futures are computed using equation (2) and are presented for three, six and nine-month contracts, and also for the daily cross-sectional average of all three underlying assets. Implied rates from put-call parity relations are computed using equation (3) and are averaged for each index for maturities ranging from 30 to 90 days, 90 to 180 days, 180 to 270 days, and 270 to 365 days. For options, only contracts with a volume greater than 10 contracts for calls and puts at the same strike and on the same day are included. Also, all calls and puts for a given strike for which the absolute mispricing computed using either Treasury or LIBOR was more than 50% were not included in the computations. This filter is used to avoid some extreme observations to affect the results.

As can be seen from Table 2, implied rates from futures and options are very similar. The difference between the three-month rate obtained from all futures differs only 2 bp from the same rate obtained from all options, while the same difference is 10 and 5 bp for the six and nine-month implied rate. Also, both implied rates are on average between Treasury and LIBOR rates, although the implied rate is closer to LIBOR (Borrowing) than lending (Treasury). This fact is consistent with the original finding in Brenner and Galai (1986) using a different time period and option contracts.

Daily implied interest rates, either from futures or option contracts, are very noisy, specially at short maturities. It is possible to avoid part of the noise in implied rates by taking averages during long periods of time. Figure 3 shows monthly averages from January 1998 to December 2007 for the interest rates described in Table 2. The top panel displays three-month interest rates. As can be seen from the figure, implied interest rates follow closely the other two market rates and on average stay within those bounds, as is expected by the results already presented in Table 2. However, there are periods of time during which *both* rates move in the same direction away from Treasury or LIBOR rates. This is surprising since both implied rates are obtained from different sources. The bottom panel shows that twelve-month implied rates are less noisy and follow more closely the LIBOR rate.

The fact that daily implied interest rates are noisy might occur if futures contracts are mispriced, or if the put-call parity relation is violated. I analyze this issue in the following subsection.

2.4 Borrowing Constraints

The notion of risk-free rate measures the price of borrowing *and* lending *unlimited* units of the numeraire at any given point in time. In real markets, however, agents face borrowing constraints that prevent them to borrow unlimited amounts of funds (see e.g. Black, 1972) or other securities (see e.g. D'Avolio, 2002). Therefore, in real markets the risk-free rate represents a latent variable over which borrowing costs have to be incorporated to obtain the final cost of a trading strategy.

Using only Treasury or LIBOR rates can actually be misleading. Treasury bills and bonds are issued by the U.S. government and might be considered closer to a lending rate from the point of view of an investor. In addition to this, the literature suggests that there is an additional convenience yield that accrues to the holder of the Treasury security. Similarly, LIBOR rates are usually used to measure the cost of borrowing for banks and other financial institutions which makes them closer to a borrowing rate. In addition to borrowing costs for funds, agents might also have to pay lending fees if the want to borrow a stock.

In the presence of different borrowing and lending rates, or if lending fees are charged to sell short the risky asset, no-arbitrage arguments only provide an interval within the price of the futures admits no-arbitrage opportunities. In order to determine the futures price, it is necessary to know the agent's preferences and the quantity to be bought or sold. For example, an agent that wants to hedge a relatively large short position in a futures contract will have to buy the risky asset and short the riskless asset. If there are borrowing costs, the futures price should be higher than in an otherwise equivalent frictionless economy to incentive the agent to hold the position. Similarly, an agent who wants to hedge a long position in a futures contract will have to short the risky asset and long the riskless asset. If there are lending fees, the agent will require the futures to be priced lower relative to the price in an otherwise equivalent frictionless economy.

In equilibrium, however, the impact of borrowing costs on the price of the derivative will depend on the demands by all agents. For example, if an agent wants to go long a futures contract because of hedging purposes, and another agent wants to go short the same contract for exactly the same reason, the equilibrium price might actually be the no-arbitrage price in a frictionless economy. On the other hand, if an agent wants to go long a futures but there are no other counterparts with the same need, she might be willing to pay more for the long position than the cost-of-carry price.

Therefore, the presence of borrowing costs for either the risk-free or the risky asset will have an impact on the price of the derivative if there is demand pressure for long or short positions. In the next section I formalize these ideas and derive an equilibrium model to price futures contracts.

3 A Model with Borrowing Costs

3.1 Economic Environment

3.1.1 Economy

I consider a continuous-time economy with a finite horizon T in which is defined a probability space $(\Omega, \mathcal{F}, P)$, and a filtration $\{\mathcal{F}_t\}_{t=0}^{T}$ satisfying the usual conditions (see e.g. Karatzas and Shreve, 1998, Section 2.7). There is a single consumption good that acts as the numeraire. The economy is populated by a continuum of two types of agents: *arbitrageurs* and *traders*. The economy is competitive, all agents are price takers and there exists a representative agent of each type.

The arbitrageur can buy or sell in any amount a risky asset S_t, a locally risk-free asset M_t which can be thought as a money-market account, and futures contracts $F_t(\tau)$ written on the risky asset for a continuum of maturities $\tau \in (0, T]$. Because the arbitrageur is a price taker, from her perspective the prices of the risk-free and the risky asset are exogenously given.

Futures contracts are financial instruments that are traded in organized exchanges. A particular feature of futures contracts is that they are continuously marked-to-market[8], so any profit or loss accrues immediately to the owner of the contract. Thus, at any point in time the futures price is such that the cost of entering the contract for both parties is zero.

Another feature of futures contracts is that market participants are required to keep a margin account at the exchange. An initial margin is required to start trading and traders receive margin calls whenever their margin account drops below a certain predetermined level. In this model, however, I abstract from margin requirements because the arbitrageur is constrained to maintain a strictly positive wealth at all times, as will be shown in the next subsection.

Traders are investors who have an exogenous demand for futures contracts which might be driven by either hedging or speculative reasons. For example, traders could be pension funds, insurance

[8] See, for example, Cox, Ingersoll, and Ross (1981), Jarrow and Oldfield (1981), Duffie and Stanton (1992).

companies, hedge funds, asset managers or investment banks wishing to decrease or increase its exposure to the risky asset. In this respect, they are similar to option's "end users" in Garleanu et al. (2007) or "clientele investors" in Vayanos and Vila (2007). While it is possible to motivate why investors might want to hold call and put options, or risk-free bonds of a certain maturity, it is more difficult to motivate the demand for futures contracts since their payoffs are essentially linear in the risky asset. Therefore, in this model traders do not have any specific reasons to trade in the futures market. I show later that for S&P 500 index futures the empirical evidence suggests that "traders" in that market are indeed large speculators that trade because of sentiment reasons, i.e. they take long positions when they are bullish and short positions when they are bearish.

In this model, traders' demand for futures contracts is inversely related to the arbitrageur's funding cost. The arbitrageur provides liquidity to the traders by taking the opposite position in the futures market. The equilibrium futures price for all maturities $\tau \in (0, T]$ is such that in equilibrium the market clears, i.e. the aggregate demand for futures contracts is zero.

3.1.2 Arbitrageurs

The arbitrageur is a fully rational risk-averse agent that derives utility from consumption and is compensated for the risks and costs associated with providing liquidity to other investors. In contrast with traders that might want to speculate or hedge only part of their exposure to the risky asset through the use of futures contracts, the arbitrageur holds an optimal portfolio that allows her to optimally hedge her position in the futures market.

In this model, the arbitrageur also has the role of a liquidity provider. The reason I call this agent an arbitrageur is that in equilibrium she is the only agent that "arbitrages away" any price discrepancy by either increasing or decreasing her demand for futures contracts. However, as opposed to the traditional literature on limits to arbitrage, in this paper the arbitrageur does not make any abnormal profit since by construction the market is competitive and there are no arbitrage opportunities. Therefore, in this economy the arbitrageur will price the futures contracts at the marginal cost of carrying forward her portfolio.

The arbitrageur is endowed with a strictly positive wealth W_0 and consumes continuously at a rate C_t up to an arbitrary time T. At each point in time the arbitrageur holds a self-financing portfolio consisting of $\{Z_t^f(\tau)\}_{\tau \in (0,T]}$ futures contracts for each maturity τ, Z_t^m units of the riskless asset, and Z_t^s units of the risky asset. Since futures contracts are continuously marked-to-market and the gains or losses are credited to the arbitrageur's money-market account, the futures position is not included in the arbitrageur's wealth computation. The market value of the arbitrageur's wealth at time t is given by:

$$W_t = Z_t^m M_t + Z_t^s S_t. \tag{4}$$

Since the arbitrageur's portfolio is self-financing, the wealth evolution equation of the arbitrageur satisfies:

$$dW_t = \int_0^T Z_t^f(\tau) dF_t(\tau) d\tau + Z_t^m dM_t + Z_t^s dS_t + Z_t^s D_t dt - C_t dt. \tag{5}$$

The first term in (5) describes the mark-to-market feature of futures contracts. After entering into a futures position of $Z_t^f(\tau)$ contracts for each maturity, all gains or loses $dF_t(\tau)$ over the next time interval are credited to the arbitrageur's account, resulting in a total cash-flow of $Z_t^f(\tau)dF_t(\tau)$ for all $\tau \in (0, T]$. The second and third terms are standard and describe how the wealth changes when the price of the riskless and the risky asset change. The fourth term reflects the net dividend that accrues to the holder of the risky asset but not to the holder of a futures contract. Finally, the last term reflects the drop in wealth caused by intermediate consumption.

In order to rule out arbitrage opportunities, some restrictions need to be imposed on the array of trading strategies available to the arbitrageur . If trading strategies are unrestricted, it can be shown (e.g. Harrison and Kreps, 1979) that an investor can achieve any desired level of wealth without incurring in any costs by following, for example, a standard doubling strategy.

One way to rule out arbitrage opportunities is to directly impose constraints on the trading strategies, as shown by Harrison and Kreps (1979) and Harrison and Pliska (1981). An alternative way to achieve the same goal, which is the one I follow in this paper, is to require the arbitrageur to keep her wealth strictly positive at all times, i.e. $W_t > 0$ for all $t \in [0, T]$ (Dybvig and Huang, 1988). This positivity constraint on wealth rules out arbitrage opportunities by indirectly limiting the array of trading strategies available to the arbitrageur. It also rules out the possibility of bankruptcy, which is consistent with the fact that in this paper the arbitrageur is not required to maintain a margin account in the futures exchange. Finally, it simplifies the portfolio choice analysis because it is then possible to work with returns on wealth rather than the level of wealth itself.

Using equation (4) and because of the positivity constraint on wealth, it is possible to rewrite (5) as

$$\frac{dW_t}{W_t} = \int_0^T z_t^f(\tau)\frac{dF_t(\tau)}{F_t(\tau)}d\tau + (1 - z_t^s)\frac{dM_t}{M_t} + z_t^s\left(\frac{dS_t}{S_t} + \delta dt\right) - c_t dt, \tag{6}$$

where $z_t^f(\tau) = \dfrac{Z_t^f(\tau)F_t(\tau)}{W_t}$ is the total value of the futures position per dollar of wealth, $z_t^s = \dfrac{Z_t^s S_t}{W_t}$ is the percentage of wealth invested in the risky asset, and $c_t = \dfrac{C_t}{W_t}$ is the proportion of wealth that is continuously consumed.

The arbitrageur's problem is to dynamically choose a consumption rate c_t and a portfolio $\phi_t = \{\{z_t^f(\tau)\}_{\tau\in(0,T]}, z_t^s\}$, for all $t \in [0, T]$, so as to maximize

$$E_0\left(\int_0^T e^{-\rho t}\log(C_t)dt\right). \tag{7}$$

It is important to note that the problem is well-defined because the wealth is required to be strictly positive at all times. If this was not the case, the arbitrageur could borrow funds to finance any desired consumption stream and pay her debt by using a doubling strategy. This would allow her to achieve any desired level of wealth in a finite amount of time. Later, it is shown that if the problem admits a solution, the arbitrageur's risk aversion is inversely related to the level of wealth. As the wealth approaches zero, the absolute risk aversion goes to infinity. In this case, one might be inclined to conclude that whenever the wealth of the arbitrageur approaches zero, she will trade in such a way so as to keep her wealth positive at all times. However, it is important to note that the positivity of wealth is actually necessary to solve the problem in the first place, and not a consequence of the arbitrageur having a logarithmic utility.

The choice of this preference structure in dynamic portfolio choice problems is standard (see e.g. Xiong, 2001, Liu and Longstaff, 2004, Vayanos and Vila, 2007). With logarithmic utility, the arbitrageur has a constant relative risk aversion which is equal to one, and it is a well known fact that her portfolio decisions are myopic. In spite of this two restrictions, the choice of this utility function provides some additional intuition on the model and closed-form solutions for futures prices, which greatly simplifies the econometric estimation of the model.

3.1.3 Traders

In this economy, each trader has a demand for futures contracts at one or several maturities. The demand for each contract can be either positive or negative, depending on weather she wants to take a long or a short position, respectively. Therefore, the aggregate demand from traders includes the netting out of opposite positions. For example, if one investor wants to take a long position in a contract with maturity τ, and a second investor wants to take an opposite position in the same contract, the aggregate futures demand from those two investors is zero.

In this paper I model the aggregate demand for futures contracts by traders as

$$H(X_t, \tau) = \lambda(\tau) X_t, \tag{8}$$

where $\lambda(\tau)$ is a deterministic function of maturity and X_t is an unobserved factor. The function $\lambda(\tau)$ can be thought as the term structure of the demand for futures contracts. This demand function is general enough to accommodate for different demand term-structures, although the demand is driven by a single factor, implying that demand shocks at all maturities are perfectly correlated. This restriction, however, can be easily relaxed by allowing for more factors in equation (8).

In this model, the demand factor X_t is stochastically cointegrated with arbitrageurs wealth W_t such that

$$X_t = x_t W_t, \tag{9}$$

where x_t is an unobserved factor that follows a mean-reverting process

$$dx_t = -\kappa_x x_t dt + \sigma_x dw_t^x, \tag{10}$$

and w_t^x is a Brownian motion under P.

This cointegrating relationship is necessary to obtain a stationary equilibrium. If the demand for futures contracts is too high, futures prices will increase so as to accommodate the excess demand, making it profitable for other arbitrageurs to enter the market and provide liquidity to end users. This will have the long-run effect of increasing the representative arbitrageur's wealth hence keeping the ratio X_t/W_t stationary. Similarly, if the demand is too low, it might be profitable for some arbitrageurs to leave the market, hence decreasing the wealth of the representative arbitrageur. Because by construction the arbitrageur's wealth is strictly positive, the demand factor x_t is mathematically well-defined.

3.1.4 Assets

The price S_t of the risky asset follows a geometric Brownian motion

$$\frac{dS_t}{S_t} = \mu_s dt + \sigma_s dw_t^s, \tag{11}$$

where w_t^s is a Brownian motion under P correlated with w_t^x such that $(dw_t^s)(dw_t^x) = \rho_{sx}dt$. In order to simplify the results I denote by $v_{ij} = \rho_{ij}\sigma_i\sigma_j, \forall i,j \in \{s,x\}$, and $u_{ij} = [\Sigma^{-1}]_{ij}, \forall i,j \in \{s,x\}$, where $\Sigma = (v_{ij})_{i,j\in\{s,x\}}$ is a covariance matrix.

The risky asset in this model could represent any traded asset like a stock market index, a currency or a commodity. The risky asset pays a continuous dividend $D_t = S_t\delta$, where the dividend yield δ is a constant. Even though in many cases the assumption of a constant dividend yield might not seem adequate, it greatly simplifies the results. Nevertheless, the model could easily be modified to incorporate a stochastic dividend yield process.

The value M_t of the risk-free asset at time t solves, for a strictly positive initial investment M_0,

$$\frac{dM_t}{M_t} = \widetilde{r}_t dt. \tag{12}$$

where $\widetilde{r}_t$ is the arbitrageur's net funding cost. The risk-free asset is a bank account where the arbitrageur can make deposits or borrow funds as needed. I work with the usual convention that the arbitrageur's demand for funds M_t is positive when the arbitrageur makes a deposit and negative when she needs to borrow funds.

If the risky asset is an index, there are costs associated with selling short the index constituents, namely lending fees (see e.g. D'Avolio, 2002). There are also costs associated with borrowing funds, as borrowing rates are usually higher than lending rates (see e.g. Black, 1972). There might be other costs associated with either borrowing funds or selling short the risk asset, like information asymmetry, reputation or liquidity. In order to keep the model simple, the funding cost faced by the arbitrageur is included in the return on the money-market account such that

$$\widetilde{r}_t = r + \vartheta x_t, \tag{13}$$

where r is the risk-free rate. The intuition behind this modeling choice is as follows. If traders take a long position in futures contracts, the arbitrageur will take a short position in the futures market. In order to hedge the additional risk carried by the futures, the arbitrageur will simultaneously take a long position in the risky asset. If this position is sufficiently large, she will have to borrow funds inducing borrowing costs and possibly other costs like borrowing constraints. Equivalently, if traders short the futures, the arbitrageur will have to take the opposite position in the futures market and hedge by selling the risky asset. If the shorting demand by traders is sufficiently large, the arbitrageur will have to sell short the risky asset, incurring in lending fees and possibly other costs associated with shorting the risky asset, like short selling restrictions.

In this model borrowing costs depend on the traders' demand x_t and not on the arbitrageur's own demand for the risky asset z_t^s. The reason for this is that I consider a fully competitive market and do not allow for strategic behavior from the part of the arbitrageur. The analysis, however, could be modified to account for strategic behavior, although the results are somewhat more complicated in that case.

3.2 Equilibrium

3.2.1 Portfolio Choice Problem

The optimal consumption and portfolio choice is solved using standard dynamic programming methods. Because the borrowing cost of the arbitrageur depends on an additional state variable x_t, her value function $J(W_t, x_t, t)$ depends current wealth W_t and x_t. The Bellman equation for the arbitrageur's problem is

$$\max_{\{c_t, \phi_t\}} e^{-\rho t} \log(C_t)dt + E_t dJ_t = 0, \tag{14}$$

such that $J(W_T, x_T, T) = 0$. It is a well known fact (e.g. Merton, 1971) that under logarithmic utility the consumption and portfolio choice is independent of other state variables. The following lemma just makes explicit this fact in the context of this particular application.

Lemma 1 *If the arbitrageur's problem is given by (14), then her portfolio choice problem ϕ_t for $t \in [0, T]$ is equivalent to solving*

$$\max_{\phi_t} \frac{1}{dt} E_t \left(\frac{dW_t}{W_t} - \frac{1}{2} \left(\frac{dW_t}{W_t} \right)^2 \right). \tag{15}$$

Proof. See the appendix. ■

The intuition behind Lemma 1 is simple. Under logarithmic utility, the arbitrageur's value function is separable, as shown in equation (A.2). This means that the optimal consumption and portfolio choice depend only on what happens with the evolution of wealth. The arbitrageur's problem is then equivalent to maximizing a simple quadratic utility function over instantaneous returns on wealth. A similar result is obtained if the arbitrageur only consumes at time T (see e.g. Liu and Longstaff, 2004).

As shown in the appendix, by solving equation (15) and imposing the equilibrium condition that the aggregate demand for futures contracts for all maturities should be zero in equilibrium, it is possible to derive the equilibrium futures price, as stated in the following proposition.

Proposition 1 *If the function $\lambda(\tau)$ is such that β is the unique solution to*

$$\beta = \int_0^T \lambda(\tau) \xi(\widetilde{\kappa}_x, \tau) d\tau, \tag{16}$$

where $\widetilde{\kappa}_x = \kappa_x + \vartheta \left(\frac{u_{xs} - \beta}{u_{xx}} \right)$, and $\xi(\kappa, \tau) = \frac{1 - e^{-\kappa\tau}}{\kappa}$, then the economy admits a unique equilibrium futures price given by

$$F(S_t, x_t, \tau) = S_t \exp\left(a(\tau) + \vartheta \xi(\widetilde{\kappa}_x, \tau) x_t\right), \tag{17}$$

where

$$a(\tau) = (r - \delta)\tau + \left(v_{sx} + \widetilde{\kappa}_x \widetilde{\theta}_x\right) \left(\frac{\vartheta}{\widetilde{\kappa}_x}\right) (\tau - \xi(\widetilde{\kappa}_x, \tau)) + \frac{1}{2} v_{xx} \left(\frac{\vartheta}{\widetilde{\kappa}_x}\right)^2 (\tau - 2\xi(\widetilde{\kappa}_x, \tau) + \xi(2\widetilde{\kappa}_x, \tau)), \tag{18}$$

and $\widetilde{\theta}_x = \left(\frac{1}{\widetilde{\kappa}_x}\right) \left(\frac{u_{xs}}{u_{xx}}\right) (\mu_s + \delta)$.

Proof. See the appendix. ■

The existence and uniqueness of the equilibrium futures price relies on the assumption that there exists a unique solution β for the equation $\beta = \int_0^T \lambda(\tau)\xi(\widetilde{\kappa}_x, \tau)d\tau$. Even though this assumption may seem arbitrary, the function $\lambda(\tau)$ is given exogenously and many functional forms for this function will generate a solution for this equation.

It is easy to see that if $\vartheta = 0$ then the equilibrium futures prices reduces to the well-known cost of carry formula. Intuitively, if there are no frictions, regardless of the demand for futures contracts by traders, the arbitrageur can always hedge her position. In a competitive market the cost of carrying forward this position is just $r - \delta$. I state this observation in the following corollary.

Corollary 1 *If there are no frictions, i.e. if $\vartheta = 0$, then the futures price is given by $F(S_t, \tau) = S_t e^{(r-\delta)\tau}$.*

3.2.2 Risk-Neutral Pricing

It is a well-known fact that, at least in the finite probability case (Harrison and Pliska, 1981), that the absence of arbitrage opportunities implies the existence of a pricing kernel. In a setting in which the arbitrageur can borrow and short the risky asset without incurring in any costs, Cox et al. (1981) show that the futures price is equal to the expected value of the spot price under a risk-neutral measure Q in which the risky asset drifts at $r - \delta$. It turns out that a similar result can be obtained in this economy since the positivity of wealth rules out arbitrage opportunities. The following proposition makes this statement precise and shows that in this economy there exists an equivalent measure Q^x such that the futures price is equal the expected spot price under Q^x.

Proposition 2 *Let $\eta_t^s = \frac{1}{\sigma_s}\left[(\mu_s + \delta - r - \vartheta x_t\right]$ and $\eta_t^x = \frac{1}{\sigma_x}\left[-\left(\frac{u_{xs}}{u_{xx}}\right)(\mu_s + \delta) + \vartheta\left(\frac{u_{xs} - \beta}{u_{xx}}\right)x_t\right]$, where β is defined as the unique solution to equation (16) in Proposition 1. Define the measure Q^x by its Radon-Nikodym derivative such that $\frac{dQ^x}{dP} = \exp\left(-\int_0^T \eta_t dw_t - \frac{1}{2}\int_0^T \eta_t \cdot \eta_t dt\right)$, where $\eta_t = (\eta_t^s, \eta_t^x)'$ and $w_t = (w_t^s, w_t^x)'$. Then, the futures price in this economy is given by the expectation of the spot price under Q^x, i.e.*

$$F(S_t, x_t, \tau) = E_t^{Q^x}(S_{t+\tau}), \tag{19}$$

where

$$\frac{dS_t}{S_t} = (\widetilde{r}_t - \delta)dt + \sigma_s d\widetilde{w}_t^s, \quad (20)$$

$$dx_t = \widetilde{\kappa}_x(\widetilde{\theta}_x - x_t)dt + \sigma_x d\widetilde{w}_t^x, \quad (21)$$

$\widetilde{r}_t$ is defined in equation (13), $\widetilde{\kappa}_x$ and $\widetilde{\theta}_x$ are defined in Proposition 1, and $\widetilde{w}_t = (\widetilde{w}_t^s, \widetilde{w}_t^x)'$ is a Brownian motion under Q^x.

Proof. See the appendix. ■

Thus, it is possible to recover a result similar to the traditional frictionless economy in which η_t^s and η_t^x as the arbitrageur's local market prices of risk of w_t^s and w_t^x, respectively. In this economy, however, the market prices of risk of both innovations $\widetilde{w}_t^s$ and $\widetilde{w}_t^x$ are time-varying and depend linearly on the state variable x_t. The risk adjustment of the risky asset is intuitive. In a risk neutral world, the risky asset should drift at the cost of carrying forward a long position in the asset. This cost includes the cost of capital, given by the risk-free rate r, plus any other cost associated with borrowing funds or selling short the risky asset. The risk-adjustment of the demand factor is more subtle but is one of the main points of this paper. There exists a unique equilibrium futures price such that the market for futures contracts clears. For this to happen the arbitrageur needs to be compensated for the costs and risks associated with the hedging of the futures positions.

3.2.3 Stochastic Interest Rates

A natural extension of the model that is relevant in empirical applications is when interest rates are stochastic. In this case, the arbitrageur faces an additional source of risk that can be hedged if she is allowed to trade in risk-free bonds, Eurodollar futures or any other claim that is contingent on the risk-free rate. For expositional simplicity I consider the case where the arbitrageur hedges her exposure to interest rate risk using an interest rate futures.

In order to keep the model tractable, I consider the case where the risk-free rate follows a Gaussian one factor mean-reverting process as in Vasicek (1977). Therefore, the general model under the historical measure P is defined as

$$ds_t = \left(\mu_s - \frac{1}{2}\sigma_s^2\right) dt + \sigma_s dw_t^s, \tag{22}$$

$$dx_t = -\kappa_x x_t dt + \sigma_x dw_t^x, \tag{23}$$

$$dr_t = \kappa_r(\theta_r - r_t)dt + \sigma_r dw_t^r, \tag{24}$$

where $s_t = \log(S_t)$, w_t^s, w_t^x and w_t^r are Brownian motions such that $(dw_t^s)(dw_t^x) = \rho_{sx}dt$, $(dw_t^x)(dw_t^r) = \rho_{xr}dt$, and $(dw_t^r)(dw_t^s) = \rho_{rs}dt$. Again, in order to simplify the results I denote by $v_{ij} = \rho_{ij}\sigma_i\sigma_j, \forall i,j \in \{s,x,r\}$, and $u_{ij} = [\Sigma^{-1}]_{ij}, \forall i,j \in \{s,x,r\}$, where $\Sigma = (v_{ij})_{i,j\in\{s,x,r\}}$ is a covariance matrix.

The arbitrageur's funding cost now depends on r_t and x_t such that $\widetilde{r}_t = r_t + \vartheta x_t$. A small modification of the argument used in the proof of Lemma 1 shows that in this case the arbitrageur's portfolio choice problem is to dynamically choose a portfolio $\phi_t = \{\{z_t^f(\tau)\}_{\tau\in(0,T]}, z_t^s, z_t^r\}$ so as to solve (15).

As shown in the appendix, the wealth evolution equation of the arbitrageur now depends on three sources of risk. Since innovations to the demand factor are correlated with changes in the risky asset and risk-free rates, the arbitrageur can diversify away part of the risk generated by the demand factor. These correlations also affect the convexity adjustment of the futures prices.

In order to allow for additional flexibility in the risk premia induced by the model, however, I assume that traders' demand for futures is given by:

$$H(X_t, \tau) = (\lambda_c(\tau) + \lambda_x(\tau)x_t + \lambda_r(\tau)r_t)\, W_t. \tag{25}$$

Under this setup, it is possible to find the equilibrium futures price as stated in the following proposition.

Proposition 3 *If the function $\lambda_i(\tau)$ is such that β_i is the unique solution to*

$$\beta_i = \int_0^T \lambda_i(\tau)\xi(\widetilde{\kappa}_x, \tau)d\tau, i \in \{c,x,r\} \tag{26}$$

where $\widetilde{\kappa}_x = \kappa_x + \vartheta\left(\frac{u_{xs} - \beta_x}{u_{xx}}\right)$, *and* $\xi(\kappa, \tau) = \frac{1 - e^{-\kappa\tau}}{\kappa}$, *then the economy admits a unique equilibrium futures price given by*

$$F(S_t, x_t, r_t, \tau) = S_t \exp\Big(a(\tau) + \vartheta\xi(\widetilde{\kappa}_x, \tau)x_t + \Big((1+\phi)\,\xi(\widetilde{\kappa}_r, \tau) + \phi e^{-\widetilde{\kappa}_x\tau}\xi(\widetilde{\kappa}_r - \widetilde{\kappa}_x, \tau)\Big)\, r_t\Big), \quad (27)$$

where $\phi = \vartheta\left(\frac{\beta_r - u_{xs} + u_{xr}\varphi_1^r\sigma_r}{u_{xx}}\right)$, $\widetilde{\kappa}_r = \kappa_r + \eta_1^r\sigma_r$ *and* $a(\tau)$ *is defined in equation (A.70) in the appendix.*

Like in the constant risk-free rate case, the existence and uniqueness of the equilibrium relies on the fact that β_i is a unique solution to $\beta_i = \int_0^T \lambda_i(\tau)\xi(\widetilde{\kappa}_x, \tau)d\tau$, for each $i \in \{c, x, r\}$. If $\vartheta = 0$, the futures price is equivalent to the frictionless case like in Schwartz (1997). Also, there is an analog result for Proposition 2.

Proposition 4 *Let* $\eta_t^s = \frac{1}{\sigma_s}(\mu_s + \delta - r - \vartheta x_t))$, $\eta_t^x =$, *and* $\eta_t^r =$ *be the market prices of risk of* w_t^s, w_t^x *and* w_t^r, *respectively. Define the measure* Q^x *by its Radon-Nikodym derivative such that* $\frac{dQ^x}{dP} = \exp\left(-\int_0^T \eta_t dw_t - \frac{1}{2}\int_0^T \eta_t \cdot \eta_t dt\right)$, *where* $\eta_t = (\eta_t^s, \eta_t^x, \eta_t^r)'$ *and* $w_t = (w_t^s, w_t^x, w_t^r)'$. *Then, the futures price in this economy is given by the expectation of the spot price under* Q^x, *i.e.*

$$F(S_t, x_t, \tau) = E_t^{Q^x}(S_{t+\tau}), \quad (28)$$

where

$$\frac{dS_t}{S_t} = (\widetilde{r}_t - \delta)dt + \sigma_s d\widetilde{w}_t^s, \quad (29)$$

$$dx_t = \widetilde{\kappa}_x(\widetilde{\theta}_x - x_t + \phi r_t)dt + \sigma_x d\widetilde{w}_t^x, \quad (30)$$

$$dr_t = \widetilde{\kappa}_r(\widetilde{\theta}_r - r_t)dt + \sigma_r d\widetilde{w}_t^r, \quad (31)$$

$\widetilde{\kappa}_x$, $\widetilde{\kappa}_r$ *and* $\widetilde{\theta}_r$ *are defined as in Proposition 3, and* $\widetilde{w}_t = (\widetilde{w}_t^s, \widetilde{w}_t^x, \widetilde{w}_t^r)'$ *is a Brownian motion under* Q^x.

4 Model Estimation

I estimate the model defined by equations (22), (23), (24), (29), (30) and (31) using daily S&P 500 index futures prices from January 1989 to December 2007. I choose to estimate the model using S&P 500 index futures because this contract has the longest time-series of prices among all index futures and options.

Daily data on the S&P 500 index, futures prices, and dividends are obtained from Bloomberg. Dividends are computed as described in section 2.2.1. I start my estimation in 1989 because detailed daily data on S&P 500 dividends starts only in January 1988. I use a total of 4788 time-series observations.

There is an important reason to use daily data in the estimation. The no-arbitrage relation between futures and spot prices could be violated from purely mechanical reasons, like asynchronous trading between the underlying cash index and the futures contract. Miller et al. (1994) show that asynchronous trading in the constituents of an index can produce mean-reversion of the basis, generating the illusion of futures mispricing. Also, the closing futures price is measured at 4:15pm, whereas the closing index price is measured at 4:00pm[9]. Part of these problems should be mitigated by the use of daily data since state variables estimates using the Kalman filter are essentially weighted-averages of current and past observations, and as such behave like moving averages.

The model is cast in state-space form and estimated using the Kalman filter[10]. I use the spot price, and the three, six, and nine-month futures in my estimation. The measurement equation is given by

$$\log\left(F(S_t, x_t, r_t, \tau)\right) = a(\tau) + s_t + \vartheta\xi(\widetilde{\kappa}_x, \tau)x_t + \Big((1+\phi)\,\xi(\widetilde{\kappa}_r, \tau) + \phi e^{-\widetilde{\kappa}_x\tau}\xi(\widetilde{\kappa}_r - \widetilde{\kappa}_x, \tau)\Big)\, r_t, \quad (32)$$

where τ is the maturity of the futures, and all the other variables and parameters are defined in propositions 3 and 4. It is important to note that I use the observed value for the index as one of my observations by setting the maturity of the futures to zero.

[9] In a recent paper, Richie, Daigler, and Gleason (2008) show that price discrepancies exist even when the S&P 500 Exchange Traded Fund (ETF) is used as the underlying index. The ETF should mitigate staleness and transaction costs problems, as well as the effects of volatility associated with the staleness of the index.

[10] The Kalman filter has been used extensively in the finance literature. See, for example, Schwartz (1997), Babbs and Nowman (1999), Duffee (1999) for applications of the Kalman filter to the estimation of commodity, term-structure, and corporate bond models, respectively. For a detailed discussion of state-space models and the Kalman filter see, for example, Harvey (1990, Chap. 3).

In order to identify the model, I set the parameter $\vartheta = 1$. Thus, the demand factor x_t is a scaled version of the futures demand, as specified by the model.

The transition equations are the time-discretized versions of equations (22), (23), (24):

$$\begin{pmatrix} s_{t+\Delta t} \\ x_{t+\Delta t} \\ r_{t+\Delta t} \end{pmatrix} = \begin{pmatrix} \left(\mu_s - \frac{1}{2}\sigma_s^2\right)\Delta t \\ 0 \\ \kappa_r \theta_r \Delta t \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\kappa_x \Delta t & 0 \\ 0 & 0 & 1-\kappa_r \Delta t \end{pmatrix} \begin{pmatrix} s_t \\ x_t \\ r_t \end{pmatrix} + \begin{pmatrix} \epsilon_t^s \\ \epsilon_t^x \\ \epsilon_t^r \end{pmatrix}, \quad (33)$$

where $s_t = \log(S_t)$, ϵ_t^s, ϵ_t^x and ϵ_t^r are standarized normal innovations such that $E\epsilon_t^i \epsilon_t^j = \rho_{ij}\sigma_i\sigma_j\Delta t, \forall i, j \in \{s, x, r\}$.

Since the model only have three state variables but there are four prices, the estimation is over-identified and I assume that all prices are estimated with some error with standard deviation σ_m. In total, the reduced form of the model has 16 parameters to be estimated: μ_s, σ_s, κ_x, σ_x, θ_r, κ_r, σ_r, ρ_{sx}, ρ_{xr}, ρ_{rs}, $\widetilde{\kappa}_x$, $\widetilde{\theta}_x$, ϕ, $\widetilde{\kappa}_r$, $\widetilde{\theta}_r$, and σ_m.

Parameter estimates are obtained by maximizing the likelihood function of log-price innovations. Since the measurement equation is linear in the state-variables and the transition equations follow a Gaussian process, the estimation is efficient. Standard errors of all parameters are computed by inverting the information matrix. An estimate of the information matrix is obtained by computing the outer-product of the gradient (see, for example, Hamilton, 1994, p. 143).

Table 3 shows parameter estimates. Most parameter are significant due to the large number of observations used in the estimation. There are two important results to notice from the table. The first is that there exists a strong correlation between the demand factor and innovations in returns. This means that threre is some diversification benefit between the demand factor and the underlying cash index. Second, the mean reverting parameter of the demand factor is different under the P and the Q–measure, suggesting the existence of a premium. I test for this difference and find that it is statistically significant at the 1% level.

The top panel of Figure 4 shows the implied spot-rate and the target overnight fed funds rate from January 1989 to December 2007. Note that this period is different from the one analyzed in Section 2 since I have nine more years of data. The figure shows that these two rates are very closely related, implying that the model works well in extracting a meaningful short-term rate from the futures data. The bottom panel of the same figure shows that the three-month implied rate also follows very closely Treasury and LIBOR rates. In general, the implied interest rate is between Treasury and LIBOR, specially in periods when the TED spread widens. However, during other periods of time the implied rate is either below Treasury or above LIBOR. This is shown in Figure 5, where I plot the spread of the implied rate with respect to Treasury and LIBOR rates for three, six and twelve-months.

Because of the credit crisis of 2007-08, in the months following the summer of 2007 there has been an increasing interest in looking at alternative measures of the risk-free rate (Brunnermeier, 2008, Michaud and Upper, 2008). In particular, the OIS rate has become a widely followed proxy from which to measure credit spreads from LIBOR, and liquidity spreads from Treasury rates. Figure 6 shows three-month implied and OIS interest rates. It can be seen that the implied rate in general follows very closely the OIS rate.

Table 4 presents summary statistics for market and implied interest rates for this extended period. I present results for two different types of implied rates. First, simple implied rates are obtained by inverting the cost-of-carry formula, as was done in Section 2. I also present results for interest rates implied by the model which are computed as

$$R_t(\tau) = \left(\widetilde{\theta}_r \left(1 - \zeta(\widetilde{\kappa}_r, \tau)\right) - \frac{1}{2}\left(\frac{\sigma_r}{\widetilde{\kappa}_r}\right)^2 \left(1 - 2\zeta(\widetilde{\kappa}_r, \tau) + \zeta(2\widetilde{\kappa}_r, \tau)\right)\right) + \zeta(\widetilde{\kappa}_r, \tau) r_t, \tag{34}$$

where $\zeta(\kappa, \tau) = \dfrac{1 - e^{-\kappa\tau}}{\kappa\tau}$.

There are several important results from this table. First, the overnight implied rate is on average 15 bp higher than the federal funds rate. Second, all simple implied rates have higher means than model implied interest rates. Third, all implied rates are on average between Treasury and LIBOR rates, for all maturities

Another important implication of the model is that in general, or over long periods of time, the implied interest rate should at least not increase the futures mispricing and put-call parity violations. I define the futures mispricing, or "basis", as

$$\textit{Futures Basis} = \frac{F_t(\tau) e^{-(r_t(\tau) - \delta_t(\tau))\tau} - S_t}{\textit{Margin} \cdot S_t}. \tag{35}$$

This definition is standard although I scale the usual "basis" by a margin requirement to reflect the fact that a futures position is in fact a leveraged transaction. If the futures price is mispriced, this would be the total return per unit of capital deployed by an arbitrageur. In my empirical applications, I set the margin equal to 5% which is consistent with historical margin information for S&P 500 futures (Brunnermeier and Pedersen, 2008, Figure 1). Note, however, that the margin is just a constant scaling factor and does not affect the results of the paper. In the absence of frictions and arbitrage opportunities, the basis should always be equal to zero.

Table 5 presents summary statistics for the basis and the absolute basis computed using market and model implied rates. Interest rates for each maturity are obtained by linearly interpolating continuously-compounded Constant Maturity Treasury rates (CMTs) and LIBOR rates, respectively. The interpolation is anchored to the effective overnight U.S. Federal Funds rate. The use of Treasury rates in computing the basis is standard in the finance literature (see e.g. Roll et al., 2007), while it is common practice in industry to use LIBOR rates Hull (2002, p. 94). Dividends are computed following the procedure described in section 2.2.1. It can be seen from the table that the absolute basis is smaller when computed using the implied rate rather than the other two market rates. The difference can be particularly important for the case of Treasury rates.

In summary, this section show s that the empirical data is consistent with some testable implications of the model. First, I find that the demand factor is priced in the cross-section, implying the existence of basis risk. Second, the overnight implied rate is in general closely related to the fed funds rate. Third, the mispricing of futures contracts using the implied risk-free rate from the model is smaller than the mispricing obtained when the basis is computed using Treasury or LIBOR rates.

5 Mispricing and Demand

[IN REVISION]

6 Concluding Remarks

[IN REVISION]]

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A Proofs

In some proofs I use the following function defined as

$$\xi(\kappa,\tau) = \frac{1-e^{-\kappa\tau}}{\kappa} \tag{A.1}$$

A.1 Proof of Lemma 1

We start by guessing that the value function is given by

$$J(W_t,x_t,t) = e^{-\rho t}\left((1/\rho)\log(W_t) + G(x_t,t)\right), \tag{A.2}$$

where $G(x_t,t) \in C^{2,1}(\mathbb{R}\times[0,T])$. Applying Ito's lemma to (A.2) gives

$$\begin{aligned} dJ_t = e^{-\rho t}\Bigg(&\frac{1}{\rho}\left(\frac{dW_t}{W_t} - \left(\frac{dW_t}{W_t}\right)^2\right) - \log(W_t)dt \\ &+G_x(x_t,t)dx_t + \frac{1}{2}G_{xx}(x_t,t)(dx_t)^2 + (G_t(x_t,t) - \rho G(x_t,t))dt\Bigg). \end{aligned} \tag{A.3}$$

The Bellman equation (14) can then be rewritten as

$$\begin{aligned} \max_{\{c_t,\phi_t\}}\Bigg\{&\log(c_t) + \log(W_t) + \frac{1}{dt}E_t\left(\frac{1}{\rho}\left(\frac{dW_t}{W_t} - \left(\frac{dW_t}{W_t}\right)^2\right) - \log(W_t)dt\right. \\ &\left.+G_x(x_t)dx_t + \frac{1}{2}G_{xx}(x_t)(dx_t)^2 + (G_t(x_t,t) - \rho G(x_t,t))dt\right)\Bigg\} = 0, \end{aligned} \tag{A.4}$$

where we have used the fact that $C_t = c_tW_t$. This is equivalent to

$$\begin{aligned} &\max_{c_t}\left\{\log(c_t) + \frac{1}{\rho}\max_{\phi_t}\left\{\frac{1}{dt}E_t\left(\frac{dW_t}{W_t} - \left(\frac{dW_t}{W_t}\right)^2\right)\right\}\right\} \\ &+ \frac{1}{dt}E_t\left(G_x(x_t)dx_t + \frac{1}{2}G_{xx}(x_t)(dx_t)^2 + (G_t(x_t,t) - \rho G(x_t,t))dt\right) = 0, \end{aligned} \tag{A.5}$$

so the portfolio choice problem is equivalent to solving

$$\max_{\phi_t}\left\{\frac{1}{dt}E_t\left(\frac{dW_t}{W_t} - \left(\frac{dW_t}{W_t}\right)^2\right)\right\}. \tag{A.6}$$

A.2 Proof of Proposition 1

To solve for the equilibrium futures price, I start by guessing that the futures price is an exponential-affine function of the form

$$F(S_t, x_t, \tau) = S_t e^{a(\tau)+b(\tau)x_t} \tag{A.7}$$

where $a(\tau)$ and $b(\tau)$ are functions that only depend on maturity τ such that $a(0) = 0$ and $b(0) = 0$.These conditions imply that the futures price at expiration is just the spot price, which prevents arbitrage opportunities at maturity. Applying Ito's lemma to (A.7) gives

$$\frac{dF_t}{F_t} = \mu_t^f(\tau)dt + \sigma_s dw_t^s + \sigma_x b(\tau)dw^x, \tag{A.8}$$

where

$$\mu_t^f(\tau) = \mu_s + (v_{sx} - \kappa_x x_t)b(\tau) + \frac{1}{2}v_{xx}b^2(\tau) - (a'(\tau) + b'(\tau)x_t). \tag{A.9}$$

By plugging in (A.8) in (6) we find

$$\begin{aligned}\frac{dW_t}{W_t} = &\left(\int_0^T z_t^f(\tau)\mu_t^f(\tau)d\tau + (1 - z_t^s)\widetilde{r}_t + z_t^s(\mu_s + \delta) - c_t\right) dt \\ &+ \sigma_s \left(\int_0^T z_t^f(\tau)d\tau + z_t^s\right) dw_t^s + \sigma_x \left(\int_0^T b(\tau)z_t^f(\tau)d\tau\right) dw_t^x. \end{aligned}\tag{A.10}$$

Using the wealth evolution equation (A.10) into equation (15) we find that the first order conditions of this problem are

$$v_{ss}\pi_s + v_{sx}\pi_x = \mu_s + \delta - \widetilde{r}_t \tag{A.11}$$

$$(v_{ss} + v_{sx}b(\tau))\pi_s + (v_{xs} + v_{xx}b(\tau))\pi_x = \mu_t^f(\tau) \tag{A.12}$$

where $\pi_s = \left(\int_0^T z_t^f(\tau)d\tau + z_t^s\right)$ and $\pi_x = \left(\int_0^T z_t^f(\tau)b(\tau)d\tau\right)$. It is then possible to rewrite these two equations as a linear system

$$\underbrace{\begin{pmatrix} v_{ss} & v_{sx} \\ v_{xs} & v_{xx} \end{pmatrix}}_{\Sigma} \underbrace{\begin{pmatrix} \pi_t^s \\ \pi_t^x \end{pmatrix}}_{\Pi} \underset{=}{=} \underbrace{\begin{pmatrix} \mu_s + \delta - \widetilde{r}_t \\ \frac{\mu_t^f(\tau) - (\mu_s + \delta - \widetilde{r}_t)}{b(\tau)} \end{pmatrix}}_{R^e}, \tag{A.13}$$

which can then be solved by inverting Σ as

$$\underbrace{\begin{pmatrix} \pi_t^s \\ \pi_t^x \end{pmatrix}}_{\Pi} \underset{=}{=} \underbrace{\begin{pmatrix} u_{ss} & u_{sx} \\ u_{xs} & u_{xx} \end{pmatrix}}_{\Sigma^{-1}} \underbrace{\begin{pmatrix} \mu_s + \delta - \widetilde{r}_t \\ \frac{\mu_t^f(\tau) - (\mu_s + \delta - \widetilde{r}_t)}{b(\tau)} \end{pmatrix}}_{R^e}. \tag{A.14}$$

This provides an explicit solution for π_t^x such that

$$\pi_t^x = u_{xs}\left(\mu_s + \delta - \widetilde{r}_t\right) + u_{xx}\left(\frac{\mu_t^f(\tau) - (\mu_s + \delta - \widetilde{r}_t)}{b(\tau)}\right). \tag{A.15}$$

Applying the equilibrium condition, we find that

$$\left(\vartheta\beta x_t + u_{xs}\left(\mu_s + \delta - \widetilde{r}_t\right)\right)\frac{b(\tau)}{u_{xx}} + \left(\mu_t^f(\tau) - \left(\mu_s + \delta - \widetilde{r}_t\right)\right) = 0, \tag{A.16}$$

where $\beta = \frac{1}{\vartheta}\int_0^T \lambda(\tau)b(\tau)d\tau$. Since equation (A.16) should be valid for all x_t, by grouping terms we find that the following system of ordinary differential equations should hold:

$$a'(\tau) = (r - \delta) + \left(v_{sx} + \frac{u_{xs}(\mu_s + \delta)}{u_{xx}}\right)b(\tau) + \frac{1}{2}v_{xx}b^2(\tau), \tag{A.17}$$

$$b'(\tau) = \vartheta - \left(\kappa_x + \vartheta\left(\frac{u_{xs} - \beta}{u_{xx}}\right)\right)b(\tau). \tag{A.18}$$

Solving these equations subject to $a(0) = 0$, and $b(0) = 0$ we find that

$$a(\tau) = (r - \delta)\tau + \left(v_{sx} + \frac{u_{xs}(\mu_s + \delta)}{u_{xx}}\right)\int_0^\tau b(t)dt + \frac{1}{2}v_{xx}\int_0^\tau b^2(t)dt, \tag{A.19}$$

$$b(\tau) = \vartheta\xi(\widetilde{\kappa}_x, \tau), \tag{A.20}$$

where

$$\widetilde{\kappa}_x = \kappa_x + \vartheta \left(\frac{u_{xs} - \beta}{u_{xx}} \right), \tag{A.21}$$

$$\int_0^\tau b(t)dt = \left(\frac{\vartheta}{\widetilde{\kappa}_x} \right) (\tau - \xi(\widetilde{\kappa}_x, \tau)), \tag{A.22}$$

$$\int_0^\tau b^2(t)dt = \left(\frac{\vartheta}{\widetilde{\kappa}_x} \right)^2 (\tau - 2\xi(\widetilde{\kappa}_x, \tau) + \xi(2\widetilde{\kappa}_x, \tau)). \tag{A.23}$$

A.3 Proof of Proposition 2

Let

$$\eta_t^s = \varphi_0^s + \varphi_1^s x_t, \tag{A.24}$$

$$\eta_t^x = \varphi_0^x + \varphi_1^x x_t \tag{A.25}$$

and define $\eta_t = (\eta_t^s, \eta_t^x)'$ and $w_t = (w_t^s, w_t^x)'$. It follows from Girsanov's Theorem (see e.g. Karatzas and Shreve, 1998, Section 3.5) that $\widetilde{w}_t^s = w_t^s + \int_0^t \eta_u^s du$ and $\widetilde{w}_t^x = w_t^x + \int_0^t \eta_u^x du$ are also Brownian motions under the measure Q^x defined by its Radon-Nikodym derivative as

$$\frac{dQ^x}{dP} = \exp\left(- \int_0^T \eta_t dw_t - \frac{1}{2} \int_0^T \eta_t \cdot \eta_t dt \right). \tag{A.26}$$

Therefore, under Q^x we have that

$$\frac{dS_t}{S_t} = (\mu_s - \varphi_0^s \sigma_s - \varphi_1^s \sigma_s x_t)dt + \sigma_s d\widetilde{w}_t^s, \tag{A.27}$$

$$dx_t = (-\varphi_0^x \sigma_x - (\kappa_x + \varphi_1^x \sigma_x) x_t)dt + \sigma_x d\widetilde{w}_t^x. \tag{A.28}$$

Let $M(S_t, x_t, \tau) = E_t^{Q^x}(S_{t+\tau})$. It is easy to see that $M(S_t, x_t, \tau)$ is a Q^x–martingale so its drift under Q^x is zero. By applying Ito's lemma,we find that $M(S_t, x_t, \tau)$ satisfies the following partial differential equation

$$(\mu_s - \varphi_0^s \sigma_s - \varphi_1^s \sigma_s x_t) S_t M_s + \frac{1}{2} v_{ss} S_t^2 M_{ss} + (-\varphi_0^x \sigma_x - (\kappa_x + \varphi_1^x \sigma_x) x_t) M_x + \frac{1}{2} v_{xx} M_{xx} + v_{sx} M_{sx} - M_\tau = 0. \tag{A.29}$$

The idea now is to show that there exists η_t^s and η_t^x such that

$$F(S_t, x_t, \tau) = E_t^{Q^x}(S_{t+\tau}), \tag{A.30}$$

i.e. that η_t^s and η_t^x are the market prices of risk of w_t^s and w_t^x, respectively. The function $M(S_t, x_t, \tau) = S_t e^{a(\tau)+b(\tau)}$ satisfies this partial differential equation if

$$a'(\tau) = \mu_s - \varphi_0^s \sigma_s + (v_{sx} - \varphi_0^x \sigma_x) b(\tau) + \frac{1}{2} v_{xx} b^2(\tau), \tag{A.31}$$

$$b'(\tau) = -\varphi_1^s \sigma_s - (\kappa_x + \varphi_1^x \sigma_x) b(\tau). \tag{A.32}$$

By comparing these two equations with (A.17) and (A.18), we find that

$$\varphi_0^s = \frac{1}{\sigma_s}(\mu_s + \delta - r), \tag{A.33}$$

$$\varphi_1^s = -\frac{\vartheta}{\sigma_s}, \tag{A.34}$$

$$\varphi_0^x = -\frac{1}{\sigma_x}\left(\frac{u_{xs}}{u_{xx}}\right)(\mu_s + \delta), \tag{A.35}$$

$$\varphi_1^x = \frac{\vartheta}{\sigma_x}\left(\frac{u_{xs} - \beta}{u_{xx}}\right). \tag{A.36}$$

Thus

$$\eta_t^s = \frac{1}{\sigma_s}\left[(\mu_s + \delta - r - \vartheta x_t\right], \tag{A.37}$$

$$\eta_t^x = \frac{1}{\sigma_x}\left[-\left(\frac{u_{xs}}{u_{xx}}\right)(\mu_s + \delta) + \vartheta\left(\frac{u_{xs} - \beta}{u_{xx}}\right) x_t\right] \tag{A.38}$$

are the market prices for risk in this economy. It is then possible to rewrite the processes for S_t and x_t under Q in more familiar terms as

$$\frac{dS_t}{S_t} = (\widetilde{r}_t - \delta)dt + \sigma_s d\widetilde{w}_t^s, \tag{A.39}$$

$$dx_t = \widetilde{\kappa}_x(\widetilde{\theta}_x - x_t)dt + \sigma_x d\widetilde{w}_t^x, \tag{A.40}$$

where $\widetilde{\kappa}_x = \kappa_x + \dfrac{u_{xs} - \beta}{u_{xx}}$ and $\widetilde{\theta}_x = \left(\dfrac{1}{\widetilde{\kappa}_x}\right)\left(\dfrac{u_{xs}}{u_{xx}}\right)(\mu_s + \delta)$.

A.4 Bond Pricing

Lemma A.1 (Vasicek, 1977) *If the instantaneous risk-free rate is driven by (24), and the market price of risk of w_t^r is equal to $\eta_t^r = \varphi_0 + \varphi_1 r_t$, then the risk-free zero rate $R_t(\tau)$ with maturity τ is given by*

$$R_t(\tau) = \left(\widetilde{\theta}_r \left(1 - \zeta(\widetilde{\kappa}_r, \tau)\right) - \frac{1}{2}\left(\frac{\sigma_r}{\widetilde{\kappa}_r}\right)^2 \left(1 - 2\zeta(\widetilde{\kappa}_r, \tau) + \zeta(2\widetilde{\kappa}_r, \tau)\right)\right) + \zeta(\widetilde{\kappa}_r, \tau) r_t, \tag{A.41}$$

where $\widetilde{\kappa}_r = \kappa_r + \sigma_r \eta_1^r$, $\widetilde{\theta}_r = \dfrac{\kappa_r \theta_r - \sigma_r \eta_0^r}{\kappa_r + \sigma_r \eta_1^r}$ *and* $\zeta(\kappa, \tau) = \dfrac{1 - e^{-\kappa\tau}}{\kappa\tau}$. *Moreover, the price of a futures contract $F_t^r(\tau)$ with maturity τ written on the spot rate and with a notional of one unit of the numeraire is $F_t^r(\tau) = (1 - e^{-\widetilde{\kappa}_r \tau})\widetilde{\theta}_r + e^{-\widetilde{\kappa}_r \tau} r_t$. The instantaneous return of taking a long position in such a contract is*

$$dF_t^r(\tau) = \eta_t^r \sigma_r e^{-\widetilde{\kappa}_r \tau} dt + \sigma_r e^{-\widetilde{\kappa}_r \tau} dw_t^r. \tag{A.42}$$

Proof. This proof is standard (see e.g. Duffie and Kan, 1996) but is giving here for completeness. Let Q^r be an equivalent measure given by its Radon-Nikodym derivative as

$$\frac{dQ^r}{dP} = \exp\left(-\int_0^T \eta_t^r dw_t^r - \frac{1}{2}\int_0^T \eta_t^r \cdot \eta_t^r dt\right), \tag{A.43}$$

where $\eta_t^r = \varphi_0^r + \varphi_1^r r_t$ is the market price of risk of w_t^r. Then it is possible to rewrite (24) as

$$dr_t = \widetilde{\kappa}_r(\widetilde{\theta}_r - r_t)dt + \sigma_r d\widetilde{w}_t^r, \tag{A.44}$$

where $\widetilde{w}_t^r = w_t^r + \int_0^t \eta_u^r du$ is a Brownian motion under Q^r, and

$$\widetilde{\kappa}_r = \kappa + \sigma_r \varphi_1^r, \tag{A.45}$$

$$\widetilde{\theta}_r = \frac{\kappa_r \theta_r - \sigma_r \varphi_0^r}{\kappa_r + \sigma_r \varphi_1^r}. \tag{A.46}$$

Using standard no-arbitrage methods (see e.g. Duffie and Kan, 1996, p. 384), it is easy to show that the price of a zero-coupon bond with maturity τ is

$$B(r_t, \tau) = \exp(a(\tau) + b(\tau) r_t), \tag{A.47}$$

where

$$a(\tau) = -\widetilde{\theta}_r \left(\tau - \xi(\widetilde{\kappa}_r, \tau)\right) + \frac{1}{2}\left(\frac{\sigma_r}{\widetilde{\kappa}_r}\right)^2 \left(\tau - 2\xi(\widetilde{\kappa}_r, \tau) + \xi(2\widetilde{\kappa}_r, \tau)\right), \tag{A.48}$$

$$b(\tau) = -\xi(\widetilde{\kappa}_r, \tau). \tag{A.49}$$

The yield $R_t(\tau)$ on a zero-coupon bond with maturity τ is then given by

$$R_t(\tau) = -\frac{1}{\tau}(f(\tau) + g(\tau)r_t), \tag{A.50}$$

where I have used the fact that $R_t(\tau) = -\frac{1}{\tau}\log\left(B(r_t, \tau)\right)$. The interest rate futures price $F_t^r(\tau)$ with maturity τ and with a notional value of one units of the numeraire is equal to the conditional expectation of the spot rate r_t under Q. By solving the stochastic differential equation (A.44) it follows that (see e.g. Karatzas and Shreve, 1998, Section 5.6)

$$F_t^r(\tau) = E_t^Q(r_{t+\tau}) = (1 - e^{-\widetilde{\kappa}_r\tau})\widetilde{\theta}_r + e^{-\widetilde{\kappa}_r\tau} r_t. \tag{A.51}$$

Moreover, the instantaneous return of taking a long position on a futures contract under P is given by

$$dF_t^r(\tau) = \sigma_r e^{-\widetilde{\kappa}_r\tau} d\widetilde{w}_t^r = \eta_t^r \sigma_r e^{-\widetilde{\kappa}_r\tau} dt + \sigma_r e^{-\widetilde{\kappa}_r\tau} dw_t^r, \tag{A.52}$$

which ends the proof. ■

A.5 Proof of Proposition 3

As in the proof of Proposition 1, I start by guessing that the futures price is an exponential-affine function of the demand factor x_t and the risk-free rate r_t such that

$$F(S_t, x_t, r_t, \tau) = S_t e^{a(\tau) + b(\tau)x_t + c(\tau)r_t}. \tag{A.53}$$

We can then apply Ito's lemma to (A.53) to find

$$\frac{dF_t}{F_t} = \mu_t^f(\tau)dt + \sigma_s dw_t^s + \sigma_x b(\tau) dw^x + \sigma_r c(\tau) dw^r, \tag{A.54}$$

where

$$\mu_t^f(\tau) = \mu_s - \kappa_x b(\tau)x_t + \kappa_r c(\tau)(\theta_r - r_t) + \frac{1}{2}v_{xx}b^2(\tau) + \frac{1}{2}v_{rr}c^2(\tau) + v_{sx}b(\tau) + v_{xr}b(\tau)c(\tau) + v_{rs}c(\tau) - (a'(\tau) + b'(\tau)x_t + c'(\tau)r_t). \quad \text{(A.55)}$$

The arbitrageur's wealth evolution equation in this case is

$$\frac{dW_t}{W_t} = \int_0^T z_t^f(\tau)\frac{dF_t(\tau)}{F_t(\tau)}d\tau + (1 - z_t^s)\frac{dM_t}{M_t} + z_t^s\left(\frac{dS_t}{S_t} + \delta(t)dt\right) + z_t^r dF_t^r(\tau_r) - c_t dt, \quad \text{(A.56)}$$

where $z_t^f(\tau)$ is the total value of the futures position per dollar of wealth, z_t^s is the percentage of wealth invested in the risky asset, z_t^r is the the notional value of the interest rate futures position with an arbitrary maturity τ_r per dollar of wealth, c_t is the proportion of wealth that is consumed by the arbitrageur, and $dF_t^r(\tau_r)$ is computed in Lemma A.1. By plugging in (A.54) in (A.56) we find

$$\frac{dW_t}{W_t} = \left(\int_0^T z_t^f(\tau)\mu_t^f(\tau)d\tau + (1 - z_t^s)\widetilde{r}_t + z_t^s(\mu_s + \delta) + z_t^r\eta_t^r\sigma_r g(\tau_r) - c_t\right)dt + \sigma_s\left(\int_0^T z_t^f(\tau)d\tau + z_t^s\right)dw_t^s + \sigma_x\left(\int_0^T b(\tau)z_t^f(\tau)d\tau\right)dw_t^x + \sigma_r\left(\int_0^T c(\tau)z_t^f(\tau)d\tau + g(\tau_r)z_t^r\right)dw_t^r, \quad \text{(A.57)}$$

where $g(\tau) = e^{-\widetilde{\kappa}_r\tau}$ and τ_r is the maturity of the interest rate futures contract chosen by the arbitrageur to hedge the interest-rate risk.

Using the wealth evolution equation (A.57) into equation (15) we find that the first order conditions of this problem are

$$v_{ss}\pi_s + v_{sx}\pi_x + v_{sr}\pi_r = \mu_s + \delta - \widetilde{r}_t, \quad \text{(A.58)}$$

$$(v_{ss} + v_{sx}b(\tau) + v_{sr}c(\tau))\,\pi_s + (v_{xs} + v_{xx}b(\tau) + v_{xr}c(\tau))\,\pi_x + (v_{rs} + v_{rx}b(\tau) + v_{rr}c(\tau))\,\pi_r = \mu_t^f(\tau), \quad \text{(A.59)}$$

$$v_{rs}\pi_s + v_{rx}\pi_x + v_{rr}\pi_r = \eta_t^r\sigma_r, \quad \text{(A.60)}$$

where $\pi_s = \left(\int_0^T z_t^f(\tau)d\tau + z_t^s\right)$, $\pi_x = \left(\int_0^T z_t^f(\tau)b(\tau)d\tau\right)$ and $\pi_r = \left(\int_0^T c(\tau)z_t^f(\tau)d\tau + g(\tau_r)z_t^r\right)$. It is then possible to rewrite these three equations as a linear system

$$\underbrace{\begin{pmatrix} v_{ss} & v_{sx} & v_{sr} \\ v_{xs} & v_{xx} & v_{xr} \\ v_{rs} & v_{rx} & v_{rr} \end{pmatrix}}_{\Sigma} \underbrace{\begin{pmatrix} \pi_s \\ \pi_x \\ \pi_r \end{pmatrix}}_{\Pi} \underset{=}{=} \underbrace{\begin{pmatrix} \mu_s + \delta - \widetilde{r}_t \\ \frac{\mu_t^f(\tau) - (\mu_s + \delta - \widetilde{r}_t) - \eta_t^r \sigma_r c(\tau)}{b(\tau)} \\ \eta_t^r \sigma_r \end{pmatrix}}_{R^e}, \tag{A.61}$$

which can then be solved by inverting Σ as

$$\underbrace{\begin{pmatrix} \pi_s \\ \pi_x \\ \pi_r \end{pmatrix}}_{\Pi} \underset{=}{=} \underbrace{\begin{pmatrix} u_{ss} & u_{sx} & u_{sr} \\ u_{xs} & u_{xx} & u_{xr} \\ u_{rs} & u_{rx} & u_{rr} \end{pmatrix}}_{\Sigma^{-1}} \underbrace{\begin{pmatrix} \mu_s + \delta - \widetilde{r}_t \\ \frac{\mu_t^f(\tau) - (\mu_s + \delta - \widetilde{r}_t) - \eta_t^r \sigma_r c(\tau)}{b(\tau)} \\ \eta_t^r \sigma_r \end{pmatrix}}_{R^e}. \tag{A.62}$$

This provides an explicit solution for Z_x such that

$$\pi_t^x = u_{xs}\left(\mu_s + \delta - \widetilde{r}_t\right) + u_{xx}\left(\frac{\mu_t^f(\tau) - (\mu_s + \delta - \widetilde{r}_t) - \eta_t^r \sigma_r c(\tau)}{b(\tau)}\right) + u_{xr}\eta_t^r \sigma_r. \tag{A.63}$$

Applying the equilibrium condition, we find that

$$\left(\beta_c + \beta_x x_t + \beta_r r_t + u_{xs}\left(\mu_s + \delta - \widetilde{r}_t\right) + u_{xr}\eta_t^r \sigma_r\right)\frac{b(\tau)}{u_{xx}} + \left(\mu_t^f(\tau) - (\mu_s + \delta - \widetilde{r}_t) - \eta_t^r \sigma_r c(\tau)\right) = 0, \tag{A.64}$$

where $\beta_i = \frac{1}{\vartheta}\int_0^T \lambda_i(\tau)b(\tau)d\tau, i \in \{c, x, r\}$. This last equation should be valid for all x_t and r_t, so by grouping terms we find that the following three equations should hold simultaneously. First,

$$b'(\tau) = \vartheta - \left(\kappa_x + \vartheta\left(\frac{u_{xs} - \beta_x}{u_{xx}}\right)\right)b(\tau), \tag{A.65}$$

so that

$$b(\tau) = \vartheta\xi(\widetilde{\kappa}_x, \tau), \tag{A.66}$$

where $\widetilde{\kappa}_x = \kappa_x + \vartheta \left(\frac{u_{xs} - \beta_x}{u_{xx}} \right)$. Second,

$$c'(\tau) = 1 + (\beta_r - u_{xs} + u_{xr}\varphi_1^r\sigma_r)\, \frac{b(\tau)}{u_{xx}} - (\kappa_r + \varphi_1^r\sigma_r)\, c(\tau), \tag{A.67}$$

so that

$$c(\tau) = (1 + \phi)\, \xi(\widetilde{\kappa}_r, \tau) + \phi e^{-\widetilde{\kappa}_x \tau} \xi(\widetilde{\kappa}_r - \widetilde{\kappa}_x, \tau), \tag{A.68}$$

where $\phi = \vartheta \left(\frac{\beta_r - u_{xs} + u_{xr}\varphi_1^r\sigma_r}{u_{xx}} \right)$ and $\widetilde{\kappa}_r = \kappa_r + \varphi_1^r\sigma_r$. Finally,

$$\begin{aligned} a'(\tau) = -\delta + \left(v_{sx} + \frac{\beta_c + u_{xs}(\mu_s + \delta) + u_{xr}\varphi_0^r\sigma_r}{u_{xx}} \right) b(\tau) + (v_{rs} + \kappa_r\theta_r - \varphi_0^r\sigma_r)\, c(\tau) \\ + \frac{1}{2} v_{xx} b^2(\tau) + v_{xr} b(\tau) c(\tau) + \frac{1}{2} v_{rr} c^2(\tau), \end{aligned} \tag{A.69}$$

so that

$$\begin{aligned} a(\tau) = -\delta\tau + \left(v_{sx} + \frac{\beta_c + u_{xs}(\mu_s + \delta) + u_{xr}\varphi_0^r\sigma_r}{u_{xx}} \right) \int_0^\tau b(t)dt \\ + (v_{rs} + \kappa_r\theta_r - \varphi_0^r\sigma_r) \int_0^\tau c(t)dt + \frac{1}{2} v_{xx} \int_0^\tau b^2(t)dt \\ + v_{xr} \int_0^\tau b(t)c(t)dt + \frac{1}{2} v_{rr} \int_0^\tau c^2(t)dt, \end{aligned} \tag{A.70}$$

where

$$\int_0^\tau b(t)dt = \left(\frac{\vartheta}{\widetilde{\kappa}_x}\right)(\tau - \xi(\widetilde{\kappa}_x, \tau)), \tag{A.71}$$

$$\int_0^\tau c(t)dt = \left(\frac{1+\phi}{\widetilde{\kappa}_r}\right)(\tau - \xi(\widetilde{\kappa}_r, \tau)) + \left(\frac{\phi}{\widetilde{\kappa}_r - \widetilde{\kappa}_x}\right)(\xi(\widetilde{\kappa}_x, \tau) - \xi(\widetilde{\kappa}_r, \tau)), \tag{A.72}$$

$$\int_0^\tau b^2(t)dt = \left(\frac{\vartheta}{\widetilde{\kappa}_x}\right)^2 (\tau - 2\xi(\widetilde{\kappa}_x, \tau) + \xi(2\widetilde{\kappa}_x, \tau)), \tag{A.73}$$

$$\begin{aligned}\int_0^\tau b(t)c(t)dt = {} & \vartheta\left(\frac{1+\phi}{\widetilde{\kappa}_x\widetilde{\kappa}_r}\right)(\tau - \xi(\widetilde{\kappa}_x, \tau) - \xi(\widetilde{\kappa}_r, \tau) + \xi(\widetilde{\kappa}_x + \widetilde{\kappa}_r, \tau)) \\ & + \vartheta\left(\frac{\phi}{\widetilde{\kappa}_x(\widetilde{\kappa}_r - \widetilde{\kappa}_x)}\right)(\xi(\widetilde{\kappa}_x, \tau) - \xi(\widetilde{\kappa}_r, \tau) - \xi(2\widetilde{\kappa}_x, \tau) + \xi(\widetilde{\kappa}_x + \widetilde{\kappa}_r, \tau)),\end{aligned} \tag{A.74}$$

$$\begin{aligned}\int_0^\tau c^2(t)dt = {} & \left(\frac{1+\phi}{\widetilde{\kappa}_r}\right)^2 (\tau - 2\xi(\widetilde{\kappa}_r, \tau) + \xi(2\widetilde{\kappa}_r, \tau)) \\ & + \left(\frac{\phi}{\widetilde{\kappa}_r - \widetilde{\kappa}_x}\right)^2 (\xi(2\widetilde{\kappa}_x, \tau) - 2\xi(\widetilde{\kappa}_x + \widetilde{\kappa}_r, \tau) + \xi(2\widetilde{\kappa}_r, \tau)) \\ & + 2\left(\frac{(1+\phi)\phi}{\widetilde{\kappa}_r(\widetilde{\kappa}_r - \widetilde{\kappa}_x)}\right)(\xi(\widetilde{\kappa}_x, \tau) - \xi(\widetilde{\kappa}_r, \tau) - \xi(\widetilde{\kappa}_x + \widetilde{\kappa}_r, \tau) + \xi(2\widetilde{\kappa}_r, \tau)).\end{aligned} \tag{A.75}$$

A.6 Proof of Proposition 4

[TO BE WRITTEN]

B Commitments of Traders Reports

The CFTC publishes at a weekly frequency the Commitments of Traders (COT) reports which can be found at the CFTC's website[11]. The COT reports provide a breakdown of each Tuesday's open interest for markets in which 20 or more traders hold positions equal to or above the reporting levels established by the CFTC. Each report shows the total open interest, for both long and short positions, for two types of traders: reportable positions or "large traders", and non-reportable positions or "small traders". Large traders are identified as those that hold positions above specific reporting levels set by CFTC regulations. According to the CFTC's website, the aggregate of all traders' positions represents 70 to 90 percent of the total open interest in any given market.

Large traders are subsequently subdivided between commercial traders or "large hedgers“, and non-commercial traders or "large speculators“. All of a trader's reported futures positions in

[11] http://www.cftc.gov/marketreports/commitmentsoftraders/index.htm.

a commodity are classified as commercial if the trader uses futures contracts in that particular commodity for hedging[12]. A trading entity generally is classified as a "commercial" trader if she declares and files to the CFTC that she "[...] is engaged in business activities hedged by the use of the futures or option markets." In any case the CFTC reserves the right to re-classify a trader. A trader may be classified as a commercial trader in some commodities and as a non-commercial trader in other commodities. Even though a single trading entity cannot be classified as both a commercial and non-commercial trader in the same commodity, a financial organization trading in financial futures may have a banking entity whose positions are classified as commercial and have a separate money-management entity whose positions are classified as non-commercial. The positions of small traders can be derived from subtracting the long and short reportable positions from the total open interest. Whether a small trader is a speculator or a hedger is unknown.

Historical data on COT reports is provided at a bimonthly frequency from 1986 to 1992, and at a weekly frequency from 1993 to present. Table 6 describes the open interest for S&P 500 futures contracts by type of trader from January 1988 to December 2007. The table shows that on average, large speculators hold the largest proportion of open interest, for both long (69.77%) and short (68.29%) positions. They are followed by small traders that account for 23.70% of long and 20.49% of short positions. Finally, large speculators contribute to only 5.83% of long and 10.52% of short positions.

Even though large speculators have the smallest contribution to the total open interest, they have the largest absolute "net position", defined as the difference between the long and short positions for each type of trader. On average, large speculators are net short (-4.69%) whereas small traders (3.21%) and large hedgers are net long (1.47%). Although large hedgers have on average the smallest absolute net position, their position exhibit the largest volatility (9.47%) among the three groups.

C Liquidity Measures

I compute all liquidity measures using daily data on NYSE stocks. I include only stocks with a price greater than five dollars on a given day. All measures are computed daily from January 1989

[12] As defined in CFTC Regulation 1.3(z), 17 CFR 1.3(z).

to December 2007.

The first control that I include in my analysis is Amihud (2002) illiquidity measure which for a single stock at a single day is defined as:

$$Illiq_t^i = \frac{|R_t^i|}{DollarVolume_t^i}, \tag{C.1}$$

where R_t^i, and $DollarVolume_t^i$ are the daily return and the dollar volume at time t for security i, respectively. A stock is illiquid if the price moves significantly when there is low volume. Amihud (2002) shows that this measure of illiquidity is related to measures of price impact and fixed trading costs. In order to obtain a market measure of illiquidity, I compute the cross-sectional average of all NYSE stocks satisfying the price filter defined previously. One concern about Amihud (2002) illiquidity measure when used in a time-series setting is that the denominator is a possibly non-stationary variable. In order to correct for this non-stationarity, I follow Acharya and Pedersen (2005) and multiply the illiquidity measure by the ratio of the market capitalization of all NYSE stocks at time t and the market portfolio at the beginning of the sample.

The second control I use is a variant of Amihud (2002) measure in which I account for the non-stationarity by using share turnover in the denominator:

$$IlliqTurn_t^i = \frac{|R_t^i|}{TurnOver_t^i}. \tag{C.2}$$

The idea of this measure is to normalize the illiquidity measure for each security before computing the market measure of illiquidity. The market illiquidity measure is the average of this measure across stocks for a given day.

The third control that I use for illiquidity follows Lesmond, Ogden, and Trzcinka (1999) and is based on the proportion of zero returns on any given day. The idea is that a higher proportion of zero returns on any given day will reflect that investors are not trading because the costs of trading represents a threshold that needs to be surpassed before a security return reflects new information.

The fourth illiquidity proxy is the effective cost measure of the Roll (1984) model, which is computed as:

$$EffCost_t^i = \sqrt{-Cov_t(R_t^t, R_{t-1}^i)}, \tag{C.3}$$

where the covariance is computed using a moving 20 days window. For each day and stock, if the covariance is positive then it is set equal to zero. The market illiquidity measure is obtained by averaging each day all NYSE stocks satisfying the price filter.

Finally, I also include share turnover defined as the daily volume over the number of shares, as a measure of liquidity. Thus, turnover is inversely related to all the other measures since it measures liquidity as opposed to illiquidity.

Using all these measures, I perform a principal components analysis (PCA) and extract the first factor. The PCA is run on the daily illiquidity measures from October 1989 to December 2007. The factor is standardized such that its positively correlated with illiquidity measures. The first factor explains almost 60% of the total variation, while the first three factors explain a little more than 95% of the total variation.

Table 7 reports summary statistics for the illiquidity measures and for the illiquidity factor, as well as pair-wise correlations among these variables. The table also reports the factor loadings of the illiquidity factor on the other measures. All *illiquidity* measures are positively correlated and negatively correlated with turnover which is a *liquidity* measure. In particular, the illiquidity factor achieves a high correlation with all other measures. The illiquidity factor loads more on the modified Amihud (2002) measure, and less on the Roll (1984) measure. All measures load on the illiquidity factor with the expected sign.

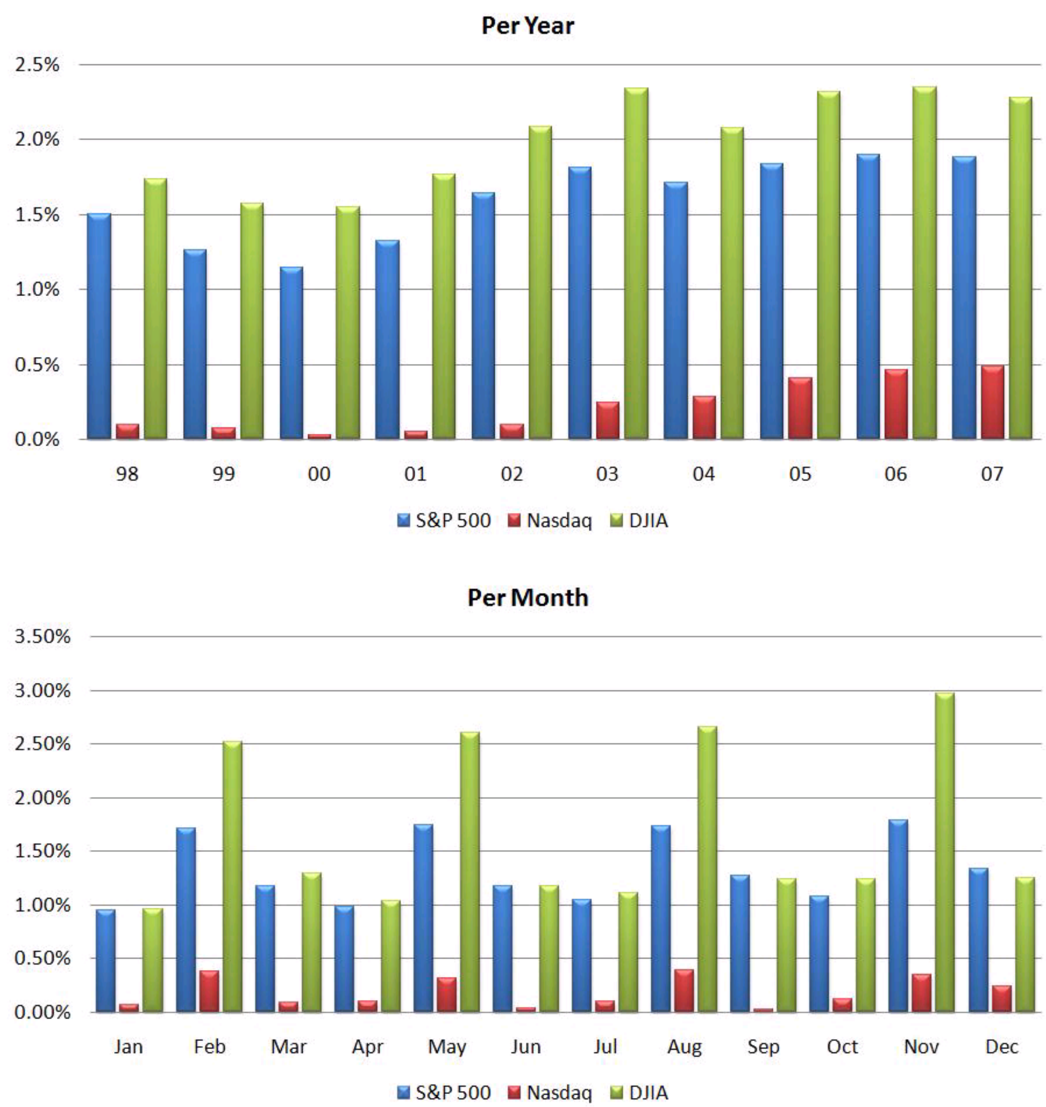

Figure 2: Dividend Yield

Notes: The figure shows dividend yield averages for S&P 500, Nasdaq and DJIA indexes. The period is January 1988 to December 2007. Averages are computed using daily data. Daily dividend payments for each index is obtained from Bloomberg. The top panel displays the average dividend yield per year by index. The bottom panel shows average dividend yields per calendar month of the year by index for the same period.

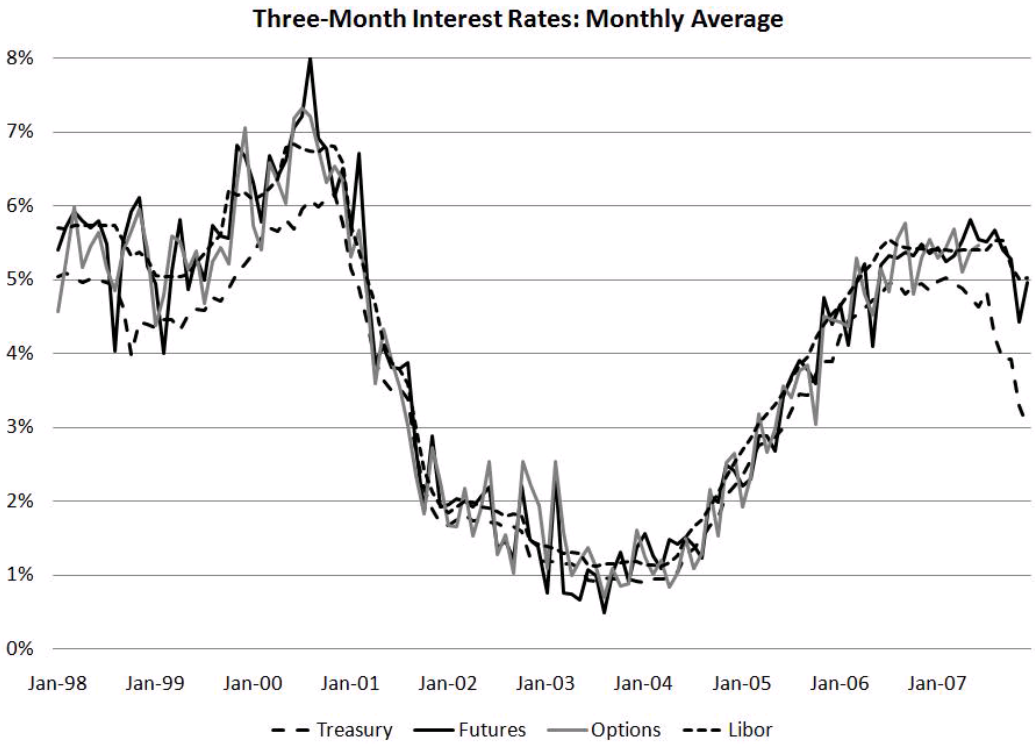

Figure 3: Monthly Implied Rates

Notes: The figure shows monthly averages for three-month Treasury, LIBOR and Implied rates obtained from index (S&P 500, Nasdaq, and DJIA) futures and options, for the period January 1998 to December 2007. Averages are computed using daily data. All rates are continuously compounded. Treasury rates are CMTs provided by the U.S. Department of the Treasury. Implied index futures and option rates are computed from equations (2) and (3). Implied rates are averaged daily for all underlyings for the respective maturity. For options, only contracts with a volume greater than 10 contracts for calls and puts are included. Also, all contracts for a given strike for which the absolute mispricing computed using either Treasury or LIBOR was greater than 50% are not included in the computations.

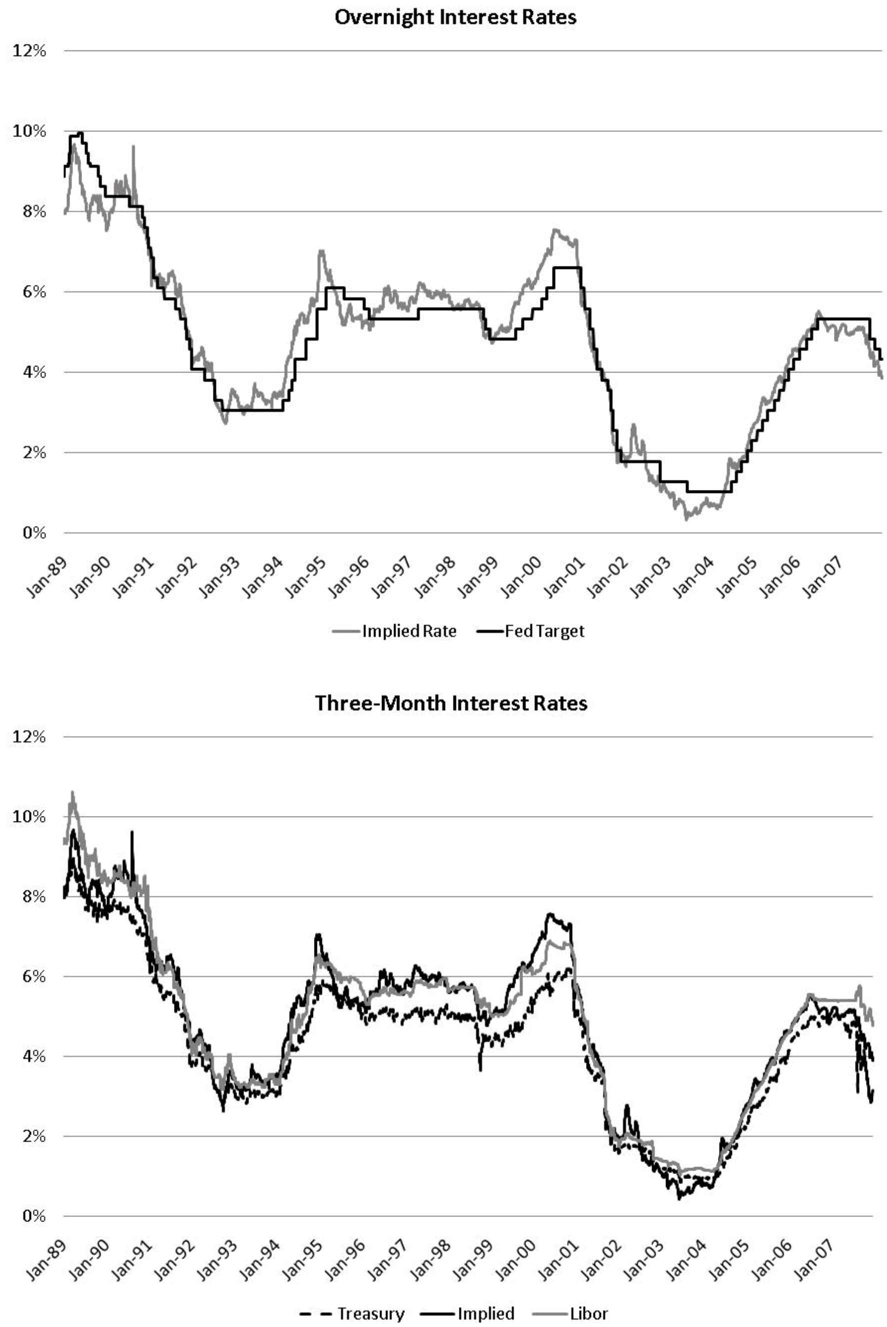

Figure 4: Model Implied Rates

Notes: The top panel of the figure shows daily fed funds target and model implied overnight rates. The bottom panel shows three-month Treasury, LIBOR and model implied interest rates. The period is January 1989 to December 2007. All rates are continuously compounded. Treasury rates are CMTs provided by the U.S. Department of the Treasury. Model implied interest rates for maturity τ are computed using equation (34) and the parameter values given in Table 3.

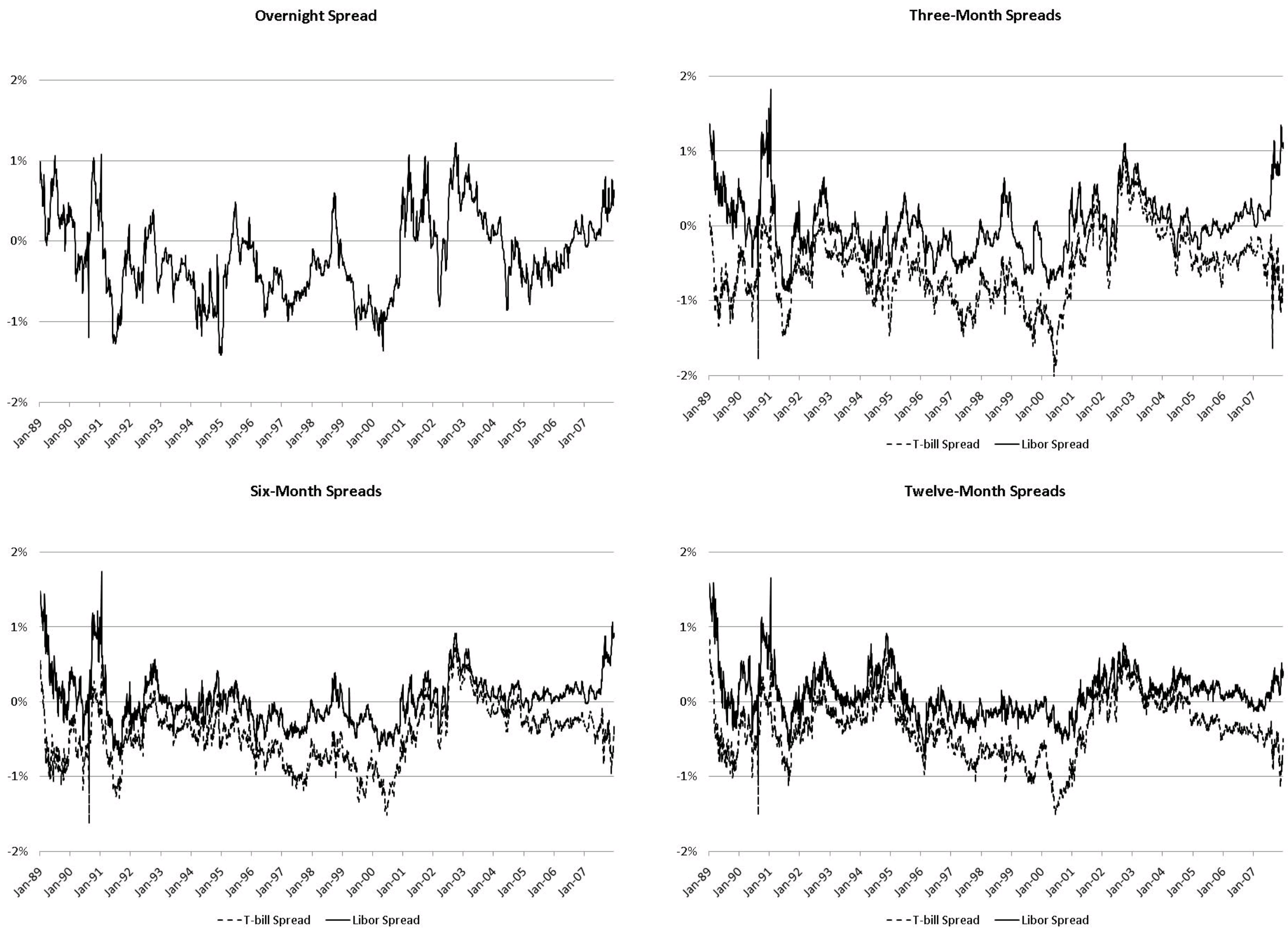

Figure 5: Spreads

Notes:

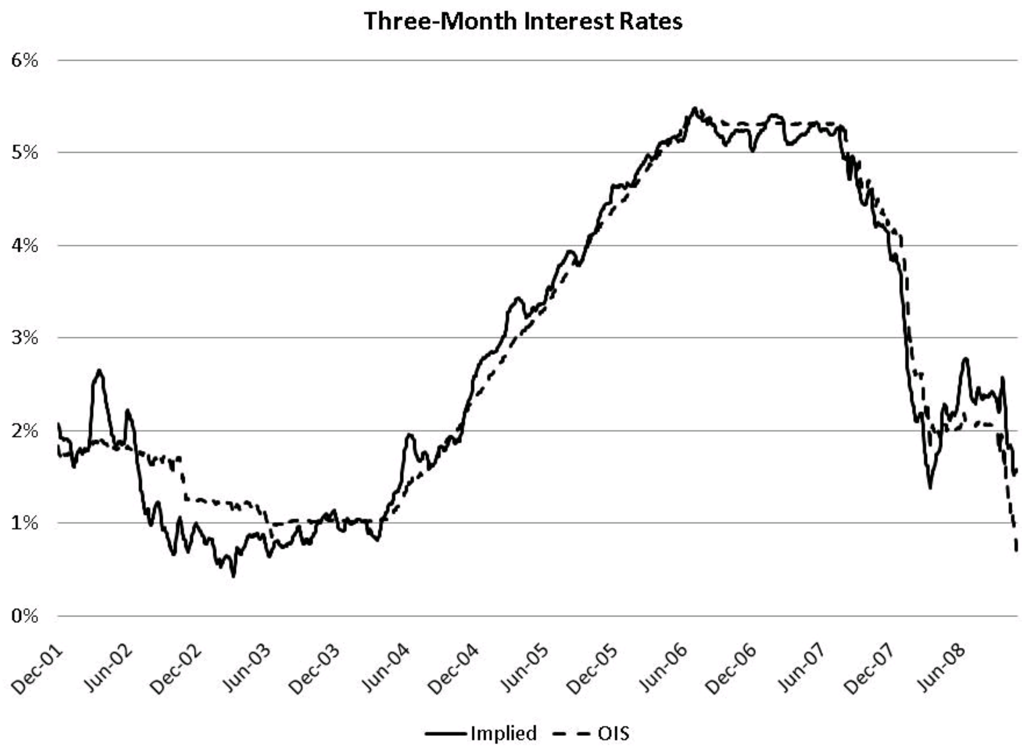

Figure 6: Model Implied and OIS Interest Rate

Notes: The figure shows the three month Overnight Index Swap and model implied overnight rates. The period is December 2001 to December 2007.

Table 1: Open Interest

Market	Contract	Mean	Median	St. Dev.
		Panel A: Futures		
S&P 500	3-month	94.82%	96.14%	4.50%
	6-month	4.31%	2.93%	4.18%
	9-month	0.68%	0.53%	0.63%
	12-month	0.19%	0.09%	0.42%
Nasdaq	3-month	98.77%	99.86%	4.75%
	6-month	0.84%	0.12%	2.12%
	9-month	0.38%	0.00%	3.38%
	12-month	0.00%	0.00%	0.00%
DJIA	3-month	94.30%	97.45%	9.16%
	6-month	3.86%	1.93%	5.49%
	9-month	1.22%	0.03%	3.12%
	12-month	0.62%	0.00%	3.54%
		Panel B: Options		
S&P 500	3-month	51.90%	56.42%	21.22%
	6-month	29.04%	27.06%	14.96%
	9-month	14.02%	10.02%	11.65%
	12-month	6.57%	3.62%	7.68%
Nasdaq	3-month	83.09%	90.28%	19.66%
	6-month	17.29%	11.36%	18.20%
	9-month	5.97%	2.36%	9.15%
	12-month	3.46%	0.81%	6.60%
DJIA	3-month	57.63%	59.88%	23.90%
	6-month	28.66%	23.31%	20.41%
	9-month	12.18%	6.46%	13.87%
	12-month	7.87%	2.08%	11.75%

Notes: This table presents summary statistics for the open interest in S&P 500, Nasdaq and DJIA futures and options during the period January 1998 to December 2007. For each day, contract, and maturity, the proportion of open interest by maturity with respect to the total open interest for the contract on that day is computed. Mean, median and standard deviations are computed over those daily proportions, for both futures and options.

Table 2: Observed and Implied Interest Rates

	Maturity	Mean	Median	St. Dev.
	Panel A: Market Rates			
Treasury Bills	3-Month	3.50%	3.92%	1.66%
	6-Month	3.63%	4.18%	1.67%
	12-Month	3.71%	4.14%	1.56%
Libor	3-Month	4.03%	4.92%	1.88%
	6-Month	4.07%	4.86%	1.85%
	12-Month	4.17%	4.77%	1.75%
	Panel B: Futures			
S&P 500	3-Month	4.06%	4.35%	2.49%
	6-Month	4.18%	4.77%	2.07%
	9-Month	4.23%	4.83%	2.02%
Nasdaq	3-Month	3.83%	4.13%	3.39%
	6-Month	3.84%	4.37%	1.97%
	9-Month	3.81%	4.42%	1.85%
Dow Jones Industrial Average	3-Month	4.04%	4.29%	2.46%
	6-Month	4.18%	4.69%	2.06%
	9-Month	4.23%	4.72%	2.02%
All Futures	3-Month	3.98%	4.25%	2.78%
	6-Month	4.07%	4.61%	2.04%
	9-Month	4.09%	4.66%	1.96%
	Panel C: Options			
S&P 500	[30, 90]	4.20%	4.51%	2.18%
	[90, 180]	4.09%	4.60%	1.88%
	[180, 270]	4.04%	4.63%	1.78%
	[270, 365]	4.09%	4.67%	1.70%
Nasdaq	[30, 90]	3.79%	4.26%	2.65%
	[90, 180]	3.96%	4.44%	1.67%
	[180, 270]	4.03%	4.67%	1.52%
	[270, 365]	4.82%	5.11%	0.93%
Dow Jones Industrial Average	[30, 90]	3.78%	4.03%	2.35%
	[90, 180]	3.80%	4.24%	1.94%
	[180, 270]	3.70%	4.27%	1.84%
	[270, 365]	3.68%	4.42%	1.86%
All Options	[30, 90]	4.00%	4.35%	2.36%
	[90, 180]	3.99%	4.47%	1.86%
	[180, 270]	3.97%	4.58%	1.75%
	[270, 365]	4.04%	4.69%	1.73%

Notes: This table presents summary statistics for Treasury, LIBOR and Implied rates obtained from index (S&P 500, Nasdaq, and DJIA) futures and options, for the period January 1998 to December 2007. Averages, medians and standard deviations are computed using daily data. All rates are continuously compounded. Treasury and LIBOR rates are presented for three, six and twelve-month maturities. Treasury rates are CMTs provided by the U.S. Department of the Treasury. Implied index futures rates are computed as

$$R_t(\tau) = \frac{1}{\tau} \log\left(\frac{F_t(\tau)}{S_t}\right) + \delta_t(\tau),$$

where τ is the exact maturity of the contract, $F_t(\tau)$ is the closing futures price, S_t is the closing spot price, and $\delta_t(\tau)$ is the expected dividend yield from time t up to time $t+\tau$. Expected dividends are computed according to the procedure described in Section 2.2.1. Implied rates from futures are computed from three, six and nine-month S&P 500, Nasdaq and DJIA index futures contracts. Implied rates from option are computed according to

$$R_t(\tau) = -\frac{1}{\tau} \log\left(\frac{S_t e^{-\delta_t(\tau)} - C_t(K,\tau) + P(K,\tau)}{K}\right),$$

where $C_t(K,\tau)$ and $P_t(K,\tau)$ are the prices of call and put options with maturity τ and strike K, respectively. These implied rates are averaged daily for each underlying for maturities ranging from 30 to 90 days, 90 to 180 days, 180 to 270 days, and 270 to 365 days. For options, only contracts with a volume greater than 10 contracts for calls and puts are included. Also, all contracts for a given strike for which the absolute mispricing computed using either Treasury or LIBOR was greater than 50% are not included in the computations.

Table 3: Parameter Estimates Reduced-Form Model

Parameter	Estimate	St. Error	P-Value
μ	0.0967	0.0577	0.09
σ_s	0.2053	0.0010	0.00
κ_x	268.1081	3.2757	0.00
σ_x	1.8127	0.0956	0.00
θ_r	0.0280	0.0251	0.26
κ_r	0.1804	0.0851	0.03
σ_r	0.0116	0.0011	0.00
ρ_{sx}	0.1290	0.0070	0.00
ρ_{xr}	-0.1956	0.0259	0.00
ρ_{rs}	-0.0004	0.0255	0.99
$\widetilde{\kappa}_x$	48.4843	2.5507	0.00
$\widetilde{\phi}$	-0.0275	0.0925	0.77
$\widetilde{\theta}_x$	0.0002	0.0051	0.98
$\widetilde{\kappa}_r$	0.1760	0.0084	0.00
$\widetilde{\theta}_r$	0.0607	0.0032	0.00
$\widetilde{\sigma}_m$	0.0003	0.0000	0.00

Notes: This table presents parameter estimates of the theoretical model presented in Section 3. The model is estimated with the Kalman filter using daily S&P 500 cash index, and three, six, and nine-month futures contracts. The measurement equation is given by

$$\log\left(F(S_t, x_t, r_t, \tau)\right) = a(\tau) + s_t + \xi(\widetilde{\kappa}_x, \tau)x_t + \Big((1+\phi)\,\xi(\widetilde{\kappa}_r, \tau) + \phi e^{-\widetilde{\kappa}_x \tau}\xi(\widetilde{\kappa}_r - \widetilde{\kappa}_x, \tau)\Big)\, r_t,$$

where τ is the maturity of the futures, and all the other variables and parameters are defined in propositions 3 and 4. The transition equations are the time-discretized versions of equations (22), (23), (24):

$$\begin{pmatrix} s_{t+\Delta t} \\ x_{t+\Delta t} \\ r_{t+\Delta t} \end{pmatrix} = \begin{pmatrix} \left(\mu_s - \frac{1}{2}\sigma_s^2\right)\Delta t \\ 0 \\ \kappa_r \theta_r \Delta t \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1-\kappa_x \Delta t & 0 \\ 0 & 0 & 1-\kappa_r \Delta t \end{pmatrix} \begin{pmatrix} s_t \\ x_t \\ r_t \end{pmatrix} + \begin{pmatrix} \epsilon_t^s \\ \epsilon_t^x \\ \epsilon_t^r \end{pmatrix},$$

where $s_t = \log(S_t)$, ϵ_t^s, ϵ_t^x and ϵ_t^r are standarized normal innovations such that $E\epsilon_t^i \epsilon_t^j = \rho_{ij}\sigma_i\sigma_j\Delta t, \forall i,j \in \{s,x,r\}$. All prices are assumed to be estimated with measurement error with standard deviation $\widetilde{\sigma}_m$. In total, there are 16 parameters to be estimated: μ_s, σ_s, κ_x, σ_x, θ_r, κ_r, σ_r, ρ_{sx}, ρ_{xr}, ρ_{rs}, $\widetilde{\kappa}_x$, $\widetilde{\theta}_x$, ϕ, $\widetilde{\kappa}_r$, $\widetilde{\theta}_r$, and σ_m. Parameter estimates are obtained by maximizing the likelihood function of measurement innovations. Standard errors of all parameters are computed by inverting the information matrix, which is estimated by computing the outer-product of the gradient.

Table 4: Model Implied Rates

	Maturity	Mean	Median	St. Dev.
Fed Funds	Overnight	4.72%	5.14%	2.13%
Treasury Bill	3-Month	4.33%	4.71%	1.86%
	6-Month	4.48%	4.88%	1.86%
	12-Month	4.61%	4.86%	1.82%
LIBOR	3-Month	4.91%	5.39%	2.09%
	6-Month	4.96%	5.36%	2.05%
	12-Month	5.07%	5.30%	1.97%
Implied (Simple)	3-Month	4.95%	5.13%	2.61%
	6-Month	5.08%	5.41%	2.15%
	9-Month	5.13%	5.45%	2.09%
Implied (Model)	Overnight	4.88%	5.23%	2.20%
	3-Month	4.90%	5.25%	2.15%
	6-Month	4.93%	5.26%	2.10%
	12-Month	4.97%	5.30%	2.01%

Notes: This table presents Treasury, LIBOR and implied rates. The period is January 1989 to December 2007. Averages, medians and standard deviations are computed using daily data. All rates are continuously compounded. Treasury and LIBOR rates are presented for three, six and twelve-month maturities. Treasury rates are CMTs provided by the U.S. Department of the Treasury. Implied rates are computed using two methods. Simple implied rates $R_t(\tau)$ for maturity τ are computed as

$$R_t(\tau) = \frac{1}{\tau}\log\left(\frac{F_t(\tau)}{S_t}\right) + \delta_t(\tau).$$

Model implied interest rates for same maturity are computed as

$$R_t(\tau) = \left(\widetilde{\theta}_r\left(1 - \zeta(\widetilde{\kappa}_r, \tau)\right) - \frac{1}{2}\left(\frac{\sigma_r}{\widetilde{\kappa}_r}\right)^2 \left(1 - 2\zeta(\widetilde{\kappa}_r, \tau) + \zeta(2\widetilde{\kappa}_r, \tau)\right)\right) + \zeta(\widetilde{\kappa}_r, \tau) r_t,$$

where $\zeta(\kappa, \tau) = \frac{1-e^{-\kappa\tau}}{\kappa\tau}$, and all parameters were estimated as described in Section 4. Parameter values are given in Table 3. Simple implied rates are estimated for three, six, and nine-month maturities. Model implied rates are estimated for overnight, three, six, and twelve-month maturities.

Table 5: S&P 500 Futures Basis

	Contract	Mean	Median	St. Dev.
		Panel A: Basis		
Treasury Bills	3-Month	1.92%	1.80%	4.12%
	6-Month	4.54%	4.28%	4.92%
	9-Month	6.33%	5.91%	5.95%
	All	4.29%	3.80%	5.39%
Libor	3-Month	0.35%	0.44%	3.83%
	6-Month	0.25%	0.46%	4.51%
	9-Month	-0.04%	0.24%	5.37%
	All	0.18%	0.39%	4.61%
Model	3-Month	0.36%	0.48%	3.85%
	6-Month	0.47%	0.73%	4.14%
	9-Month	0.72%	0.90%	3.94%
	All	0.53%	0.70%	3.99%
		Panel B: Absolute Basis		
Treasury Bills	3-Month	3.43%	2.70%	2.98%
	6-Month	5.40%	4.58%	3.95%
	9-Month	7.12%	6.10%	4.98%
	All	5.34%	4.26%	4.35%
Libor	3-Month	2.80%	2.18%	2.63%
	6-Month	3.37%	2.65%	3.01%
	9-Month	3.94%	3.00%	3.65%
	All	3.37%	2.59%	3.15%
Model	3-Month	2.84%	2.24%	2.62%
	6-Month	3.12%	2.50%	2.76%
	9-Month	3.00%	2.41%	2.66%
	All	3.00%	2.39%	2.69%

Notes: This table presents summary statistics for the basis and absolute basis in S&P 500 futures contracts. The period is January 1989 to December 2007. Averages, medians and standard deviations are computed using daily data. The futures basis is defined as

$$\textit{Futures Basis} = \frac{F_t(\tau)e^{-(r_t(\tau)-\delta_t(\tau))\tau} - S_t}{\textit{Margin} \cdot S_t},$$

where the margin is set to 5%. The basis is estimated using Treasury, LIBOR and model implied interest rates. For Treasury and LIBOR rates, $r_t(\tau)$ is obtained by linearly interpolating continuously-compounded CMTs and LIBOR rates, respectively. The interpolation is anchored to the effective overnight U.S. Federal Funds rate. Model implied rates are obtained from

$$R_t(\tau) = \left(\widetilde{\theta}_r \left(1 - \zeta(\widetilde{\kappa}_r, \tau)\right) - \frac{1}{2}\left(\frac{\sigma_r}{\widetilde{\kappa}_r}\right)^2 \left(1 - 2\zeta(\widetilde{\kappa}_r, \tau) + \zeta(2\widetilde{\kappa}_r, \tau)\right)\right) + \zeta(\widetilde{\kappa}_r, \tau) r_t,$$

where $\zeta(\kappa, \tau) = \frac{1-e^{-\kappa\tau}}{\kappa\tau}$, and all parameters were estimated as described in Section 4. Parameter values are given in Table 3.

Table 6: Open Interest by Trader Type

Type of Trader	Position	Mean	Median	St. Dev.
Large Speculators	Long	5.89%	5.38%	2.39%
	Short	10.61%	9.89%	4.31%
	Net Position	-4.72%	-4.49%	5.03%
Large Hedgers	Long	69.74%	69.69%	3.30%
	Short	68.38%	69.16%	7.82%
	Net Position	1.36%	0.69%	9.55%
Small Traders	Long	23.67%	23.36%	2.97%
	Short	20.31%	20.99%	5.20%
	Net Position	3.36%	2.77%	6.18%

Notes: This table presents summary statistics for open interest by trader type. The period is January 1989 to December 2007. Averages, medians and standard deviations are computed using bi-monthly data from January 1989 to October 1992, and weekly data from November 1992 to December 2007. The data source are Commitments of Traders (COT) reports which can be found at the CFTC's website http://www.cftc.gov/marketreports/commitmentsoftraders/index.htm. Each report shows the total open interest, for both long and short positions, for two types of traders: reportable positions or "large traders", and non-reportable positions or "small traders". Large traders are identified as those that hold positions above specific reporting levels set by CFTC regulations. Large traders are subsequently subdivided between commercial traders or "large hedgers", and non-commercial traders or "large speculators".

Table 7: Illiquidity Measures

				Correlations						PCA
Measure	**Mean**	**Median**	**St. Dev.**	**Illiq**	**IlliqTurn**	**Zero**	**EffCost**	**Turnover**	**IlliqFactor**	**Loadings**
Illiq	0.2332	0.2125	0.1127	1.0000	0.4506	0.0078	0.4501	-0.3585	0.5315	0.3139
				(0.0000)	(0.0000)	(0.5887)	(0.0000)	(0.0000)	(0.0000)	
IlliqTurn	10.9622	11.5117	5.1879	0.4506	1.0000	0.7370	0.4320	-0.7829	0.9375	0.5537
				(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	
ZeroReturns	0.1164	0.0957	0.0879	0.0078	0.7370	1.0000	0.2091	-0.7517	0.7896	0.4663
				(0.5887)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	
EffCost	0.0055	0.0055	0.0013	0.4501	0.4320	0.2091	1.0000	-0.2423	0.5482	0.3237
				(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	
Turnover	0.0047	0.0039	0.0023	-0.3585	-0.7829	-0.7517	-0.2423	1.0000	-0.8842	-0.5222
				(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	
IlliqFactor	0.0000	0.5759	1.6933	0.5315	0.9375	0.7896	0.5482	-0.8842	1.0000	1.0000
				(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	(0.0000)	

Note: This table presents summary statistics for five liquidity measures: Illiq, IlliqTurn, ZeroReturns, EffCost and Turnover. The definition of each liquidity measure can be found in Appendix C. The period is January 1989 to December 2007. Averages, medians, standard deviations and correlations are computed using daily data. The measure IlliqFactor is the first principal component of all measures.