Asset Pricing with Matrix Affine Jump Diffusions*

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ABSTRACT

This paper introduces a new class of matrix-valued affine jump diffusions that are convenient for modeling multivariate risk factors in many financial and econometric problems. We provide an analytical transform analysis for this class of models, leading to an analytical treatment of a broad class of multivariate valuation and econometric problems. Examples of potential applications include fixed-income problems with stochastically correlated risk factors and default intensities, multivariate option pricing with general volatility and correlation leverage structures, and dynamic portfolio choice with jumps in returns, volatilities or correlations.

JEL classification: D51, E43, G13, G12

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Introduction

In this paper, we introduce a new analytically tractable, yet flexible, multivariate framework, which allows stochastic volatilities, stochastic correlations, and jumps to be consistently modeled by means of a matrix-valued affine jump diffusion (AJD) process. With this approach, we propose a new way to conveniently model many of the most salient features of financial data in a multi-dimensional setting. Since we specify the model by means of a matrix-valued process that is affine, we are able to retain a high degree of analytical tractability. This tractability is very useful for studying various important financial problems with a unifying methodology, such as option pricing, term structure modeling, and portfolio allocation.

We start with the specification of a new class of matrix AJD processes convenient for our purposes. This class includes the Wishart pure diffusion process of Bru (1991) and the pure-jump matrix Ornstein-Uhlenbeck subordinator in Barndorff-Nielsen and Stelzer (2007). We then provide analytical transform analysis for this class of models, which allows us to study in a tractable way a large class of new multivariate asset pricing models, in which stochastic volatilities, stochastic correlations and stochastic intensities can arise together with discontinuous price processes, as well as discontinuous multivariate second moments and leverage effects.

From a methodological point of view, our approach extends the transform analysis of AJD processes in Duffie, Pan, and Singleton (2000) from a state space $\mathbb{R}^{m+} \times \mathbb{R}^{n-m}$ to another one including the convex cone of symmetric positive definite matrices. This approach has several advantages, in that it can naturally specify covariance matrix processes for multivariate asset pricing and at the same time it allows the formulation of multifactor models with a rich conditional dependence structure between factors. Examples of natural application fields of our framework cover are fixed-income problems with stochastically correlated risk factors and default intensities, multivariate option pricing with general volatility and correlation leverage structures, dynamic portfolio choice with jumps in returns, volatilities or correlations, the pricing of credit derivatives on a basket of defaultable assets and, more generally, multi-asset option pricing with quanto, rainbow, basket and spread based pay-offs. We illustrate our matrix AJD approach by deriving explicit asset pricing implications for three concrete asset pricing applications to multifactor option pricing, term structure modeling and dynamic multivariate portfolio choice.

A large part of the early literature in finance has adopted the assumption that prices are driven by diffusion processes, mostly geometric Brownian motion. However, the behavior of many financial time series differs significantly from what we would expect from such an assumption. Two main adjustments have been made in the literature to account for these deviations. The first stream of literature introduces the concept of stochastic volatility, by modeling volatility with a separate diffusion process. These models are of great analytical convenience and have been applied in many areas, such as derivatives pricing, where they have been shown to be able to account for part of the cross-sectional and time-series properties of implied volatilities. Early contributions include Hull and White (1987) and Heston (1993), among many others.

However, there is considerable discussion in the literature about whether diffusive stochastic volatility can be consistent with the extreme movements sometimes observed in financial time series, or with the cross-sectional smile features of some option markets. Therefore, a second stream of literature has introduced processes with discontinuous sample paths, such as jump diffusions. Jump diffusions share the intuitive appeal of pure diffusion models, because they let prices change smoothly most of the time, but at the same time they allow for larger infrequent jumps that might be difficult to explain with a diffusion model. Jump diffusion models have a long and rich history in financial economics dating back at least to Merton (1976). They often come at the expense of analytical tractability.

Convincing empirical evidence for jumps in interest rates and asset prices was given as early as in Ball and Torous (1985) and Jorion (1988), among many others, making jumps an essential modeling device for many modern asset pricing models. The more recent literature has extended these early works by providing models with stochastic volatility and jumps in returns or in returns and volatility. Andersen, Benzoni, and Lund (2002) conclude that a reasonable continuous-time model for equity index returns must include both stochastic volatility as well as jumps in the return process. Further support for their conclusion using option data is provided in Bakshi, Cao, and Chen (1997), Bates (2000), Pan (2002), and Broadie, Chernov, and Johannes (2007), among others. Finally, Chernov, Gallant, Ghysels, and Tauchen (2003), Eraker, Johannes, and Polson (2003), Eraker (2004) find evidence that both price and volatility dynamics exhibit discontinuities in their time series behavior and that these models fare better in fitting options and returns data simultaneously.[1] Jones (2003) shows that the volatilities implied by the square-root volatility process are too smooth to be reconciled with the empirical evidence. These finding are confirmed also by Duan and Yeh (2007), who use data from the VIX volatility index, instead of option implied

[1] An stylized model with jumps in both asset prices and volatilities has been already presented in Naik (1993). In this model, the volatility switches between different discrete states, while the stock price follows a jump diffusion process.

volatilities, to estimate different stochastic volatility models with jumps in the price process only. They show that the Heston (1993) square-root stochastic volatility process performs poorly, even when jumps in the price process are allowed. In addition to providing a good description of the time series properties of returns and volatility, univariate jump diffusion processes have been successfully proposed to explain the cross-sectional patterns of the smile observed in some option markets. Particularly, their ability to generate steep negative skews at short maturities have been exploited convincingly in explaining the large asymmetric skew of index options implied volatilities at short maturities; see, e.g., Duffie, Pan, and Singleton (2000) and Carr, Geman, Madan, and Yor (2003), among others.

The usefulness of models with stochastic volatility and discontinuous trajectories for many applications is well supported by the empirical evidence. At the same time, multivariate or multi-factor models are key for many areas in finance, such as option pricing, term structure modeling and dynamic portfolio choice, to mention just a few. As noted, in the option pricing literature it is well documented that single-factor stochastic volatility models with jumps can generate smiles and smirks; see, e.g., Duffie, Pan, and Singleton (2000). However, many single-factor models implicitly impose quite a restrictive relationship between volatility and the level and slope of the smile. In particular, they fail to capture salient features of option prices in some markets, such as the variability of the skew of the smile; see Carr and Wu (2006) for recent evidence on the topic in currency option markets.[2] Such a behavior of volatility smiles is observed also for index and single stock options, even if these markets highlight quite different pricing patters, with index option smiles that are typically much steeper than those of single stock options; see, e.g., Bakshi and Madan (2000). Some of these tensions can be potentially addressed by means of multi-factor stochastic volatility models. Skiadopoulos, Hodges, and Clewlow (2000) find that the level and the variation of option implied volatilities are well explained by the first two principal components of the option implied volatility surface. Egloff, Leippold, and Wu (2007) show that at least two factors are needed to explain the term structure of variance swaps on the S&P500 index. Related evidence has been produced by Christoffersen, Heston, and Jacobs (2007), who propose an extended Heston model with two independent factors to generate a stochastic leverage. Even if the stochastic leverage structure implied by their setting is restricted by two artificial boundaries, their empirical findings show that two-factor stochastic volatility models improve substantially on single-factor models in explaining the cross-sectional and time series patterns of index option implied volatilities. Given

[2] Implied volatility data also suggest that, in addition to the slope, the curvature of the smile is also stochastic, however to a much lesser extent.

that the differential pricing of index and single stock options is related to a correlation risk premium for a stochastic correlation component among stocks, as shown recently by Driessen, Manhout, and Vilkov (2008), multi-factor option pricing models with stochastically correlated factors and a positive jump intensity offer a natural setting for studying more consistently the pricing of these derivatives. In some simple pricing exercises, we illustrate the extent to which models with factors following a matrix AJD process can improve the pricing performance of standard AJD option pricing models.

Multi-factor diffusion models with stochastically correlated factors and a positive jump intensity also offer a potentially convenient framework for term structure modeling. Standard affine term structure models typically impose restrictive assumptions on the factor dependence structure, in order to guarantee admissibility and econometric identification of the latent state variables. Dai and Singleton (2000), for instance, emphasize the implied trade off between factors' dependence and their stochastic volatilities, as well as the arising drawbacks for explaining the empirical yield curve regularities. In order to match the physical dynamics of the yield curve, these models have introduced progressively richer specifications of the market price of risk, starting from the early "completely affine" models[3], to the "essentially affine" extension class in Duffee (2002), and concluding with the most general "extended affine" specification in Cheridito, Filipovic, and Kimmel (2007), which allows the market price of risk of all factors to be inversely proportional to their volatilities. Recently, Buraschi, Cieslak, and Trojani (2007) developed a completely affine yield curve model in which factors follow a matrix affine diffusion process that grants a new flexibility in the simultaneous modeling of stochastic volatilities and correlations of factors. In their empirical analysis, this modeling approach is shown to provide a unified and parsimonious answer to several empirical term structure regularities, such as the deviations from the expectation hypothesis, the persistence of yield volatilities, and the humped term structure of cap implied volatilities. The inclusion of a jump component in this setting can prove useful in several other dimensions, also because it can generate naturally an incomplete bond market. Ball and Torous (1999) and Chen and Scott (2001), among others, find that innovations in interest rate levels are largely uncorrelated with innovations in the volatility of interest rates. In a related vein, Heidari and Wu (2003) document that interest rate factors explaining most of the variation in the yield curve, can explain only little of the variation in swaption implied volatilities: A large portion of the variation of option prices seems to be driven by factors that are uncorrelated with those driving the underlying swap

[3] These models have been systematically characterized by Dai and Singleton (2000).

rates. Collin-Dufresne and Goldstein (2002) regress changes in straddle prices of caps and floors on changes in swap rates and find very modest R^2-values, which are are in sharp contrast with the high R^2 of 90% they obtain when using an estimated affine diffusion model. This is the so-called unspanned volatility, or incomplete bond market, phenomenon, which has been further investigated more recently in Han (2007) and Joslin (2007), among others. Affine models with diffusion factors are able to generate incomplete markets by means of parameter restrictions that typically further restrict the model flexibility. Affine jump diffusion models can generate a more general form of incompleteness, due to unspanned jump risk. Wright and Zhou (2007) note explicitly that a potentially important source of unspanned volatility may be the presence of jump risk in bond returns. They additionally show empirically that a good fraction of the predictability of bond returns can be due to a time varying jump size and jump intensity. We calibrate a simple completely affine yield curve model in which factors follow a matrix AJD process and we show that such a setting is also able to account for these further stylized facts of bond returns.

A sufficiently flexible yet tractable model for returns and their stochastic conditional second moments is potentially very important for applications to dynamic portfolio choice. For instance, Ball and Torous (2000) examine the correlation process of a number of international stock market indices and find an estimated correlation structure that is changing dynamically over time. They argue that the stochastic nature of the inter-relation between these markets may follow from different responses to shifts in government policy and other fundamental economic changes, and conclude that ignoring the stochastic component of correlation can lead to erroneous portfolio allocations and risk management. Buraschi, Porchia, and Trojani (2007) extend the seminal work of Merton (1971) and propose a general multivariate portfolio choice model with stochastic volatilities and stochastic correlations driven by a matrix diffusion process. They derive in closed form the hedging portfolio against volatility and correlation risk, estimate the model in some real data applications, and find that the hedging demand in the multivariate setting can be be substantially larger than the volatility hedging demand implied by univariate models. Matrix AJD processes usefully extend this multivariate portfolio choice setting by accounting for the empirical evidence of a non smooth dynamics of returns and second moments, which cannot be easily neglected in some concrete applications. For instance, Huang and Tauchen (2005) find strong evidence for the presence of jumps in S&P Index data, even using a test with likely low power, which account for about 4.5 to 7.0 percent of the total daily variance of the S&P Index, cash or futures. However, they leave open the question about the economic significance of the contribution of jumps to the total daily variance of returns. An interesting answer along this dimension is provided by Das and Uppal (2004), who study a multivariate jump diffusion model with a diffusion part that follows a geometric Brownian motion and a jump part driven by a common Poisson process with constant intensity. They show that if jumps are neglected when selecting the optimal portfolio, the potential for conditional diversification is substantially overstated. Our matrix AJD approach allows us to go beyond these portfolio choice settings, and to study further aspects relevant for dynamic portfolio choice. We study the intertemporal hedging demand against stochastic variance covariance risk in a matrix AJD setting in which second moments of returns have a jump component driven by a stochastic intensity. In this setting, jumps in second moments increase the sensitivity of the marginal utility of wealth to variance covariance shocks, leading to a larger and economically significant hedging demand against variance covariance risk than in the pure diffusion model of Buraschi, Porchia, and Trojani (2007).

The extension of stochastic volatility models to a multivariate setting poses significant challenges, mainly because a covariance matrix processes has so satisfy well-known properties like symmetry and positive definiteness. Bru (1991) has proposed the Wishart matrix diffusion process as a convenient process of symmetric positive definite matrices. Following this work, Gourieroux and Sufana (2003) and Gourieroux and Sufana (2004) have first used the Wishart diffusion process to model multivariate risk in financial applications. We go one step further in this direction and introduce matrix AJD processes, which include the Wishart diffusion as special case, study their main properties and implications for financial purposes, and consider several application possibilities. In order to introduce a jump component with an affine stochastic intensity, we specify directly the jump part of matrix AJD as a process taking values in the state space of positive definite symmetric matrices. This simple approach is very different from the one followed by Cheng and Scaillet (2007), who specify linear-quadratic jump diffusions for standard state spaces by means of a squared affine jump diffusion with an affine intensity, and avoids some of the restrictions necessary in their setting to preserve both positivity of the intensity and the affine structure of the whole process.

The plan of the paper is as follows. Section I introduces the matrix-valued affine jump diffusion process. In Section II, we focus on derivative pricing and we additionally include jumps in the return process. We specify our model both in a multi-factor setting as well as for multivariate return processes and we derive tractable solutions for option pricing. Section III proposes a term structure model based on our AJD process. We investigate to what extent such a term structure model can account for unspanned volatility. Section IV derives the optimal portfolio allocation in the presence of jumps in the covariance process and we discuss the potential impact of these jumps on the intertemporal hedging demand. Section V concludes.

I. Transform Analysis of AJD State Matrices

In this section, we introduce our matrix valued state-process and the general solutions to the different transforms needed to solve various asset pricing problems. We also derive closed-form transform formulas for all cases in which the jump intensity of the state process is constant.

A. Matrix Affine Jump-Diffusions

We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and a filtration $\{\mathcal{F}_t\}$ satisfying the usual conditions. We suppose that X is a Markov process with respect to $\{\mathcal{F}_t\}$, taking values in some state space $D \subset \mathcal{S}_n^+$, with $\mathcal{S}_n^+$ the positive cone of symmetric positive semi definite $n \times n$ matrices. We also denote by $\mathcal{S}_n^{++}$ the strictly positive cones of positive definite matrices. e_i, $i = 1, \ldots, n$, denotes the $i - th$ unit vector in $\mathbb{R}^n$. In comparison to the standard affine literature, the main distinction of our approach lies in the choice of the state space D. Instead of working with a subset of $\mathbb{R}^{+m} \times \mathbb{R}^{n-m}$, where $n \geq m$, we use the cone of symmetric positive semi definite matrices. As we show below, this approach has several convenient features for modeling multivariate sources of diffusive and/or jump risk in finance.

Assumption 1 *The Markov process X solves the stochastic differential equation:*

$$dX_t = \left(\Omega\Omega' + MX_t + X_t M'\right) dt + \sqrt{X_t} dB_t Q + Q' dB_t' \sqrt{X_t} + dJ_t \ , \ X_0 = x \in \mathcal{S}_n^+ \tag{1}$$

where $\Omega, M, Q \in \mathbb{R}^{n\times n}$, B is a matrix of standard Brownian motions in $\mathbb{R}^{n\times n}$, and J is a pure jump process taking values in $\mathcal{S}_n^+$. Jump sizes ξ^X are IID and follow a finite probability distribution ν^X on $\mathcal{S}_n^+$. Jumps are realized with an intensity $\{\lambda_X(X_t) : t \geq 0\}$, where the function $\lambda_X : D \to \mathbb{R}^+$ is affine,

$$\lambda_X(x) = \lambda_{X,0} + tr(\lambda_{X,1} x) \ , \tag{2}$$

with $\lambda_{X,0} \geq 0$ and $\lambda_{X,1} \in \mathcal{S}_n^+$. Finally, we denote by Θ^X the Laplace transform of the jump size

ξ^X:

$$\Theta^X(\Gamma) = \int_{\mathcal{S}_n^+} \exp(tr(\Gamma x))\nu^X(dx) \tag{3}$$

for $\Gamma \in \mathcal{S}_n^+$.

The restriction that $\Omega\Omega' \gg Q'Q$ guarantees that X_t is positive definite. The positivity of $\lambda_X(X_t)$ then follows directly, since $\lambda_{X,0} \geq 0$ and both matrices $\lambda_{X,1}$ and X_t are symmetric positive semi definite. In the case where the jump intensity is zero, we obtain the (pure diffusion) Wishart process introduced by Bru (1991) and studied in Gouriéroux and Sufana (2004). For $\Omega\Omega' = kQ'Q$, $k > n-1$, the transition density of this process is Wishart (see, e.g., Muirhead (1982), p. 44). If $\Omega\Omega'$ and Q are both matrices of zeros, then X is a pure jump process in the class of Ornstein-Uhlenbeck matrix subordinators analyzed by Barndorff-Nielsen and Stelzer (2007). The closed form solution for this process is given by

$$X_t = \exp(tM)X_0\exp(tM') + \int_0^t \exp((t-s)M)dJ_s\exp((t-s)M') \quad . \tag{4}$$

Note that there are many candidate distributions for the jump sizes of J, which we specify as a pure jump process taking values in $\mathcal{S}_n^+$. These distributions include e.g., the Wishart, Inverse Wishart, and matrix-variate Gamma, Beta, Dirichlet, Liouville, and confluent Hypergeometric distributions of kind 1 and 2.

Under regularity conditions, the Lévy infinitesimal generator $\mathcal{L}_X$ of the matrix Markov process X is defined for bounded C^2 functions $f : D \to \mathbb{R}$ by:

$$\mathcal{L}_X f(x) = tr[(\Omega\Omega' + Mx + xM')\mathcal{D} + 2x\mathcal{D}Q'Q\mathcal{D}]f(x) + \lambda_X(x)\int_{\mathcal{S}_n^+}(f(x+z) - f(x))d\nu^X(z), \tag{5}$$

where $\mathcal{D}$ is a $n \times n$ matrix of differential operators with ij-component given by $\frac{\partial}{\partial x_{ij}}$. The affine dependence of this generator on state $x \in \mathcal{S}_n^+$ emphasizes the fact that the process defined by (1) is an AJD with state space $\mathcal{S}_n^+$. This feature implies that many transforms used to solve several important problems in asset pricing are computable explicitly as exponentially affine functions of the current state of process X.

B. Correlation, Volatility and Intensity Processes

Matrix AJD are convenient to specify the asset returns process of many asset pricing models. Due to their positive semi definite state space, they are the natural processes to model stochastic covariance matrices having potentially discontinuous trajectories, with jumps that are realized according to a stochastic intensity. Moreover, matrix AJD are useful to specify flexible multivariate risk structures, in which both positive and negative factors can feature stochastic co-volatility and co-jumps. In this section, we illustrate these properties focusing on the implied correlation processes $\rho_{ij} := X_{ij}/\sqrt{X_{ii}X_{jj}}$, $1 \leq i \leq j \leq n$.

Proposition 1 *The dynamics of the $ij-th$ correlation coefficient implied by the positive semi definite matrix AJD process X in Assumption 1 is given by:*

$$\begin{aligned} d\rho_{ijt} &= m(\rho_{ijt})dt + \frac{e_i'\sqrt{X_t}dB_tQ_j + e_j'\sqrt{X_t}dB_tQ_i}{\sqrt{X_{iit}X_{jjt}}} - \rho_{ijt}\left(\frac{e_i'\sqrt{X_t}dB_tQ_i}{X_{iit}} + \frac{e_j'\sqrt{X_t}dB_tQ_j}{X_{jjt}}\right) \\ &\quad +\rho_{ijt}\zeta_{ij}^X dJ_t^\rho , \end{aligned}$$

where Q_i and Q_j are the $i-th$ and $j-th$ column of matrix Q, respectively, and J^ρ is a pure jump process with intensity $\lambda_X(X_t)$ and IID (percentage) correlation jump size given by:

$$\zeta_{ij}^X = \frac{1 + \frac{\xi_{ij}^X}{X_{ijt}}}{\sqrt{\left(1 + \frac{\xi_{ii}^X}{X_{iit}}\right)\left(1 + \frac{\xi_{jj}^X}{X_{jjt}}\right)}} - 1 \ . \tag{6}$$

$m(\rho_{ijt}) = A_{ijt}\rho_{ijt}^2 + B_{ijt}\rho_{ijt} + C_{ijt}$ is a quadratic drift function with stochastic coefficients, given explicitly in Appendix A, which depend only on X_{kk} and ρ_{kl}, $1 \leq k, l \leq n$. The instantaneous conditional variance of the correlation process has the form:

$$\frac{1}{dt}\mathbb{E}_t\left(d\rho_{ijt}^2\right) = v^2(\rho_{ijt}) + \lambda_X(X_t)\mathbb{E}\left({\zeta_{ij}^X}^2\right), \tag{7}$$

where

$$v^2(\rho_{ijt}) = (\rho_{ijt}^2 - 1)\left(2\rho_{ijt}\frac{Q_i'Q_j}{\sqrt{X_{iit}X_{jjt}}} - \frac{Q_i'Q_i}{X_{iit}} - \frac{Q_i'Q_j}{X_{jjt}}\right) \ . \tag{8}$$

The correlation drift $m(\rho_{ijt})$ is a quadratic function of ρ_{ijt} with stochastic coefficients A_{ijt}, B_{ijt} and C_{ijt}. From the explicit expressions in Appendix A, the coefficient A_{ijt} is a function only of the conditional variances X_{iit} and X_{jjt}, but coefficients B_{ijt} and C_{ijt} depend on all conditional variances X_{kk}, $1 \leq k \leq n$, and all other correlations ρ_{ik}, ρ_{jk}, $k \neq i, j$. The conditional variance of $d\rho_{ijt}$ is the sum of $v(\rho_{ijt})$ and $\lambda(X_t)\mathbb{E}[(\zeta_{ij}^X)^2]$. The first term, which is the contribution of the continuous part of the correlation process to the conditional variance, depends only on the correlation ρ_{ijt} and the corresponding variance terms X_{iit} and X_{jjt}. The contribution of the discontinuous part of the process to the conditional variance of the correlation depends, instead, on all correlations and variances ρ_{ijt}, X_{iit} and X_{jjt}, where $1 \leq i, j \leq n$, via the stochastic intensity $\lambda(X_t)$. This last feature generates potentially very rich dynamics in conditional second moments, which are not generally mappable into a low–dimensional factor structure.

To investigate the dynamic properties of our matrix AJD setting, we simulate a 3×3 matrix AJD and illustrate the resulting co-volatility, co-jump and intensity dynamics. The parameters of the process are arbitrarily chosen and listed in Panel A of Table III. Figure 2 presents simulated trajectories for volatilities $\sqrt{X_{11}}$, $\sqrt{X_{22}}$, $\sqrt{X_{33}}$ and correlations ρ_{12}, ρ_{13}, ρ_{23} in the 3×3 matrix AJD model. All volatilities and correlations feature rich mean reversion features, stochastic second moments and volatility/correlation clustering patterns.

[Figure 1 about here.]

Figure 2, top panel, illustrates the correlation co-jump dynamics.

[Figure 2 about here.]

Correlations can feature both positive and negative jumps, with a large variation in jumps sizes, which can range range between -0.15 to approximately 0.33. Jumps tend to cluster in dependence of the value of the intensity $\lambda(X_t)$, plotted in the bottom panel of Figure 2. Realized intensities vary stochastically over time between approximately 0.15% and 0.7% and feature clustering patterns, with high (low) jump intensities that tend to be followed by high (low) future jump intensities.

C. Exponentially Affine Transforms

We now analyze the Laplace transform of the matrix AJD process. To this end, let $\Psi^X(\Gamma, X_t, t, T)$ denote the discounted Laplace transform of X_T:

$$\Psi^X(\Gamma, X_t, t, T) = \mathbb{E}\left[\exp(-\int_t^T R(X_s)ds + tr(\Gamma X_T))|\mathcal{F}_t\right] , \tag{9}$$

where $\Gamma \in \mathcal{S}_n^+$ and $\{R(X_t) : t \geq 0\}$ is the short interest rate process. This transform is exponentially affine if the short rate process $R(X_t)$ is affine.

Assumption 2 *The short rate process is affine: $R(x) = \rho_0 + tr(\rho_1 x)$, where $\rho_0 \geq 0$ and $\rho_1 \in \mathcal{S}_n^+$.*

Assumption 2 directly implies a positive short rate, which is convenient to ensure existence of a well-defined transform Ψ^X. Given Assumptions 1 and 2, the discounted Laplace transform of X_T is exponentially affine, with coefficients that are obtained by solving a corresponding system of matrix Riccati differential equations.

Proposition 2 *Let Assumptions 1, 2 and additional regularity conditions be satisfied. Then, the discounted Laplace transform (9) is exponentially affine:*

$$\Psi^X(\Gamma, X_t, t, T) = \exp\left(B(T-t) + tr(A(T-t)X_t)\right) \tag{10}$$

with functions $B(u) \in \mathbb{R}$ and $A(u) \in \mathcal{S}_n^+$ that solve the system of matrix Riccati equations:

$$\begin{aligned} \frac{dA(\tau)}{d\tau} &= -\rho_1 + M'A(\tau) + A(\tau)M + 2A(\tau)Q'QA(\tau) + \lambda_{X,1}\left[\Theta^X(A(\tau)) - 1\right], &(11)\\ \frac{dB(\tau)}{d\tau} &= -\rho_0 + tr\left(A(\tau)\Omega\Omega'\right) + \lambda_{X,0}\left[\Theta^X(A(\tau)) - 1\right], &(12) \end{aligned}$$

subject to terminal conditions $B(0) = 0$ and $A(0) = \Gamma$.

Despite the multivariate state space structure, closed-form solutions for functions $A(u)$ and $B(u)$ in Proposition 2 are available in the case where the intensity $\lambda_X(X_t)$ is constant ($\lambda_{X,1} = 0$). In models with stochastic intensity, accurate asymptotic approximations can be developed starting from the closed form solution for the case $\lambda_{X,1} = 0$. This analytical approach circumvents the need for numerical solutions, which are unavoidable in many standard multivariate affine models with state space $\mathbb{R}^{+m} \times \mathbb{R}^{n-m}$, $n \geq m$. The closed form transform solution for the constant intensity case is given next.

Corollary 1 *Let Assumptions 1, 2 and additional regularity conditions be satisfied. Assume further that $\lambda_{X,1} = 0$. Then, the closed form expressions for function $A(\tau)$ in Proposition 2 is as follows:*

$$A(\tau) = C_{22}(\tau)^{-1}C_{21}(\tau) \ , \tag{13}$$

where $C_{12}(\tau)$ and $C_{22}(\tau)$ are $n \times n$ blocks of the following matrix exponential:

$$\begin{pmatrix} C_{11}(\tau) & C_{12}(\tau) \\ C_{21}(\tau) & C_{22}(\tau) \end{pmatrix} := \exp\left[\tau \begin{pmatrix} M & -2Q'Q \\ -\rho_1 & -M' \end{pmatrix}\right]. \tag{14}$$

Given the solution for $A(\tau)$, the coefficient $B(\tau)$ follows by direct integration:

$$B(\tau) = -\rho_0\tau - \frac{k}{2} tr[\ln C_{22}(\tau) + \tau M'] + \lambda_{X,0}\left[\int_0^\tau \Theta^X(A(s))ds - \tau\right] \tag{15}$$

D. Transform Inversion Formula

Given the solutions in Corollary 1, we can efficiently perform the transform analysis of matrix AJD models and apply it to several multivariate financial and econometric problems, in which state variables are modeled by a matrix AJD process. For instance, for option pricing purposes we can consider for any $A, B \in \mathcal{S}_n$ the following transform, which is the matrix AJD version of the transform used in Duffie, Pan, and Singleton (2000) to price several types of European options in affine models with state space $D = \mathbb{R}^{+m} \times \mathbb{R}^{n-m}$:

$$G_{A,B}(y; X_0, T) := \mathbb{E}\left[\exp[-\int_0^T R(X_s)ds] \exp(tr(AX_T))\mathbb{I}_{tr(BX_t)\leq y}\right] \tag{16}$$

where $\mathbb{I}_{\mathcal{C}}$ is the indicator function of event $\mathcal{C}$, $y \in \mathbb{R}$, and $A, B \in \mathcal{S}_n^+$. The Fourier-Stieltjes transform of $G_{A,B}$, if well defined, is given by:

$$\begin{aligned} \mathcal{G}_{A,B}(v; X_0, T) &= \int_{\mathbb{R}} \exp(ivy) dG_{A,B}(y; X_0, T) \\ &= \mathbb{E}\left[\exp[-\int_0^T R(X_s)ds] \exp[tr((A + ivB))X_t]\right] \\ &= \Psi^X(A + ivB, X_0, 0, T) \end{aligned} \tag{17}$$

where $v \in \mathbb{R}$ and $i = \sqrt{-1}$. The inversion formula for $G_{A,B}(y, X_0, T)$ then immediately follows (see Duffie, Pan, and Singleton (2000)).

Corollary 2 *Under regularity conditions, the transform in equation (16) is given by:*

$$G_{A,B}(y; X_0, T) = \frac{\Psi^X(A, X_0, 0, T)}{2} - \frac{1}{\pi}\int_0^\infty \frac{Im\left[\Psi^X(A + ivB, X_0, 0, T)\exp(-ivy)\right]}{v} dv \tag{18}$$

where $Im(c)$ is the imaginary part of $c \in \mathbb{C}$.

Formula (18) and Proposition 2 allow us to extend the range of known transform solutions in Duffie, Pan, and Singleton (2000) to general matrix AJD processes of the type (1). For $R(X_t) = 0$ and $A = 0$, formula (18) gives us the conditional probability distribution of $tr(BX_T)$ given X_0. The corresponding density of $tr(BX_T)$ follows by differentiation of $G_{A,B}(\cdot; X_0, T)$.

E. Stochastic Discount Factor and Risk Neutral Pricing

Let Assumption 1 be satisfied and $\mathbb{P}$ be the physical probability measure. An exponentially affine stochastic discount factor is a process $\xi = \{\xi_t : 0 \leq t \leq T\}$ defined by

$$\xi_t = \exp\left(-\int_0^t R(X_s)ds\right)\exp(\alpha(t,T) + tr(\beta(t,T)X_t)) \ , \tag{19}$$

such that $\{\xi_t \exp\left(\int_0^t R(X_s)ds\right) : 0 \leq t \leq T\}$ is a martingale process under $\mathbb{P}$, for given continuous functions $\alpha(\cdot, T) : [0,T] \to \mathbb{R}$ and $\beta(\cdot, T) : [0,T] \to \mathcal{S}_n^+$ with $\alpha(0,T) = 0$ and $\beta(0,T) = 0$. It follows that we can define an equivalent martingale measure $\mathbb{P}^*$ by the density $\frac{d\mathbb{P}^*}{d\mathbb{P}} := \xi_T \exp(\int_0^T R(X_s)ds)$. Therefore, the arbitrage-free time$-t$ price V_t of any $T-$measurable contingent claim $V_T := V(X_T, T)$ satisfies the risk neutral valuation formula:

$$V_t = \frac{1}{\xi_t}\mathbb{E}_t\left[\xi_T V_T\right] = \mathbb{E}_t^*\left[\exp\left(-\int_t^T R(X_s)ds\right)V_T\right] \tag{20}$$

where $\mathbb{E}_t^*[\cdot]$ denotes conditional expectation with respect to probability $\mathbb{P}^*$. Under these conditions, X is a matrix AJD, but with time-dependent coefficients, also with respect to the risk neutral probability.

Proposition 3 *The dynamics of matrix AJD process X in Assumption 1 with respect to the risk neutral probability $\mathbb{P}^*$ takes the form:*

$$dX_t = (\Omega\Omega' + M^*(t)X_t + X_t M^{*\prime}(t))dt + \sqrt{X_t}dB_t^* Q + Q' dB_t^{*\prime}\sqrt{X_t} + dJ_t \tag{21}$$

where B^* is a $n \times n$ standard Brownian motion and J is a pure jump process with value in $\mathcal{S}_n^+$ and having independent jump sizes ξ_t^X with distribution $\nu^{X*}(t)$ and affine intensity $\lambda_X^*(X_t, t) = \lambda_{X,0}^*(t) + tr(\lambda_{X,1}^*(t)X_t)$. The parameters of the $X-$dynamics (21) under risk neutral probability $\mathbb{P}^*$ are $M^*(t) = M + 2Q'Q\beta(t,T)$, $\lambda_{X,0}^*(t) = \lambda_{X,0}\Theta^X(\beta(t,T))$, $\lambda_{X,1}^*(t) = \lambda_{X,1}\Theta^X(\beta(t,T))$ and $\Theta^{X*}(\Gamma, t) = \Theta^X(\Gamma + \beta(t,T))/\Theta^X(\beta(t,T))$.

The main implication of Proposition 5 is that under the risk neutral probability $\mathbb{P}^*$, implied by the exponentially affine density $\frac{d\mathbb{P}^*}{d\mathbb{P}} = \exp(\int_0^T R(X_s)ds)\xi_T$, the discounted Laplace transform (9) is again exponentially affine. The coefficients in the exponential of this transform satisfy the same system of matrix Riccati differential equations as in Proposition 2, but with (time dependent) parameters $M^*(t)$, $\lambda^*_{X,0}(t)$, $\lambda^*_{X,1}(t)$ and $\Theta^{X*}(\Gamma, t)$. Hence, with the exponentially affine stochastic discount factor (38), the matrix AJD structure of process X is preserved both under the physical and the risk neutral probabilities.

II. Derivatives Pricing

Our approach extends in a natural way the single asset transform analysis in Duffie, Pan and Singleton (2000) from the classical affine state space $D \subset \mathbb{R}^{+m} \times \mathbb{R}^{n-m}$ to settings in which state variables can be driven by matrix AJD. We first present the results for the single asset case and then we move on to the multi-asset case, for which the advantage of our modeling approach becomes particularly obvious.

A. *Multi Factor Double-Jump Option Pricing Models*

A convenient feature of our matrix AJD model is that it can easily model flexible factor co-volatilities together with a double-jump structure in assets returns and their multi factor volatility. To this end, we need to specify a joint jump diffusion process for some asset return and the AJD state process X.

A.1. Double-Jump Matrix AJD Process for Asset Returns

Denote by $Y_t = \log S_t$ the log return of asset S. To specify a joint AJD process for $(Y_t, X_t) \in \mathbb{R} \times \mathcal{S}_n^+$, we first introduce assumptions on the corresponding double-jump structure.

Assumption 3 *(L, J) is a pure jump process with values in $\mathbb{R} \times \mathcal{S}_n^+$. IID jump sizes $(\xi^Y, \xi^X) \in \mathbb{R} \times \mathcal{S}_n^+$ follow a finite joint probability distribution $\nu^{YX} = \nu^{Y|X}\nu^X$ on $\mathbb{R} \times \mathcal{S}_n^+$. Jumps are realized with an affine intensity $\lambda_{YX}(X_t) = \lambda_{YX,0} + tr(\lambda_{YX,1}X_t)$, where $\lambda_{YX,0} \geq 0$ and $\lambda_{YX,1} \in \mathcal{S}_n^+$. The Laplace transform Θ^{YX} of the jump size is given by:*

$$\Theta^{YX}(\gamma, \Gamma) = \int_{\mathcal{S}_n^+} \left(\int_{\mathbb{R}} \exp(\gamma y)\nu^{Y|X}(dy) \right) \exp(tr(\Gamma x))\nu^X(dx). \tag{22}$$

We use the notation $\Theta^Y(\gamma) = \Theta^{YX}(\gamma, 0)$ to denote the Laplace transform of jump sizes ξ^Y.

This jump process specification allows us to model both correlated or independent jumps between returns and volatility, in dependence of the choice of the joint jump size distributions. The affine form of $\lambda_{YX}(X_t)$ is necessary to preserve the affine form of the joint jump process for returns and (multi factor) volatility.

Remark 1 *A convenient assumption in the above model for the conditional distribution of ξ^Y given ξ^X is a normal distribution:*

$$\nu^{Y|X} \sim \mathcal{N}(\mu_Y + tr(\beta_Y \xi^X), \sigma_Y^2) \tag{23}$$

for parameters $\mu_Y \in \mathbb{R}$, $\beta_Y \in \mathcal{S}_n$ and $\sigma_Y^2 \geq 0$. The marginal distribution ν^X of jump size ξ^X can then be taken to be one among the available tractable probability distributions on $\mathcal{S}_n^+$.[4]

Given the double-jump structure in Assumption 3, we specify the AJD process for (Y_t, X_t) as follows.

Assumption 4 *The dynamics for the return process Y_t are given by:*

$$dY_t = \left[R(X_t) + \mu_e(X_t) - \frac{1}{2}tr(X_t)\right] dt + tr\left(\sqrt{X_t} dZ_t\right) + dL_t \ , \tag{24}$$

where $\mu_e(X_t) = \mu_{e,0} + tr(\mu_{e,1} X_t)$, $\mu_{e,0} \in \mathbb{R}$ and $\mu_{e,1} \in \mathcal{S}_n$, is an affine function of X_t. In equation (24), Z is a $n \times n$ standard Brownian motion:

$$Z_t = B_t P' + W_t \sqrt{I_n - PP'} \tag{25}$$

where W is another $n \times n$ standard Brownian motion, independent of B, and P is a fixed $n \times n$ matrix of correlation coefficients.

AJD dynamics for (Y_t, X_t) can be specified both under the physical or the risk neutral probability measures. In the latter case, no-arbitrage constraints have to be imposed on the functional form of the excess return process $\mu_e(X_t)$.

[4] E.g., a Wishart distribution with degrees of freedom k_X, noncentrality parameter $M_X \in \mathbb{R}^{n \times n}$ and scale parameter $\Sigma_X \in \mathcal{S}_n^+$.

Remark 2 *If the dynamics of* (Y_t, X_t) *are written with respect to the risk neutral probability measure, absence of arbitrage requires:*[5]

$$\mu_e(X_t) = -\lambda_{XY}(X_t)(\Theta^Y(1) - 1) \tag{27}$$

where the jump intensity λ_{XY} *and the jump size Laplace transform* Θ^Y *are both specified with respect to the risk neutral probability measure. In this case, the affine functional form of* $\mu_e(X_t)$ *is equivalent to the affine functional form of the intensity process* $\lambda_{YX}(X_t)$ *under the risk neutral probability measure.*

A.2. Multifactor Volatility and Stochastic Leverage Properties

A convenient feature of the matrix AJD setting introduced above is that it can model, in a tractable way, a multifactor volatility together with a stochastic leverage. We can exploit this property to approach successfully many open problems in empirical asset pricing. E.g., there is quite consistent empirical evidence for the fact that the implied volatility surface of index options is driven by more than one latent risk factor.[6] Similarly, the skew of the implied volatility smile of some option markets, e.g., exchange rate option markets, is highly stochastic, with a sign that can sometimes even change over time. Leverage is intimately linked to the skewness of asset returns and the slope of the implied volatility smile. Therefore, a model with stochastic leverage and general multifactor volatility is potentially useful to explain the cross-sectional and the time varying patterns of the skew and the term structure of implied volatility in these markets.

The first row in Table I summarizes the volatility structure of multifactor matrix AJD diffusion. In the pure diffusion case ($\lambda_{YX,0} = 0$, $\lambda_{YX,1} = 0$), the conditional variance of Y_t is $V_t = tr(X_t)$, i.e., V_t is the sum of the positive factors in the matrix AJD state X_t. Note that even if the off-diagonal factors of X_t do not impact directly on V_t, they drive the stochastic conditional correlation of the diagonal elements of X_t and, consequently, the dynamics of V_t. In the presence of jumps, V_t is increased by the affine term $\lambda_{YX}(X_t)\mathbb{E}\left[(\xi^Y)^2\right]$. This term is larger when the second moment of

[5]To see this, note that by Itò's Formula:

$$S_t - S_0 = \int_0^t S_u R(X_u)du + \int_0^t S_u tr(X_u dZ_u) + \sum_{0<u\leq t} S_{u-}(\exp(\Delta L_u) - 1) - \int_0^t S_u(\Theta^Y(1) - 1)\lambda_{YX}(X_u)du \tag{26}$$

where $\Delta L_u := L_u - L_{u-}$ denotes the jump of L at time $u > 0$. Since the last three terms on the RHS define a local martingale, we obtain the definition of the risk neutral dynamics for Y.

[6]See, e.g., Christoffersen, Heston, and Jacobs (2007) and Egloff, Leippold, and Wu (2007) for the term structure of variance swaps on the S&P 500.

the return jump size or the intensity $\lambda_{YX}(X_t)$ is larger. The last property arises in states in which X_t is larger as a positive definite matrix.

The second row in Table I summarizes the volatility of volatility structure of multifactor matrix AJD diffusions. In the pure diffusion case ($\lambda_{YX,0} = 0$, $\lambda_{YX,1} = 0$), the volatility of volatility is $\frac{1}{dt}Var_t(dV_t) = 4tr(Q'QX_t)$ and it is completely determined by parameter Q: the larger $Q'Q$ as a positive definite matrix the larger the volatility of volatility. Jumps increase the volatility of volatility in two distinct ways. First, jumps in X with size ξ^X increase the volatility of X, which in turn increases the returns volatility of volatility. This increase is driven by the matrix H, which we define as

$$H := I_n + \lambda_{YX,1}\mathbb{E}\left[(\xi^Y)^2\right].$$

Second, jumps in Y with size ξ^Y increase the sensitivity of the volatility to changes in the matrix AJD state X. E.g., when the intensity is constant, we obtain:

$$\frac{1}{dt}Var_t(dV_t) = 4tr(Q'QX_t) + \lambda_{YX,0}\mathbb{E}\left[tr(\xi^X)^2\right] \tag{28}$$

In this case, the jump size of ξ^Y has no impact on the volatility of volatility, because the return volatility itself is not affected by changes in the intensity of jumps. However, when $\lambda_{YX,1} > 0$ changes in the intensity affect V_t in a way that is proportional to the second moment $\mathbb{E}\left[(\xi^Y)^2\right]$. This feature further increases the volatility of V_t.

The last row in Table I summarizes the leverage structure of multifactor matrix AJD. In the pure diffusion case ($\lambda_{XY,0} = 0$, $\lambda_{XY,1} = 0$), the leverage is $\frac{1}{dt}Cov_t(dY_t, dV_t) = tr(PQX_t)$. Given a volatility of volatility parameterization, matrix P completely determines the leverage between returns and volatility. The multifactor structure of this leverage expression implies a stochastic correlation between return and volatility shocks,

$$Corr_t(dY_t, dV_t) = tr(PQX_t)/\sqrt{tr(X_t)tr(Q'QX_t)}.$$

The introduction of jumps can both increase or decrease the leverage between volatility and returns, depending on the joint second moments of ξ^Y and ξ^X. For instance, in the constant intensity case ($\lambda_{YX,1} = 0$), the sign of the joint second moments $\mathbb{E}[\xi^Y\xi^X_{ii}]$, $i = 1, \ldots, n$, completely determines the direction of the impact of jumps in Y and X on leverage. More generally, when the intensity is not constant, all joint second moments of ξ^Y and ξ^X_{ij}, $1 \leq i, j \leq n$, will impact on this leverage.

In addition, in the AJD case the stochastic leverage can jump itself, when the matrix AJD X has a discontinuity over time, leading to potential discontinuities in stochastic skewness of returns over time.

A.3. Transform Analysis

(Y, X) is a Markov process with values in $\mathbb{R} \times \mathcal{S}_n^+$. The Lévy infinitesimal generator $\mathcal{L}_{YX}$ of (Y, X) is defined for bounded C^2 functions $f : \mathbb{R} \times D \to \mathbb{R}$ by:

$$\begin{aligned}
\mathcal{L}_{Y,X} f(y,x) &= \left[R(x) + \mu_e(x) - \frac{1}{2} tr(x)\right] \frac{\partial f(y,x)}{\partial y} + \frac{1}{2} tr(x) \frac{\partial^2 f(y,x)}{\partial^2 y} \\
&\quad + tr[(\Omega\Omega' + Mx + xM')\mathcal{D} + (\mathcal{D}Q'P'x + xPQ\mathcal{D})\frac{\partial}{\partial y} + 2x\mathcal{D}Q'Q\mathcal{D}] f(y,x) \\
&\quad + \lambda_{YX}(x) \int_{\mathbb{R}\times\mathcal{S}_n^+} (f(y+w, x+z) - f(y,x)) d\nu^{YX}(w,z) \qquad (29)
\end{aligned}$$

where $\mathcal{D}$ is a $n \times n$ matrix of differential operators with ij-component given by $\frac{\partial}{\partial x_{ij}}$. This generator is affine in $x \in \mathcal{S}_n^+$, which implies the exponentially affine form for the Laplace transform of Y_T.

Proposition 4 *Let Assumptions 1–4 and additional regularity conditions be satisfied. Then, the discounted Laplace transform of Y_T has the exponentially affine form:*

$$\begin{aligned}
\Psi^Y(\gamma) &:= \mathbb{E}\left[\exp(-\int_t^T R(X_s) ds + \gamma Y_T)\right] \\
&= \exp(\gamma Y_t) \exp(B(t-T) + tr(A(T-t)X_t)) \qquad (30)
\end{aligned}$$

with functions $B(u) \in \mathbb{R}$ and $A(u) \in \mathcal{S}_n^+$ that solve the system of matrix differential Riccati equations:

$$\begin{aligned}
\frac{\partial A}{\partial \tau} &= (\gamma-1)\rho_1 + \gamma\mu_{e,1} + \frac{\gamma(\gamma-1)}{2} I_n + A(\tau)(M + \gamma Q'P') + (M' + \gamma PQ)A(\tau) + 2A(\tau)Q'QA(\tau) \\
&\quad + \lambda_{YX,1}[\Theta^{YX}(\gamma, A(\tau)) - 1] \qquad (31) \\
\frac{\partial B}{\partial \tau} &= (\gamma-1)\rho_0 + \gamma\mu_{e,0} + tr(\Omega\Omega' A(\tau)) + \lambda_{YX,0}[\Theta^{YX}(\gamma, A(\tau)) - 1] \qquad (32)
\end{aligned}$$

subject to terminal conditions $B(0) = 0$ and $A(0) = 0$.

The structure of the system of matrix Riccati differential equations in Proposition 4 is identical to the system of Matrix Riccati equations in Proposition 2. Therefore, the same solution approach as the one in Corollary 1 applies.

Corollary 3 *Let Assumptions 1–4 and additional regularity conditions be satisfied. Assume further that $\lambda_{YX,1} = 0$. Then, the closed form expressions for function $A(\tau)$ in Proposition 4 is as follows:*

$$A(\tau) = C_{22}(\tau)^{-1}C_{21}(\tau) \ , \tag{33}$$

where $C_{12}(\tau)$ and $C_{22}(\tau)$ are $n \times n$ blocks of the following matrix exponential:

$$\begin{pmatrix} C_{11}(\tau) & C_{12}(\tau) \\ C_{21}(\tau) & C_{22}(\tau) \end{pmatrix} := \exp\left[\tau \begin{pmatrix} M + \gamma Q'P' & -2Q'Q \\ (\gamma-1)\rho_1 + \gamma\mu_{e,1} + \frac{\gamma(\gamma-1)}{2}I_n & -(M' + \gamma PQ) \end{pmatrix}\right] . \tag{34}$$

Given the solution for $A(\tau)$, the coefficient $B(\tau)$ follows by direct integration:

$$\begin{aligned} B(\tau) &= ((\gamma-1)\rho_0 + \gamma\mu_{e,0})\tau - \frac{k}{2}tr[\ln C_{22}(\tau) + \tau(M + \gamma Q'P')] \\ &\quad + \lambda_{YX,0}\left[\int_0^\tau \Theta^{YX}(\gamma, A(s))ds - \tau\right] \end{aligned} \tag{35}$$

Given the discounted Laplace transform expression for Y_t, option prices can be again computed by standard transform methods with the results in Duffie, Pan and Singleton (2000).

Corollary 4 *Define for any $a, b \in \mathbb{R}$ the discounted Laplace transform:*

$$G_{a,b}(y; Y_0, T) := \mathbb{E}\left[\exp[-\int_0^T R(X_s)ds]\exp(aY_T)\mathbb{I}_{bY_t \leq y}\right] \tag{36}$$

It then follows, under regularity conditions:

$$G_{a,b}(y; Y_0, T) = \frac{\Psi^Y(a, Y_0, 0, T)}{2} - \frac{1}{\pi}\int_0^\infty \frac{Im\left[\Psi^Y(a + ivb, Y_0, 0, T)\exp(-ivy)\right]}{v}dv \tag{37}$$

where $Im(c)$ is the imaginary part of $c \in \mathbb{C}$.

With the formula in Corollary 4 and the results in Proposition 4, we can now price any European option on S_T by transform methods, given concrete assumptions on the distribution of the jump size (ξ^Y, ξ^X) as given, e.g., in Remark 1.

A.4. State Price Density and Risk Neutral Pricing

Let Assumption 3 and 4 be satisfied with respect to the physical probability measure. An exponentially affine stochastic discount factor is a process $\xi = \{\xi_t : 0 \leq t \leq T\}$ defined by

$$\xi_t = \exp\left(-\int_0^t R(X_s)ds\right)\exp\left(\alpha(t,T) + tr(\beta(t,T)X_t) + \gamma(t,T)Y_t\right), \tag{38}$$

such that $\{\xi_t \exp\left(\int_0^t R(X_s)ds\right) : 0 \leq t \leq T\}$ is a martingale process under $\mathbb{P}$, for given continuous functions $\alpha(\cdot,T) : [0,T] \to \mathbb{R}$, $\gamma(\cdot,T) : [0,T] \to \mathbb{R}$ and $\beta(\cdot,T) : [0,T] \to \mathcal{S}_n^+$ with $\alpha(0,T) = \gamma(0,T) = 0$ and $\beta(0,T) = 0$. It follows that we can define an equivalent martingale measure $\mathbb{P}^*$ on $\mathcal{F}_T$ by the density $\frac{d\mathbb{P}^*}{d\mathbb{P}} := \xi_T \exp(\int_0^T R(X_s)ds)$. As a consequence, the arbitrage-free $t-$time price V_t of any $T-$measurable contingent claim $V_T := V(Y_T, X_T, T)$ satisfies the risk neutral valuation formula:

$$V_t = \frac{1}{\xi_t}\mathbb{E}_t\left[\xi_T V_T\right] = \mathbb{E}_t^*\left[\exp\left(-\int_t^T R(X_s)ds\right)V_T\right] \tag{39}$$

where $\mathbb{E}_t^*[\cdot]$ denotes conditional expectation with respect to probability $\mathbb{P}^*$. Under these conditions, X is a matrix AJD, but with time-dependent coefficients, also with respect to the risk neutral probability.

Proposition 5 *The dynamics of the matrix AJD process Y in Assumption 4 with respect to the risk neutral probability $\mathbb{P}^*$ takes the form:*

$$dY_t = \left(R(X_t) - \lambda_{XY}^*(X_t,t)(\Theta^{Y*}(1) - 1)\right)dt + tr\left(\sqrt{X_t}\left(dB_t^* P' + dW_t^* \sqrt{I_n - PP'}\right)\right) + dL_t \tag{40}$$

where B^* and Z^* a two independent $n \times n$ standard Brownian motions and L is a pure jump process with value in $\mathbb{R}$ and having independent jump sizes ξ_t^Y with distribution $\nu^{Y*}(t)$ and affine intensity $\lambda_{YX}^*(X_t,t) = \lambda_{YX,0}^*(t) + tr(\lambda_{YX,1}^*(t)X_t)$. The parameters of the $Y-$dynamics (40) under risk neutral probability $\mathbb{P}^*$ are $\lambda_{YX,0}^*(t) = \lambda_{X,0}\Theta^{YX}(\gamma(t,T),\beta(t,T))$, $\lambda_{X,1}^*(t) = \lambda_{X,1}\Theta^{YX}(\gamma(t,T),\beta(t,T))$ and $\Theta^{YX*}(\gamma,\Gamma,t) = \Theta^{YX}(\gamma + \gamma(t,T), \Gamma + \beta(t,T))/\Theta^{YX}(\gamma(t,T),\beta(t,T))$.

The main implication of Proposition 5 is that under the risk neutral probability $\mathbb{P}^*$, implied by the exponentially affine density $\frac{d\mathbb{P}^*}{d\mathbb{P}} = \exp(\int_0^T R(X_s)ds)\xi_T$, the discounted Laplace transform (9) is again exponentially affine. The coefficients in the exponential of this transform satisfy the same system of matrix Riccati differential equations as in Proposition 2, but with (time dependent) parameters $M^*(t)$, $\lambda_{X,0}^*(t)$, $\lambda_{X,1}^*(t)$ and $\Theta^{X*}(\Gamma,t)$. If follows that with the exponentially affine stochastic discount factor (38) the matrix AJD structure of process X is preserved both under the physical and the risk neutral probabilities.

A.5. Application: Volatility Surface and Skew Dynamics

[TO BE COMPLETED]

B. Multivariate Return Process and Option Pricing

For univariate option pricing models, it well-known that specifications with a jump intensity in returns, volatility or both are able to obtain a better description of the steep skew of short-term index options. In multivariate settings, it has been recently recognized that time varying correlations and correlation risk premia can explain part of the differential pricing of index and single stock options. In this section, we extend the previous setting to a general AJD model for multivariate asset return processes, which can feature stochastic conditional volatilities and correlations, double-jumps in returns and conditional moments, and a multivariate stochastic leverage structure.

B.1. Double-Jump Process of Multivariate Asset Returns

Let $\{S_t := (S_{1t}, S_{2t}, \ldots, S_{nt})' \in \mathbb{R}^n \ : t \geq 0\}$ be the price process of n assets and denote by $Y_t = (\log S_{1t}, \log S_{2t}, \ldots, \log S_{nt})'$ the log return vector. To specify the AJD process for $(Y_t, X_t) \in \mathbb{R}^n \times \mathcal{S}_n^+$ we start again from the assumptions on the multivariate double-jump structure.

Assumption 5 *(L, J) is a pure jump process with values in $\mathbb{R}^n \times \mathcal{S}_n^+$. IID jump sizes $(\xi^Y, \xi^X) \in \mathbb{R}^n \times \mathcal{S}_n^+$ follow a finite joint probability distribution $\nu^{YX} = \nu^{Y|X}\nu^X$ on $\mathbb{R}^n \times \mathcal{S}_n^+$. Jumps are realized with an affine intensity $\lambda_{YX} = \lambda_{YX,0} + tr(\lambda_{YX,1}X_t)$, where $\lambda_{YX,0} \geq 0$ and $\lambda_{YX,1} \in \mathcal{S}_n^+$. The Laplace transform Θ^{YX} of the jump size is given by:*

$$\Theta^{YX}(\gamma, \Gamma) = \int_{\mathcal{S}_n^+} \left(\int_{\mathbb{R}^n} \exp(\gamma' y)\nu^{Y|X}(dy) \right) \exp(tr(\Gamma x))\nu^X(dx) \tag{41}$$

We use the notation $\Theta^Y(\gamma) = \Theta^{YX}(\gamma, 0)$ to denote the Laplace transform of jump sizes ξ^Y.

The affine form of $\lambda(X_t)$ is necessary to preserve the affine structure of the multivariate joint process for returns and their stochastic covariance matrix.

Remark 3 *A convenient assumption for the conditional distribution of the jump size* ξ^Y *in the multivariate context is:*

$$\nu^{Y|X} \sim \mathcal{N}(\mu_Y + \xi^X \beta_Y, \Sigma_Y) \tag{42}$$

with parameters $\mu_Y \in \mathbb{R}^n$, $\beta_Y \in \mathbb{R}^n$ *and* $\Sigma_Y \in \mathcal{S}_n^+$. *The marginal distribution* ν^X *of jump size* ξ^X *can be then taken within the class of tractable probability distributions on* $\mathcal{S}_n^+$, *as in the previous model settings.*

Given the multivariate jump structure in Assumption 5, we specify the following multivariate AJD for Y.

Assumption 6 *The dynamics of the return process* Y_t *under the probability* $\mathbb{P}$ *are given by:*

$$dY_t = \left[R(X_t)\mathbf{1}_n + \mu_e(X_t)\beta + X_t\eta - \frac{1}{2}\operatorname{diag}[X_t] \right] dt + \sqrt{X_t} dZ_t + dL_t \ . \tag{43}$$

where $\mu_e(X_t) = \mu_{e,0} + tr(\mu_{e,1}X_t)$, *with* $\mu_{e,0} \in \mathbb{R}$ *and* $\mu_{e,1} \in \mathcal{S}_n$, $\beta, \eta \in \mathbb{R}^n$, *and* $\operatorname{diag}[X]$ *is a* $n \times 1$ *vector with* $i-$*th component equal to* X_{ii}. *In equation (43),* Z *is a* $n \times 1$ *standard Brownian motion defined by:*

$$Z_t = B_t\rho + \sqrt{1 - \rho'\rho} W_t \ , \tag{44}$$

where W *is another* $n \times 1$ *standard Brownian motion, independent of* B, *and* $\rho = (\rho_1, \rho_2, \ldots, \rho_n)'$ *is a fixed* $n \times 1$ *correlation vector with the restriction that* $\rho'\rho \leq 1$.

The implications of the specification in Assumption 6 for the resulting multivariate expected return process are discussed below.

Remark 4 *If the dynamics of* (Y_t, X_t) *are written with respect to the risk neutral probability measure, then absence of arbitrage requires:*

$$\mu_e(X_t)\beta_i = -\lambda_{YX}(X_t)(\Theta^Y(e_i) - 1) \tag{45}$$

for any $i = 1, \ldots, n$, *where both the jump intensity* λ_{XY} *and the jump size Laplace transform* Θ^Y *are specified with respect to the risk neutral probability measure. In this case, the affine functional form of* $\mu_e(X_t)$ *is equivalent to the affine functional form of the intensity process* $\lambda_{YX}(X_t)$ *under* *the risk neutral probability measure. More generally, the process:*

$$\mu_e(X_t)\beta + X_t\eta + \lambda_{YX}(X_t)((\Theta^Y(e_1), \Theta^Y(e_2), \ldots, \Theta^Y(e_n))' - \mathbf{1}) \tag{46}$$

can be used to specify a flexible multivariate excess return process under the physical probability measure. This setting includes as a special case, when $\beta = (\Theta^Y(e_1), \Theta^Y(e_2), \ldots, \Theta^Y(e_n))'$ *and* $\mu_e(X_t) = -\lambda_{YX}(X_t)$, *the linear expected excess return case* $\mathbb{E}_t[dY_t] = R(X_t)\mathbf{1} + X_t\eta$.

B.2. Variance-Covariance, Co-Volatility and Multivariate Leverage Properties

Our matrix AJD setting can feature a variety of stochastic co-volatility and leverage structures in the multivariate return process Y. The first row in Table II highlights the variance-covariance features of matrix AJD. In the no-jump case ($\lambda_{YX,0} = \lambda_{YX,1} = 0$), we obtain a Wishart-type process for X_t and the conditional covariance matrix of the return process is

$$V_t := \frac{1}{dt} Var_t(dY_t) = X_t \tag{47}$$

In the case of jumps, the returns covariance matrix is

$$V_t = X_t + \lambda_{YX}(X_t)\mathbb{E}[\xi^Y \xi^{Y\prime}] \tag{48}$$

and is again an affine function of X_t. Every component of the returns covariance matrix depends on all elements in the state of matrix X_t. Compared to the no-jump case, the returns covariance matrix is increased by the positive definite matrix of second moments $\mathbb{E}[\xi^Y \xi^{Y\prime}]$ of return jump sizes, weighted by the stochastic intensity $\lambda_{YX}(X_t)$. The higher the conditional intensity, which is equivalent to a more positive definite matrix X_t, the larger the contribution of the jump part of process Y_t to the conditional covariance matrix of returns. Overall, the larger matrix X_t, the larger the return covariance matrix V_t.

The second row in Table II summarizes the different co-volatility structures that can arise in our matrix AJD setting. In the no-jump case, co-volatilities are affine in V_t. In particular, the co-volatility $\frac{1}{dt} Cov(dV_{iit}, dV_{jjt})$ is proportional to $V_{ijt} = X_{ijt}$, with a proportionality coefficient given by the scalar product of the $i-$th and $j-$th column of matrix Q. It follows that the instantaneous correlation between V_{iit} and V_{jjt} is given by:

$$\frac{1}{dt}Corr_t(dV_{iit}, dV_{jjt}) = \frac{Q_i'Q_j}{\sqrt{Q_i'Q_iQ_j'Q_j}}\frac{V_{ijt}}{\sqrt{V_{iit}V_{jjt}}} \tag{49}$$

and is proportional to the correlation between asset returns Y_i and Y_j. In the unconstrained AJD model, co-volatilities are affine in X_t and depend on all components of the state process X, via the stochastic intensity $\lambda_{YX}(X_t)$. This feature breaks down the strong link between covariances and co-volatilities arising in the model without jumps and implies a more general conditional correlation between V_{ii} and V_{jj}, which is not anymore proportional to the one between Y_i and Y_j.

When jumps are absent, the general (affine) leverage formula is:

$$\frac{1}{dt}Cov_t(dY_{it}, dV_{ijt}) = \rho'(X_{iit}Q_j + X_{ijt}Q_j) \ . \tag{50}$$

As a special case, the expression of the leverage between returns and volatility arises for $i = j$:

$$\frac{1}{dt}Cov_t(dY_{it}, dV_{iit}) = 2\rho'Q_iX_{iit} \ , \tag{51}$$

which implies a constant correlation between shocks in returns and their volatility,

$$corr_t(dY_{it}, dV_{iit}) = \frac{\rho'Q_i}{\sqrt{Q_i'Q_i}}. \tag{52}$$

The sign of the correlation in (52) is completely driven by the sign of the scalar product $\rho'Q_i$ between the vector ρ of correlation parameters and the $i-$th column of matrix Q. Therefore, the model cannot admit a stochastic skewness between asset returns and their volatility. When correlated jumps between Y and X are present, we obtain the more general leverage formula:

$$\begin{aligned}\frac{1}{dt}Cov_t(dY_{it}, dV_{ijt}) &= \rho'Q(H_{ij} + H_{ij}')X_te_i + \lambda_{YX}(X_t)\mathbb{E}\left[\xi_i^Y tr(H_{ij}\xi^X)\right] \\ &= \rho'(X_{iit}Q_j + X_{ijt}Q_j) + 2\rho'Q\lambda_{YX,1}X_te_i\mathbb{E}\left[\xi_i^Y\xi_j^Y\right] + \lambda_{YX}(X_t)\mathbb{E}\left[\xi_i^Y tr(H_{ij}\xi^X)\right],\end{aligned}$$

where the matrix H_{ij} is defined as

$$H_{ij} := e_je_i' + \lambda_{YX,1}\mathbb{E}\left[\xi_i^Y\xi_j^Y\right].$$

Note that the leverage depends here on all components of the state process X. For instance, the leverage between returns and volatility is:

$$\frac{1}{dt}Cov_t(dY_{it}, dV_{iit}) = 2\rho'(X_{iit}Q_i + Q\lambda_{YX,1}X_t e_i \mathbb{E}\left[(\xi_i^Y)^2\right]) + \lambda_{YX}(X_t)\mathbb{E}\left[\xi_i^Y tr(H_{ii}\xi^X)\right] \ , \tag{53}$$

which implies the following stochastic conditional correlation between shocks in returns and their volatility:

$$Corr_t(dY_{it}, dV_{iit}) = \frac{2\rho'\left(X_{iit}Q_i + Q\lambda_{YX,1}X_t e_i \mathbb{E}\left[(\xi_i^Y)^2\right]\right) + \lambda_{YX}(X_t)\mathbb{E}\left[\xi_i^Y tr(H_{ii}\xi^X)\right]}{\sqrt{(X_{ii} + \lambda_{YX}(X_t)\mathbb{E}\left[(\xi_i^Y)^2\right])(4tr(H_{ii}Q_i'QH_{ii}X_t) + \lambda_{YX}(X_t)\mathbb{E}\left[tr(H_{ii}\xi^X)^2\right])}}$$

This correlation is stochastic, because of its dependence on all components of X_t. If follows that every asset in the general matrix AJD setting can feature stochastic multifactor dynamics for the correlation between returns and their volatilities. Moreover, the correlation can jump itself with a potentially time-varying intensity $\lambda_X(X_t)$. This structural property of the model can prove useful in some applications, e.g., to consistently price multi-asset options when the return skewness of the involved assets, either under the physical or the risk neutral probability, is stochastic and potentially characterized by a non smooth behavior over time.

[Figure 3 about here.]

In Figure 3, we plot the correlation $Corr_t(dY_{it}, dV_{iit})$ for different specifications of the jump component. When there are no jumps involved, the volatility leverage is constant (dashed line). Adding a jump component into the return process gives rise to a stochastic volatility leverage (dash-dotted line). The same effect, and even stronger, occurs when we add jumps in the covariance matrix instead of the return process (dotted line). By combining the two jump specifications, we find that we can generate a stochastic leverage effect with a variability large enough to generate stochastic sign switches. From a theoretical viewpoint, the introduction of jumps into the return process may generate a sign of the volatility leverage which may change. With jumps in the covariance process only, we would fail to do so. Obviously, by combining the two jump specifications, we can generate a volatility leverage structure and hence implied volatility smiles, which is flexible enough to account for the behavior of single stock options.

B.3. Transform Analysis

Under Assumption 5 and 6, (Y,X) is a Markov process with values in $\mathbb{R}^n \times \mathcal{S}_n^+$. The Lévy infinitesimal generator $\mathcal{L}_{YX}$ of (Y,X) is defined for bounded C^2 functions $f:\mathbb{R}^n \times D \to \mathbb{R}$ by:

$$\begin{aligned}\mathcal{L}_{Y,X} f(y,x) &= \left[R(x)\mathbf{1}_n + \mu_e(x)\beta + x\eta - \frac{1}{2}\operatorname{diag}[x]\right]' \frac{\partial f(y,x)}{\partial y} + \frac{1}{2} tr\left(x \frac{\partial^2 f(y,x)}{\partial y \partial y'}\right) \\ &\quad + tr\left[(\Omega\Omega' + Mx + xM')\mathcal{D} + \mathcal{D}Q'\frac{\partial}{\partial y}\rho x + x\frac{\partial}{\partial y'}\rho' Q\mathcal{D} + 2x\mathcal{D}Q'Q\mathcal{D}\right] f(y,x) \\ &\quad + \lambda_{YX}(x)\int_{\mathbb{R}^n \times \mathcal{S}_n^+} (f(y+w,x+z) - f(y,x))d\nu^{YX}(w,z) \end{aligned} \tag{54}$$

where $\mathcal{D}$ is a $n \times n$ matrix of differential operators with ij-component given by $\frac{\partial}{\partial X_{ij}}$ and $\frac{\partial}{\partial y'}$ is the gradient operator. The affine form of the generator (54) implies a discounted Laplace transform of the return vector Y_T that is exponentially affine.

Proposition 6 *Let Assumptions 1 2, 5, 6, and additional regularity conditions be satisfied. Then, the discounted Laplace transform of Y_T has the exponentially affine form:*

$$\begin{aligned}\Psi^Y(\gamma) &:= \mathbb{E}\left[\exp(-\int_t^T R(X_s)ds + \gamma' Y_T)\right] \\ &= \exp(\gamma' Y_t)\exp(B(t-T) + tr(A(T-t)X_t)) \end{aligned} \tag{55}$$

with functions $B(u) \in \mathbb{R}$ and $A(u) \in \mathcal{S}_n^+$ that solve the system of matrix differential Riccati equations:

$$\begin{aligned}\frac{\partial A}{\partial \tau} &= (\gamma'\mathbf{1} - 1)\rho_1 + \gamma'\beta\mu_{e,1} + \eta\gamma' + \frac{1}{2}\left(\gamma\gamma' - \sum_{i=1}^n \gamma_i e_i e_i'\right) \\ &\quad + A(\tau)(M + Q'\rho\gamma') + (M' + \gamma\rho' Q)A(\tau) + 2A(\tau)Q'QA(\tau) + \lambda_{YX,1}[\Theta^{YX}(\gamma, A(\tau)) - 1] \\ \frac{\partial B}{\partial \tau} &= (\gamma'\mathbf{1} - 1)\rho_0 + \gamma'\beta\mu_{e,0} + tr(\Omega\Omega' A(\tau)) + \lambda_{YX,0}[\Theta^{YX}(\gamma, A(\tau)) - 1]\end{aligned}$$

subject to terminal conditions $B(0) = 0$ and $A(0) = 0$.

The structure of the system of matrix Riccati differential equations in Proposition 6 is again as the one of the system of equations in Proposition 2. Therefore, for the constant intensity case, the following closed form solution is obtained.

Corollary 5 *Let Assumptions 1, 2 and additional regularity conditions be satisfied. Assume further* *that* $\lambda_{YX,1} = 0$. *Then, the closed form expressions for function* $A(\tau)$ *in Proposition 2 is as follows:*

$$A(\tau) = C_{22}(\tau)^{-1} C_{21}(\tau) \ , \tag{56}$$

where $C_{12}(\tau)$ *and* $C_{22}(\tau)$ *are* $n \times n$ *blocks of the following matrix exponential:*

$$\begin{pmatrix} C_{11}(\tau) & C_{12}(\tau) \\ C_{21}(\tau) & C_{22}(\tau) \end{pmatrix} := \exp\left[\tau \begin{pmatrix} M + Q'\rho\gamma' & -2Q'Q \\ D & -M' - \gamma\rho' Q \end{pmatrix}\right] . \tag{57}$$

with $D := (\gamma'\mathbf{1} - 1)\rho_1 + \gamma'\beta\mu_{e,1} + \eta\gamma' + \frac{1}{2}(\gamma\gamma' - \sum_{i=1}^{n}\gamma_i e_i e_i')$. *Given the solution for* $A(\tau)$, *the coefficient* $B(\tau)$ *follows by direct integration:*

$$B(\tau) = ((\gamma'\mathbf{1} - 1)\rho_0 + \gamma'\beta\mu_{e,0})\tau - \frac{k}{2} tr[\ln C_{22}(\tau) + \tau(M + Q'\rho\gamma')] + \lambda_{YX,0}\left[\int_0^\tau \Theta^{YX}(\gamma, A(s))ds - \tau\right]$$

The immediate expression for the transform $G_{a,b}(y; Y_0, T)$ needed to compute the prices of European options in the multivariate context is given below (see Duffie, Pan, and Singleton (2000)).

Corollary 6 *Define for any* $a, b \in \mathbb{R}^n$ *the discounted Laplace transform:*

$$G_{a,b}(y; Y_0, T) := \mathbb{E}\left[\exp[-\int_0^T R(X_s)ds] \exp(a'Y_T)\mathbb{I}_{b'Y_t \le y}\right] \tag{58}$$

It then follows, under regularity conditions:

$$G_{a,b}(y; Y_0, T) = \frac{\Psi^Y(a, Y_0, 0, T)}{2} - \frac{1}{\pi}\int_0^\infty \frac{Im\left[\Psi^Y(a + ivb, Y_0, 0, T)\exp(-ivy)\right]}{v} dv \tag{59}$$

where $Im(c)$ *is the imaginary part of* $c \in \mathbb{C}$.

Given concrete assumptions on the distribution of the jump size (ξ^Y, ξ^X), closed form option pricing formulae for European derivatives on Y_T readily follow. Conditional normality of ξ^Y given ξ^X, as given, e.g., in Remark 3, is a convenient assumption in this multivariate return context.

B.4. State Price Density and Risk Neutral Pricing

[TO BE COMPLETED]

III. Term Structure Models

Given the matrix jump diffusion dynamics (1), a rich class of multi factor affine yield curve models with jumps can be obtained. Let dynamics (1) hold under a risk neutral measure $\mathbb{P}^*$. Then, the zero-bond prices are given by:

$$B(t,T) = \mathbb{E}^{\mathbb{P}^*}[\exp(-\int_t^T R(X_s)ds)|\mathcal{F}_t] = \Psi^X(0, X_t, t, T) \quad . \tag{60}$$

In particular, zero bond yields are affine in X_t:

$$R(t,T) = -\log \frac{B(t,T)}{T-t} = -\frac{1}{T-t}\left[B(T-t) + tr(A(T-t)X_t)\right] \tag{61}$$

and we can specify the short rate as

$$R(t) = \lim_{T \searrow t} R(t,T) = \delta + tr\left[DX_t\right],$$

Bond yields are stochastically correlated and, at the same time, they are subject to common jumps, for which the imperfectly correlated jump sizes are generated by an IID $n(n+1)/2$ dimensional random matrix ξ^X. Yields covariances are given by:

$$Cov_t\left(dR(t,T_1), dR(t,T_2)\right) = \frac{tr[A(T_1-t)Q'QA(T_2-t)X_t] + \mathbb{E}_t\left[tr(A(T_1-t)dJ_t)tr(A(T_2-t)dJ_t)\right]}{(T_1-t)(T_2-t)}$$

Suppose, for instance, that $dJ_t = dN_t\xi^X$, where $\{N(t) : t \geq 0\}$ is a Poisson process in $\mathbb{R}^+$ and the jump size ξ^X is IID distributed in $\mathcal{S}_n^+$. We can write the part of yields co-movement due to jumps in X as

$$\frac{\mathbb{E}_t\left[tr(A(T_1-t)dJ_t)tr(A(T_2-t)dJ_t)\right]}{(T_1-t)(T_2-t)} = (\lambda_{X,0} + tr(\lambda_{X,1}X_t))\frac{\mathbb{E}_t\left[tr(A(T_1-t)\xi^X)tr(A(T_2-t)\xi^X)\right]}{(T_1-t)(T_2-t)} \ ,$$

and is an affine function of X_t, via the multi factor intensity process λ^X that depends on the $n(n+1)/2$ components of the matrix AJD process X. Using concrete assumptions on the form of the distribution of the jump size,[7] this covariance structure can be made completely explicit.

[7] E.g., a Wishart distribution with degrees of freedom k_X, noncentrality parameter $M_X \in \mathbb{R}^{n\times n}$ and scale parameter Σ_X.

A. Predictability and Unspanned Volatility

To validate our modeling approach, we briefly perform a simulation exercise and benchmark our multivariate AJD term structure model against recent work on the stylized facts of term structure dynamics, particularly on predictability and unspanned volatility. Cochrane and Piazzesi (2005) showed that forward rates can predict future excess bond returns with R^2-values around 30% to 40%. At the same time, Collin-Dufresne and Goldstein (2002), argue that under the unspanned volatility hypothesis (USV), some state variables do not lie in the span of the term structure of yields. Motivated by these two findings, Wright and Zhou (2007) develop a test for the USV hypothesis following similar arguments as in Almeida, Graveline, and Joslin (2006) and Joslin (2007). As expected excess bond returns are a function of all the state variables it follows that, if the USV hypothesis is false, expected excess returns are spanned by term structure yields and any other predictor must be insignificant.

One potential source of USV may be the presence of jump risk. Identifying jumps is inherently difficult. Building on the work of Barndorff-Nielsen and Shephard (2004) and Barndorff-Nielsen and Shephard (2006), Wright and Zhou (2007) investigate a long data set of high-frequency bond data spanning the period from 1982 to 2006. They find significant evidence for jumps in the term structure dynamics. Furthermore, augmenting the regression of bond excess returns on forward rates by the realized bond jump mean greatly increases the predictability of excess bond returns. Compared to the R^2 values of around 30% using the regression specification in Cochrane and Piazzesi (2005), Wright and Zhou (2007) report R^2 values up to 60 percent when including the jump mean as explanatory variable. The inclusion of jump intensity and jump volatility does not lead to a significant increase in predictability. Also, regressions based on forward rates and realized as well as implied volatility does not alter the R^2 values. The latter finding was also reported in Collin-Dufresne and Goldstein (2002) and Andersen and Benzoni (2006), indicating the existence of unspanned risk factors.

Standard diffusion-based affine term structure models have difficulties in explaining the existence of unspanned risk factors without unreasonable assumptions on second moments of interest rates (see, Roberds and Whiteman (1999), Dai and Singleton (2000), Bansal, Tauchen, and Zhou (2004)). Only with richer specifications of market prices of risk or preferences, these models may explain some predictability in excess bond returns (see, Duffee (2002), Dai and Singleton (2002), Duarte (2004), Wachter (2006)). Therefore, we perform a simulation with a very simple specification of an AJD term structure model to investigate whether we are capable of accounting for the stylized facts of bond returns recently reported in Wright and Zhou (2007).

We follow closely the methodology in Wright and Zhou (2007) to assess the predictability of excess bond returns in our term structure model. In particular, to minimize the near-perfect collinearity problem, we use only three forward rates in the regression instead of five as in Cochrane and Piazzesi (2005). However, in contrast to Wright and Zhou (2007), we do not augment the regression with implied volatility, since they find no evidence of predictability for this measure as an explanatory variable. Writing the return on holding a zero bond with maturity T for a holding period s as

$$R^e(t,s,T) = \log B(s,T) - \log B(t,T-s) + \log B(t,s), \tag{62}$$

we consider five different regression equations, which are all nested in the specification:

$$R^e(t,s,T) = \beta_0 + \beta_1 f_1(t,1) + \beta_2 f_1(t,3) + \beta_3 f_1(t,5) + \beta_4 m_J(t,h) + \beta_5 v_J(t,h) + \beta_6 rv(t,h) + \epsilon_{t+s}, \tag{63}$$

where $f_1(t,T)$ is the T forward rate at time t with a one year period, and $m_J(t,h)$ and $v_J(t,h)$ are the realized mean and volatility of the short rate jumps measured in rolling windows of length h ending at time t. For our regression exercise, we set both s and h equal to one year. To assess the predictability of the jump components, our first regression (A) uses just the three forward rates $f_1(t,1)$, $f_1(t,3)$, and $f_1(t,5)$. The second regression (B) augments the regression by the realized mean of the interest rate jumps $m_J(t,h)$. For regression (C), we add the realized jump volatility $v_J(t,h)$ and for regression (D), we add the realized volatility of the short interest rate $rv(t,h)$. Finally, for regression (E) we use the forward rates and the realized volatility of the short interest rate, but no information from the jump measures. We simulate our model daily over a time period of 20 years and compute the jump measures and the realized volatilities using a time period of one year. We then estimate the various specifications of equation (63) based on monthly data and using Newey-West adjusted heteroscedastic-serial consistent least-squares regression with lag parameter 11.

For the simulation, we assume a time-varying jump intensity that is affine in X_t and impose a simple structure on the risk premium which is proportional to variance and affine in the jump intensity. We use the set of parameters reported in Panel B of Table III. These values are set arbitrary, but they give some reasonable term structure dynamics. We leave the estimation of AJD term structure models for future research. Since Wright and Zhou (2007) report also negative jumps in the interest rate dynamics, we have to abandon the assumption that D is positive semidefinite. However, the probability of generating negative interest rates is small and could be further offset by a sensible choice of δ. For our simulation, we find a jump mean and standard deviation of the short interest rate are 0.034% and 0.053%, respectively. The mean intensity is 0.0838. Except for the standard deviation of jumps, these values are almost identical to the values reported in Wright and Zhou (2007). They find a jump mean of 0.03%, a jump mean intensity of 0.08, and a jump volatility of 0.41%. Finally, the volatility of our simulated short rate is slightly above 2%, which is also approximately consistent with what we observe in interest rate data. As an illustration, we plot in Figure 4 the mean term structure of the simulated sample path. The mean term structure is upward sloping and shows most of its curvature at the short end. We additionally plot two arbitrary term structures generated during the simulation.

[Figure 4 about here.]

Table IV summarizes the R^2 values for our regression of bond excess returns with a one year holding period and different maturities ranging from two to five years. Running the regression of the excess bond returns on the forward rates alone gives an R^2 between 24 to 29 percent. These numbers are slightly lower than the numbers reported in Wright and Zhou (2007) (34-38 percent). In accordance with Wright and Zhou (2007), we observe that when we augment the regression based on forward rates with the jump mean, the R^2 value rises significantly to values between 37 and 43 percent. This finding indicates that the information content of the jump mean complements that of forward rates. In contrast to Wright and Zhou (2007), we also find that adding the jump volatility increases the R^2 values, although slightly less than the jump mean. However, adding realized volatility (Regression (D) and (E)), does not increase predictability at all.

Table V summarizes the coefficient estimates and the associated t-statistics for several specifications of the regression equation (63). For all regressions, the forward rates exhibit the familiar tent-shaped pattern. The coefficients are mostly significant. If we add the jump mean to the regression of excess returns on forward rates (Regression (B)), the coefficient on the jump mean (β_4) is highly negative and statistically significant for each regression specification. A negative coefficient indicates that the jump mean is a negative predictor of future excess returns, i.e., downward jumps in bond prices are followed by large positive excess returns. Adding the jump mean does not significantly alter the coefficients for the forward rates. Hence, the forward rates and the jump mean measure different components of the bond risk premia. The bond jump mean may act as an unspanned stochastic mean factor that cannot be hedged with the current yields, but can forecast excess bond returns. These findings are also consistent with the findings in Wright and Zhou (2007).

In summary, we obtain a tractable affine yield curve setting with at least two remarkable properties. First, the model implies a flexible covariance structure with stochastically correlated yields. Second, it implies a potentially significant fraction of yields variation due to multivariate jumps in the latent state process. The variation due to jumps cannot be hedged theoretically by portfolios of zero bonds, which makes the market incomplete. It follows that the model can potentially feature a flexible covariance structure and incomplete bond markets together. These two properties are well-known to be key for explaining the empirical predictability of bond returns and the main stylized facts on interest rate derivatives, such as the pricing patterns of interest rate caps and swaptions (see, e.g., Han (2007)).

IV. Portfolio Choice

Accounting for the possibility of jumps in asset returns is also important for portfolio choice problems. Das and Uppal (2004) develop a model of international equity returns using a multivariate system of jump-diffusion processes where the arrival of jumps is simultaneous across assets. In their model, the introduction of jumps in the return process reduces the international diversification gains and makes leveraged portfolios much more susceptible to large losses. However, jumps in volatilities and even more so jumps in correlations are less studied in the literature[8] and we may wonder, whether they indeed play an economically significant role.

To isolate the impact of jumps in the covariance process on portfolio allocation, we simplify our multivariate model and assume that there are no jumps in the associated price processes. We further equip an investor with a standard CRRA utility over terminal wealth W_T, who can allocate her initial wealth W_t in n risky assets with price process $\{S_t := (S_{1t}, S_{2t}, \ldots, S_{nt})' \in \mathbb{R}^n \ : t \geq 0\}$. Under the historical probability $\mathbb{P}$, the stock prices follow:

$$dS_t = \text{diag}\left[S_t\right]\left(\left[R(X_t)\mathbf{1}_n + X_t\eta\right]dt + \sqrt{X_t}dZ_t\right) \ , \tag{64}$$

where $X_t\eta$ represents the excess return vector. In equation (64), Z is a $n \times 1$ standard Brownian

[8] One notable exception is the literature based on regime switching models as, e.g., in Ang and Bekaert (2002), where they study an international portfolio allocation problem in discrete time framework.

motion defined by:

$$Z_t = B_t\rho + \sqrt{1-\rho'\rho}W_t, \quad \rho'\rho \leq 1 \ , \tag{65}$$

where W is another $n \times 1$ standard Brownian motion, independent of B, and $\rho = (\rho_1, \rho_2, \ldots, \rho_n)'$ is a fixed $n \times 1$ correlation vector.

For simplicity, we consider a money-market account with a constant interest rate $R(X_t) = r$. In this setup, the wealth dynamics follow

$$\frac{dW_t}{W_t} \;=\; rdt + w_t' X_t \eta dt + w_t' \sqrt{X_t} dZ_t, \tag{66}$$

where $w_t \in \mathbb{R}^n$ is the vector of relative wealth invested in the n assets. Assuming a constant relative risk aversion coefficient $\gamma > 0$, we can write the indirect utility function as,

$$J(t, W, X) = \sup_{w_t} \mathbb{E}\left(\frac{W_T^{1-\gamma}}{1-\gamma} |W_t = W, X_t = X \right), \quad \gamma \neq 1 \ ,$$

subject to the budget constraint in (66).

Proposition 7 *The indirect utility function has the solution*

$$J(t, W_t, X_t) = \frac{W_t^{1-\gamma}}{1-\gamma} \exp\left(tr\left[A(\tau)X_t\right] + B(\tau)\right). \tag{67}$$

and the optimal portfolio weights are

$$w^\star \;=\; \frac{1}{\gamma}\left(\eta + 2A(\tau)Q'\rho\right), \tag{68}$$

where $A(\tau)$ and $(B\tau)$ solve the following system of ordinary differential equations:

$$\begin{aligned} A'(\tau) \;&=\; A(\tau)\left(M + \frac{1-\gamma}{\gamma}Q'\rho\eta'\right) + \left(M' + \frac{1-\gamma}{\gamma}\eta\rho'Q\right)A(\tau) \\ &\quad +2A(\tau)Q'\left(I + \frac{1-\gamma}{\gamma}\rho\rho'\right)QA(\tau) - \frac{1-\gamma}{2\gamma}\eta\eta' + \lambda_{X,1}[\Theta^X(A(\tau)) - 1], \end{aligned} \tag{69}$$

$$B'(\tau) \;=\; (1-\gamma)r + tr\left[\Omega\Omega^\top A(\tau)\right] + \lambda_{X,0}[\Theta^X(A(\tau)) - 1]. \tag{70}$$

Obviously, when we have jumps in the covariance matrix only and the jump intensity is a constant, i.e., $\lambda_{X,1} = \mathbf{0}$, then jumps do not have any impact on the optimal portfolio allocation $w^\star$, but only on the level of the value function $J(t, W, X)$.

To study the potential economic significance of jumps in the covariance matrix, we briefly calculate the intertemporal hedging demand for the optimal portfolios in Proposition 7 using a simple numerical example with two assets. As input we use the estimated values in Buraschi, Porchia, and Trojani (2007) for the S&P 500 Index Futures and the 30-year Treasury Bond Futures sampled at monthly frequencies. For completeness, these values are reported in Panel C of Table III together with our baseline assumption on the jump size matrix ξ^X, for which we make an ad-hoc choice, and we set $\Omega\Omega' = kQQ'$ with $k = 10$. For the jump intensity, we set $\lambda_0 = 0$ and we choose the matrix λ_1 so that we obtain reasonable average jump probabilities.

[Figure 5 about here.]

To get a quick sense of our parameter choices for the jump components, we plot in Figure 5 the simulated processes for the return volatilities and correlations over a time period of five years. For our reference choice of λ_1, we get an average jump probability of approximately 1% per day. Given the small jump sizes, such a choice seems reasonable. Inspecting the time series of the resulting volatilities (solid lines) in the upper panel of Figure 5, there is no significant difference to the time series without jumps (dash-dotted lines).[9] At the same time, we notice that the correlation process in the lower panel exhibits more violent moves when jumps are present (solid line). From this rather heuristic comparison, we can conclude that our jump parameters represent a reasonable choice.

[Figure 6 about here.]

In Figure 6, we calculate the hedging demand in an optimal portfolio of a CRRA utility investor with relative risk aversion coefficient $\gamma = 6$ and for different investment horizons up to ten years. The left panels display the hedging demand as a fraction of the myopic portfolio for different time horizons, ranging from zero to ten years. As already argued in Buraschi, Porchia, and Trojani (2007), the presence of stochastic correlation induces a substantial hedging demand between 20% to 30% of the myopic portfolio fraction. For the given parameter set, we observe that this hedging demand is further increased, almost up to 40% of the myopic portfolio for time horizons beyond three years. Representing the jump-induced component as a fraction of the hedging portfolio, the presence of jumps may lead to a 30% increase in total hedging demand.

[9] Using the same set of variables, we simulate the volatilities for a daily time series spanning 100 years. We find that the means of the volatilities differ by -2% and 9% for the first and second asset with and without jumps, respectively. For the standard deviation of volatilities, the respective numbers are -1% and 15%.

[Figure 7 about here.]

In Figure 7, we further investigate the impact of different levels of jump intensities and jump sizes. For the jump intensities, we start with an average daily jump probability of around 0.3% per day and we increase this number up to 10%, while keeping ξ^X constant as in Panel C of Table III. For the right panels, we change the distribution of the jump sizes ξ^X, while keeping the average jump probability at 1%. In particular, we start with $\frac{1}{5}\xi^X$ and increase it up to $4\xi^X$. Depending on the different values for jump intensities and jump sizes, we observe that the hedging demand may increase to more than double that of the hedging demand implied by the diffusive part of the covariance process. At this point, we note that our choices for the jump parameters are ad hoc and the impact of jumps in covariances on the portfolio allocation warrants further empirical investigation. We leave this challenging but interesting avenue for future research and note for now that jumps in covariances may play an economically significant role for the intertemporal hedging demand.

V. Conclusions

STILL TO COME....

[....] Finally, our modeling approach allows the pricing of multi-asset options with quanto, rainbow, basket and spread based pay-offs. Various types of these multi-asset equity options recently emerged in the markets. They are either sold separately over-the-counter or as an “equity kicker” of bond-like structures, where they usually offer a certain participation in equity performance or a large coupon conditionally on a defined performance of a basket of stocks. Often, they have barrier features. Also these kind of products come with a very large lifetime (up to 10-15 years), and contain intrinsic barriers, or even some of their underlyings may be withdrawn at certain fixing dates. For instance, if the respective barrier of one of the underlying stocks was touched during its term and if the final fixing price of the worst performing underlying is equal to or lower than the strike price, the potential capital loss is the difference between the respective strike price and the value of the worst performing underlying at final fixing. The nature of these products make a multivariate and flexible modeling of the underlying risk factors indispensable.

Appendix

A. Correlation Process of Matrix AJD

Itô's Lemma implies the following dynamics for the correlation process $\rho_{ij} := X_{ij}/\sqrt{X_{ii}X_{jj}}$, $1 \leq i \leq j \leq n$, implied by the matrix AJD in Assumption 1:

$$\begin{aligned} d\rho_{ijt} \quad = \quad & \frac{dX_{ijt}}{\sqrt{X_{iit}}\sqrt{X_{jjt}}} - \frac{dX_{ijt}dX_{iit}}{2X_{iit}^{3/2}\sqrt{X_{jjt}}} - \frac{dX_{ijt}dX_{jjt}}{2X_{jjt}^{3/2}\sqrt{X_{iit}}} \\ & + \rho_{ijt}\left(\frac{dX_{iit}dX_{jjt}}{4X_{iit}X_{jjt}} - \frac{dX_{iit}}{2X_{iit}} - \frac{dX_{jjt}}{2X_{jjt}} + \frac{3dX_{iit}^2}{8X_{iit}^2} + \frac{3dX_{jjt}^2}{8X_{jjt}^2}\right) \\ & + \rho_{ijt}\left(\frac{1+\frac{\xi_{ij}}{X_{ijt}}}{\sqrt{\left(1+\frac{\xi_{ii}}{X_{iit}}\right)\left(1+\frac{\xi_{jj}}{X_{jjt}}\right)}} - 1\right) dN_t, \end{aligned}$$

where $\{N_t : t \geq 0\}$ is a Poisson process in $\mathbb{N}$ with stochastic intensity $\{\lambda(X_t) : t \geq 0\}$. Since

$$\frac{dX_{ijt}dX_{klt}}{dt} = e_i'Q'Qe_lX_{ikt} + e_i'Q'Qe_kX_{jlt} + e_j'Q'Qe_lX_{ikt} + e_j'Q'Qe_kX_{ilt} \ ,$$

where e_s denotes the $s-$th unit vector in $\mathbb{R}^n$, we can also write the correlation dynamics as:

$$\begin{aligned} d\rho_{ijt} \quad = \quad & m(\rho_{ijt})dt + \frac{e_i'\sqrt{X_t}dB_tQe_j + e_i'Q'dB_t'\sqrt{X_t}e_j}{\sqrt{X_{iit}X_{jjt}}} - \rho_{ijt}\left(\frac{e_i'\sqrt{X_t}dB_tQe_i}{X_{iit}} + \frac{e_j'\sqrt{X_t}dB_tQe_j}{X_{jjt}}\right) \\ & + \rho_{ijt}\left(\frac{1+\frac{\xi^X_{t,ij}}{X_{t,ij}}}{\sqrt{\left(1+\frac{\xi^X_{t,ii}}{X_{t,ii}}\right)\left(1+\frac{\xi^X_{t,jj}}{X_{t,jj}}\right)}} - 1\right) dN_t. \end{aligned}$$

In the last equation, the drift coefficient $m(\rho_{ijt})$ takes the form:

$$\begin{aligned} m(\rho_{ijt}) \quad = \quad & \rho_{ijt}^2\frac{e_i'Q'Qe_j}{\sqrt{X_{iit}X_{jjt}}} + \frac{e_i'(\Omega\Omega' - 2Q'Q)e_j + e_i'\left(MX_t + X_tM'\right)e_j}{\sqrt{X_{iit}X_{jjt}}} \\ & + \rho_{ijt}\left(\frac{e_i'(Q'Q - \Omega\Omega')e_i - 2e_i'MX_te_i}{2X_{iit}} + \frac{e_j'(Q'Q - \Omega\Omega')e_j - 2e_j'MX_te_j}{2X_{jjt}}\right). \end{aligned} \tag{A.1}$$

We first derive the expression for $m(\rho_{ijt})$ in terms of ρ_{kl}, X_{ll} and X_{kk}, $1 \leq k,l \leq n$. Explicit computations give:

$$
\begin{aligned}
e_i' \left(M X_t + X_t M' \right) e_j &= \sum_k (M_{ik} X_{kjt} + X_{ikt} M_{kj}') \\
&= \sum_k (M_{ik} X_{jkt} + M_{jk} X_{ikt}) \\
&= (M_{ii} + M_{jj}) X_{ijt} + M_{ji} X_{iit} + M_{ij} X_{jjt} + \sum_{k \neq i,j} (M_{ik} X_{jkt} + M_{jk} X_{ikt})
\end{aligned}
$$

In a similar way, we obtain:

$$
\begin{aligned}
e_i' M X_t e_i &= \sum_k M_{ik} X_{kit} = M_{ij} X_{ijt} + M_{ii} X_{iit} + \sum_{k \neq i,j} M_{ik} X_{ikt} \\
e_j' M X_t e_j &= \sum_k M_{jk} X_{kjt} = M_{ji} X_{ijt} + M_{jj} X_{jjt} + \sum_{k \neq i,j} M_{jk} X_{jkt}
\end{aligned}
$$

Overall, this yields:

$$
\begin{aligned}
\frac{e_i' \left(M X_t + X_t M' \right) e_j}{\sqrt{X_{iit} X_{jjt}}} &= (M_{ii} + M_{jj}) \rho_{ijt} + M_{ji} \sqrt{\frac{X_{iit}}{X_{jjt}}} + M_{ij} \sqrt{\frac{X_{jjt}}{X_{iit}}} \\
&\quad + \sum_{k \neq i,j} \left(M_{ik} \rho_{jkt} \sqrt{\frac{X_{kkt}}{X_{iit}}} + M_{jk} \rho_{ikt} \sqrt{\frac{X_{kkt}}{X_{jjt}}} \right)
\end{aligned}
$$

and

$$
\begin{aligned}
\frac{e_i' M X_t e_i}{X_{iit}} &= M_{ij} \rho_{ijt} \sqrt{\frac{X_{jjt}}{X_{iit}}} + M_{ii} + \sum_{k \neq i,j} M_{ik} \rho_{ikt} \sqrt{\frac{X_{kkt}}{X_{iit}}} \\
\frac{e_j' M X_t e_j}{X_{jjt}} &= M_{ji} \rho_{ijt} \sqrt{\frac{X_{iit}}{X_{jjt}}} + M_{jj} + \sum_{k \neq i,j} M_{jk} \rho_{jkt} \sqrt{\frac{X_{kkt}}{X_{jjt}}}
\end{aligned}
$$

It follows that $m(\rho_{ijt})$ is a quadratic polynomial in ρ_{ijt}:

$$
m(\rho_{ijt}) = A_{ijt} \rho_{ijt}^2 + B_{ijt} \rho_{ijt} + C_{ijt} \ .
$$

The coefficient of the quadratic term is:

$$
A_{ijt} = \frac{e_i' Q' Q e_j}{\sqrt{X_{iit} X_{jjt}}} - M_{ji} \sqrt{\frac{X_{iit}}{X_{jjt}}} - M_{ij} \sqrt{\frac{X_{jjt}}{X_{iit}}}. \tag{A.2}
$$

The coefficient of the linear term is:

$$B_{ijt} = \frac{e_i'(Q'Q - 2\Omega\Omega')e_i}{2X_{iit}} + \frac{e_j'(Q'Q - 2\Omega\Omega')e_j}{2X_{jjt}} - \sum_{k\neq i,j}\left(M_{ik}\rho_{ikt}\sqrt{\frac{X_{kkt}}{X_{iit}}} + M_{jk}\rho_{jkt}\sqrt{\frac{X_{kkt}}{X_{jjt}}}\right)$$

The last coefficient takes the form:

$$C_{ijt} = \frac{e_i'(\Omega\Omega' - 2Q'Q)e_j}{\sqrt{X_{iit}X_{jjt}}} + M_{ji}\sqrt{\frac{X_{iit}}{X_{jjt}}} + M_{ij}\sqrt{\frac{X_{jjt}}{X_{iit}}} + \sum_{k\neq i,j}\left(M_{ik}\rho_{jkt}\sqrt{\frac{X_{kkt}}{X_{iit}}} + M_{jk}\rho_{ikt}\sqrt{\frac{X_{kkt}}{X_{jjt}}}\right)$$

We now compute the functional form of the second conditional moment of $d\rho_{ij}$ in terms of (i) ρ_{kl}, X_{ll} and X_{kk}, $1 \leq k, l \leq n$ and (ii) the structure of the jump part of $d\rho_{ij}$. We first have:

$$\frac{1}{dt}\mathbb{E}_t(d\rho_{ijt}^2) = v^2(\rho_{ijt}) + \lambda(X_t)\mathbb{E}\left({\zeta_{ij}^X}^2\right)$$

where

$$v(\rho_{ijt})^2 = \frac{1}{dt}\mathbb{E}_t\left[\left(\frac{e_i'\sqrt{X_t}dB_tQe_j + e_i'Q'dB_t'\sqrt{X_t}e_j}{\sqrt{X_{iit}X_{jjt}}} - \rho_{ijt}\left(\frac{e_i'\sqrt{X_t}dB_tQe_i}{X_{iit}} + \frac{e_j'\sqrt{X_t}dB_tQe_j}{X_{jjt}}\right)\right)^2\right]$$

and

$$\zeta_{ij}^X = \frac{1 + \frac{\xi_{ij}^X}{X_{ijt}}}{\sqrt{\left(1 + \frac{\xi_{ii}^X}{X_{iit}}\right)\left(1 + \frac{\xi_{jj}^X}{X_{jjt}}\right)}} - 1$$

is the correlation relative jump size. To compute $v(\rho_{ijt})$, we first note that:

$$\begin{aligned}
\mathbb{E}_t\left(dB_tuv'dB_t\right) &= \mathbb{E}_t\left(dB_t'uv'dB_t'\right) = vu'dt \\
\mathbb{E}_t\left(dB_tuv'dB_t'\right) &= \mathbb{E}_t\left(dB_t'uv'dB_t\right) = v'uI_ndt
\end{aligned}$$

It then follows:

$$\begin{aligned}
\frac{1}{dt}\mathbb{E}_t\left[\left(\frac{e_i'\sqrt{X_t}dB_tQe_j + e_i'Q'dB_t'\sqrt{X_t}e_j}{\sqrt{X_{iit}X_{jjt}}}\right)^2\right] &= \frac{e_i'X_te_ie_j'Q'Qe_j + e_j'X_te_je_i'Q'Qe_i + 2e_i'X_te_je_i'Q'Qe_j}{X_{iit}X_{jjt}} \\
&= \frac{e_j'Q'Qe_j}{X_{jjt}} + \frac{e_i'Q'Qe_i}{X_{iit}} + 2\rho_{ijt}\frac{e_i'Q'Qe_j}{\sqrt{X_{iit}X_{jjt}}}\ .
\end{aligned}$$

Similarly,

$$\begin{aligned}
\frac{1}{dt}\mathbb{E}_t\left[\left(\frac{e_i'\sqrt{X_t}dB_tQe_i}{X_{iit}}+\frac{e_j'\sqrt{X_t}dB_tQe_j}{X_{jjt}}\right)^2\right] &= \frac{X_{iit}e_i'Q'Qe_i}{X_{iit}^2}+\frac{X_{jjt}e_j'Q'Qe_j}{X_{jjt}^2}+2\frac{X_{ij}e_i'Q'Qe_j}{X_{iit}X_{jjt}} \\
&= \frac{e_i'Q'Qe_i}{X_{iit}}+\frac{e_j'Q'Qe_j}{X_{jjt}}+2\rho_{ijt}\frac{e_i'Q'Qe_j}{\sqrt{X_{iit}X_{jjt}}}
\end{aligned}$$

Moreover,

$$\begin{aligned}
\frac{1}{dt}\mathbb{E}_t\left[\left(\frac{e_i'\sqrt{X_t}dB_tQe_j+e_i'Q'dB_t'\sqrt{X_t}e_j}{\sqrt{X_{iit}X_{jjt}}}\right)\frac{e_i'\sqrt{X_t}dB_tQe_i}{X_{iit}}\right] &= \frac{e_i'X_te_ie_i'Q'Qe_j+e_j'X_te_ie_i'Q'Qe_i}{X_{iit}\sqrt{X_{iit}X_{jjt}}} \\
&= \frac{e_i'Q'Qe_j}{\sqrt{X_{iit}X_{jjt}}}+\frac{\rho_{ijt}e_i'Q'Qe_i}{X_{iit}}\ ,
\end{aligned}$$

and

$$\begin{aligned}
\frac{1}{dt}\mathbb{E}_t\left[\left(\frac{e_i'\sqrt{X_t}dB_tQe_j+e_i'Q'dB_t'\sqrt{X_t}e_j}{\sqrt{X_{iit}X_{jjt}}}\right)\frac{e_j'\sqrt{X_t}dB_tQe_j}{X_{jjt}}\right] &= \frac{e_i'X_te_je_j'Q'Qe_j+e_j'X_te_je_i'Q'Qe_j}{X_{jjt}\sqrt{X_{iit}X_{jjt}}} \\
&= \frac{e_i'Q'Qe_j}{\sqrt{X_{iit}X_{jjt}}}+\frac{\rho_{ijt}e_j'Q'Qe_j}{X_{jjt}}\ .
\end{aligned}$$

Overall, we obtain:

$$\begin{aligned}
v^2(\rho_{ijt}) &= \frac{e_i'Q'Qe_i}{X_{iit}}+\frac{e_j'Q'Qe_j}{X_{jjt}}-2\rho_{ijt}\frac{e_i'Q'Qe_j}{\sqrt{X_{iit}X_{jjt}}}-\rho_{ijt}^2\left(\frac{e_i'Q'Qe_i}{X_{iit}}+\frac{e_j'Q'Qe_j}{X_{jjt}}\right)+2\rho_{ijt}^3\left(\frac{e_i'Q'Qe_j}{\sqrt{X_{iit}X_{jjt}}}\right) \\
&= (1-\rho_{ijt}^2)\left(\frac{e_i'Q'Qe_i}{X_{iit}}+\frac{e_j'Q'Qe_j}{X_{jjt}}\right)+2(\rho_{ijt}^2-1)\rho_{ijt}\frac{e_i'Q'Qe_j}{\sqrt{X_{iit}X_{jjt}}} \\
&= (\rho_{ijt}^2-1)\left(2\rho_{ijt}\frac{e_i'Q'Qe_j}{\sqrt{X_{iit}X_{jjt}}}-\frac{e_i'Q'Qe_i}{X_{iit}}-\frac{e_j'Q'Qe_j}{X_{jjt}}\right)
\end{aligned}$$

This concludes the proof.

B. Proof of Proposition 5

Let $B_t^*=B_t-2\int_0^t\sqrt{X_s}\beta(s,T)Q'ds$ for any $0\le t\le T$. Then, from Lemma 1 below, ζB^* is a local martingale in $\mathbb{R}^{n\times n}$ under probability measure $\mathbb{P}$, where $\zeta_t:=\xi_t\exp(\int_0^t R(X_s)ds)$, $0\le t\le T$. This implies that B^* is a local martingale under risk neutral measure $\mathbb{P}^*$. By Lévy's Theorem, it follows that B^* is a standard Brownian motion in $\mathbb{R}^{n\times n}$ under $\mathbb{P}^*$. Now, let process N^* be defined by:

$$N_t^* = N_t - \int_0^t \Theta^X(\beta(s,T))\lambda_X(X_s)ds \tag{A.3}$$

where N is the counting process counting the number of jumps of X. Then, using Lemma 3 in the Appendix of Duffie, Pan and Singleton (2000), ζN^* is a local martingale in $\mathbb{R}$ under $\mathbb{P}$, which implies that N^* is a local martingale under $\mathbb{P}^*$. By the martingale characterization of intensity,[10] under $\mathbb{P}^*$ process N is a counting process with intensity $\{\lambda_X^*(X_t,t) : 0 \leq t \leq T\}$ such that $\lambda_X^*(x,t) = \lambda_{X,0}^*(t) + tr(\lambda_{X,1}^*(t)X_t)$. The conditional Laplace transform of ξ^X under $\mathbb{P}^*$ is given by:

$$\Theta^{X*}(\Gamma,t) = \mathbb{E}_{t-}^*[\exp(tr(\Gamma\xi^X))] = \mathbb{E}_{t-}[\zeta_T \exp(tr(\Gamma\xi^X))]/\mathbb{E}_{t-}[\zeta_T] = \Theta^X(\Gamma + \beta(t,T))/\Theta^X(\beta(t,T)).$$

It follows that the conditional risk neutral discounted transform of X_T is exponentially affine:

$$\Psi^{X*}(\Gamma, X_t, t, T) := \mathbb{E}^*[\exp(-\int_0^T R(X_s)ds + tr(\Gamma X_T))|\mathcal{F}_t] = \exp(B^*(T-t) + tr(A^*(T-t)X_t))$$

with coefficients $B^*(\tau) \in \mathbb{R}$ and $A^*(\tau) \in \mathcal{S}_n$ that satisfy the same system of matrix Riccati equations as in Proposition 2, but with parameters M, Θ^X and $\lambda_{X,0}$, $\lambda_{X,1}$ replaced by $M^*(t)$, $\Theta^{X*}(\cdot,t)$, $\lambda_{X,0}^*(t)$ and $\lambda_{X,1}^*(t)$, respectively. This concludes the proof.

Lemma 1 *The process $\{\zeta_t B_t^* : 0 \leq t \leq T\}$ is a local martingale under $\mathbb{P}$.*

Proof of Lemma 1. For any $0 \leq t \leq T$, let:

$$B_t^* = B_t - 2\int_0^t \sqrt{X_u}\beta(u,T)Q'du \ .$$

[10] See, e.g., Brémaud (1981).

By Itô's formula, we have for any $0 \leq s \leq T$:

$$\begin{aligned}
\zeta_t B_t^* &= \zeta_s B_s^* + \int_s^t \zeta_{u-} dB_u^* + \int_s^t B_{u-}^* d\zeta_u \\
&\quad + \sum_{s<u\leq t} (\zeta_u - \zeta_{u-})(B_u^* - B_{u-}^*) + \int_s^t d[\zeta, B^*]_u^c \\
&= \zeta_s B_s^* + \int_s^t \zeta_{u-}(dB_u - 2\sqrt{X_u}\beta(u,T)Q' du) + \int_s^t B_u^* d\zeta_u \\
&\quad + \int_s^t \zeta_u d[tr(\beta(u,T)X_u^c), B_u^*] \\
&= \zeta_s B_s^* + \int_s^t \zeta_{u-}(dB_u - 2\sqrt{X_u}\beta(u,T)Q' du) + \int_s^t B_u^* d\zeta_u \\
&\quad +2 \int_s^t \zeta_u d[tr(\beta(u,T)\sqrt{X_u}B_u Q), B_u^*] \\
&= \zeta_s B_s^* + \int_s^t \zeta_{u-}(dB_u - 2\sqrt{X_u}\beta(u,T)Q' du) + \int_s^t B_u^* d\zeta_u \\
&\quad +2 \int_s^t \zeta_u d[tr(\sqrt{X_u}\beta(u,T)Q'B_u'), B_u^*]
\end{aligned}$$

Noting that

$$d[tr(\sqrt{X_u}\beta(u,T)Q'B_u'), B_u^*] = d[tr(\sqrt{X_u}\beta(u,T)Q'B_u'), B_u] = \sqrt{X_u}\beta(u,T)Q' du \ ,$$

we obtain

$$\zeta_t B_t^* = \zeta_s B_s^* + \int_s^t \zeta_{u-} dB_u + \int_s^t B_u^* d\zeta_u \ ,$$

which implies that $\{\zeta_t B_t^* : 0 \leq t \leq T\}$ is a local martingale under $\mathbb{P}$. ■

C. Proof of Proposition 7

We can write the corresponding Hamilton-Jacobi-Bellman (HJB) equation as:

$$\begin{aligned}
0 &= \sup_w \Big\{ J_t + rWJ_W + \eta' XwJ_W W + \frac{1}{2} w' XwW^2 J_{WW} + tr\left[(\Omega\Omega' + MX + XM')J_X\right] \qquad \text{(A.4)} \\
&\quad + Ww' J_{WX} Q' X\rho + \rho' XQJ_{WX} wW + 2tr\left[XJ_{XX}Q'QJ_{XX}\right] \\
&\quad + \lambda_X(X)\mathbb{E}\left[J(t,W,\tilde{X}) - J(t,W,X)\right]\Big\} \ ,
\end{aligned}$$

where $\tilde{X} = X + \xi^X$. The first-order conditions with respect to the portfolio decision w are:

$$\mathbf{0} \;=\; X\eta W J_W + XwW^2 J_{WW} + \left(XWJ_{WX}Q' + XQJ_{WX}\right)\rho W, \tag{A.5}$$

from which we obtain an implicit equation for the optimal portfolio decision:

$$w^\star \;=\; -\frac{\eta J_W}{WJ_{WW}} - \frac{1}{WJ_{WW}}\left(J_{WX}Q' + QJ_{WX}\right)\rho. \tag{A.6}$$

For a given wealth W_t and a matrix X_t, we can make the following guess for the solution of the HJB equation:

$$J(t, W_t, X_t) = \frac{W_t^{1-\gamma}}{1-\gamma}\exp\left(tr\left[A(\tau)X_t\right] + B(\tau)\right). \tag{A.7}$$

Then, the optimal portfolio weights are

$$w^\star \;=\; \frac{1}{\gamma}\left(\eta + 2A(\tau)Q'\rho\right). \tag{A.8}$$

Plugging $w^\star$ into the HJB in (A.4), we see that our guess indeed solves (A.4) and we get the following system of ordinary differential equations for $A(\tau)$ and $B(\tau)$ as displayed in (69) and (70).

D. Volatility of Volatility and Leverage in the Multifactor AJD Return Model

The volatility of returns $V_t := \frac{1}{dt}Var_t(dY_t) = tr(X_t) + \lambda(X_t)\mathbb{E}[\xi^Y\xi^Y]$ follows with straightforward calculations using the properties of the matrix AJD model. To compute the volatility of volatility, note that:

$$dV_t = tr(dX_t) + \mathbb{E}[\xi^Y\xi^Y]tr(\lambda_{YX,1}dX_t) = tr(HdX_t)\ , \tag{A.9}$$

where $H = I_n + \lambda_{YX,1}\mathbb{E}[\xi^Y\xi^Y]$. It then follows:

$$\frac{1}{dt}Var_t(dV_t) = 4tr(HQ'QHX_t) + \lambda_{YX}(X_t)\mathbb{E}\left[tr(H\xi^X)tr(H\xi^X)\right]\ . \tag{A.10}$$

In a similar way, we obtain for the leverage between return and volatility:

$$\frac{1}{dt}Cov_t(dY_t, dV_t) = 2tr(PQHX_t) + \lambda_{YX}(X_t)\mathbb{E}\left[\xi^Y tr(H\xi^X)\right]\ . \tag{A.11}$$

E. Volatility of Volatility and Leverage in the Multivariate AJD Return Model

The elements of the returns covariance matrix $V_{ijt} := \frac{1}{dt}Cov_t(dY_{it}dY_{jt}) = X_{ijt} + \lambda(X_t)\mathbb{E}[\xi_i^Y \xi_j^Y]$ follow with straightforward calculations using the properties of the matrix AJD model. To compute the volatilities of volatility, note that:

$$dV_{ijt} = tr(e_j e_i' dX_t) + \mathbb{E}[\xi_i^Y \xi_j^Y] tr(\lambda_{YX,1} dX_t) = tr(H_{ij} dX_t) \ , \tag{A.12}$$

where $H = e_j e_i' + \lambda_{YX,1}\mathbb{E}[\xi_i^Y \xi_j^Y]$. It then follows:

$$\frac{1}{dt} Var_t(dV_{ijt}) = 4tr(H_{ij}Q'QH_{ij}X_t) + \lambda_{YX}(X_t)\mathbb{E}\left[tr(H_{ij}\xi^X) tr(H_{ij}\xi^X)\right] \ . \tag{A.13}$$

To obtain the leverage between return and volatility, we first have:

$$\begin{aligned} dY_{it} &= (...)dt + e_i'\sqrt{X_t}(dB_t\rho + \sqrt{1-\rho'\rho}dW_t) + e_i' dL_t \\ &= (...)dt + tr(\sqrt{X_t}(dB_t\rho e_i' + \sqrt{1-\rho'\rho}dW_t e_i')) + e_i' dL_t \ . \end{aligned}$$

Therefore, from the independence of B and W:

$$\begin{aligned} \frac{1}{dt} Cov_t(dY_{it}, dV_{ijt}) &= \frac{1}{dt} Cov\left(tr(\sqrt{X_t} dB_t \rho e_i') + e_i' dL_t, dV_{ijt}\right) \\ &= \frac{1}{dt} Cov\left(tr(\sqrt{X_t} dB_t \rho e_i') + e_i' dL_t, tr(H_{ij} dX_t)\right) \\ &= \frac{1}{dt} Cov\left(tr(\sqrt{X_t} dB_t \rho e_i'), tr(H_{ij}\sqrt{X_t} dB_t Q) + tr(H_{ij} Q' dB' \sqrt{X_t})\right) \\ &\quad + \lambda_{YX}(X_t)\mathbb{E}\left[\xi_i^Y tr(H_{ij}\xi_X)\right] \end{aligned}$$

This yields:

$$\frac{1}{dt} Cov\left(tr(\sqrt{X_t} dB_t \rho e_i'), tr(H_{ij}\sqrt{X_t} dB_t Q)\right) = tr(H_{ij}' Q' \rho e_i' X_t) \ ,$$

and

$$\frac{1}{dt} Cov\left(tr(\sqrt{X_t} dB_t \rho e_i'), tr(H_{ij} Q' dB' \sqrt{X_t})\right) = tr(H_{ij} Q' \rho e_i' X_t) \ ,$$

which implies the expression for $\frac{1}{dt} Cov_t(dY_{it}, dV_{ijt})$.

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Table I
Volatility and Leverage Structure in the Single-Asset Model

The table summarizes the conditional variance-covariance and co-volatility structure ($V_t := Var_t(dY_t)/dt$) for the single-asset matrix AJD model together with the leverage effect. The matrix H is defined as $H := I_n + \lambda_{YX,1}\mathbb{E}\left[(\xi^Y)^2\right]$.

	$\lambda_{YX,0} = 0,\ \lambda_{YX,1} = 0$	Unconstrained
$Var_t(dY_t)/dt$:	$tr(X_t)$	$tr(X_t) + \lambda_{YX}(X_t)\mathbb{E}\left[(\xi^Y)^2\right]$
$Var_t(dV_t)/dt$:	$4tr(Q'QX_t)$	$4tr(HQ'QHX_t) + \lambda_{YX}(X_t)\mathbb{E}\left[tr(H\xi^X)^2)\right]$
$Cov_t(dY_t, dV_t)/dt$:	$2tr(PQX_t)$	$2tr(PQHX_t) + \lambda_{YX}(X_t)\mathbb{E}\left[\xi^Y tr\left(H\xi^X\right)\right]$

Table II
Variance-Covariance and Leverage Structure in Multi-Asset Model

The table summarizes the conditional variance-covariance and co-volatility structure ($V_{ijt} = Cov(dY_{it}dY_{jt})/dt$) for the multi-asset matrix AJD model, together with the leverage effect. The matrix H_{ij} is defined as $H_{ij} := e_j e_i' + \lambda_{YX,1}\mathbb{E}\left[\xi_i^Y \xi_j^Y\right]$.

	$\lambda_{YX,0} = 0,\ \lambda_{YX,1} = 0$	Unconstrained
$Cov_t(dY_{it}dY_{jt})/dt$:	X_{ijt}	$X_{ijt} + \lambda_{YX}(X_t)\mathbb{E}\left[\xi_i^Y \xi_j^Y\right]$
$Var_t(dV_{iit}dV_{jjt})/dt$:	$4X_{ijt}Q_i'Q_j$	$4tr(H_{ij}Q'QH_{ij}X_t) + \lambda_{YX}(X_t)\mathbb{E}\left[tr(H_{ij}\xi^X)^2)\right]$
$Cov_t(dY_{it}, dV_{ijt})/dt$:	$\rho'(X_{iit}Q_j + X_{ijt}Q_j)$	$\rho'Q(H_{ij} + H_{ij}')X_t e_i + \lambda_{YX}(X_t)\mathbb{E}\left[\xi_i^Y tr\left(H_{ij}\xi^X\right)\right]$

Table III
Parameter Values for Simulations

Panel A: Correlation

$$Q = \begin{pmatrix} 0.0300 & 0.2100 & 0.0625 \\ -0.1750 & 0.0325 & 0.1375 \\ -0.0300 & 0.1000 & 0.0250 \end{pmatrix} \qquad M = \begin{pmatrix} -3.6000 & 0.2100 & 0.5000 \\ 1.6790 & -1.9800 & -0.4000 \\ 0.8880 & -0.2400 & -2.4200 \end{pmatrix}$$

$$\lambda_1 = \begin{pmatrix} 0.0004 & 0.0015 & -0.0020 \\ 0.0015 & 0.0010 & 0.0003 \\ -0.0020 & 0.0003 & 0.0200 \end{pmatrix} \qquad \Sigma_X = \begin{pmatrix} 0.0004 & 0.0005 & 0.0002 \\ 0.0005 & 0.0010 & 0.0003 \\ 0.0002 & 0.0003 & 0.0010 \end{pmatrix}$$

$k = 10, \quad \lambda_0 = 0.001$

Panel B: Term structure

$$Q = \begin{pmatrix} 1.1328 & 1.2950 & -0.2410 \\ 0 & 0.0453 & 0.0667 \\ 0 & 0 & 0.6443 \end{pmatrix} \qquad M = \begin{pmatrix} -0.8506 & 0 & 0 \\ 0.2246 & -0.0787 & 0 \\ 0.2800 & 0.1250 & -0.9121 \end{pmatrix}$$

$$D = \begin{pmatrix} 0.1432 & 0.0747 & -0.0017 \\ 0.0747 & -0.0047 & 0.0390 \\ -0.0017 & 0.0390 & 0.0477 \end{pmatrix} \times 0.01 \quad \Sigma^X = \begin{pmatrix} 0.1000 & -0.0700 & 0.0800 \\ -0.0700 & 0.1000 & -0.0600 \\ 0.0800 & -0.0600 & 0.1000 \end{pmatrix}$$

$$\lambda_1 = \begin{pmatrix} 0.0008 & 0.0000 & 0.0001 \\ 0.0000 & 0.0008 & 0.0001 \\ 0.0001 & 0.0001 & 0.0002 \end{pmatrix} \times 0.01 \qquad \begin{array}{ll} k = 10 & \lambda_0 = 0.05 \\ \delta = 0.01 & \beta(t,T) = 0.002 I_n \end{array}$$

Panel C: Portfolio allocation

$$Q = \begin{pmatrix} 0.160 & 0.083 \\ -0.021 & 0.009 \end{pmatrix} \qquad M = \begin{pmatrix} -1.122 & 0.747 \\ 0.884 & -0.888 \end{pmatrix}$$

$$\xi^X = \begin{pmatrix} 0.0150 & 0.0110 \\ 0.0110 & 0.0100 \end{pmatrix} \times 0.01 \qquad \eta = \begin{pmatrix} 4.612 \\ 2.891 \end{pmatrix} \quad \rho = \begin{pmatrix} -0.279 \\ -0.247 \end{pmatrix}$$

Table IV
R^2-values for regression equation for excess returns

The table reports R^2-values for different regression equations for excess returns (regressions (A) through (E)). The estimations are based on monthly data spanning a time period of 20 years. The calculations are based on Newey-West adjusted heteroscedastic-serial consistent least-squares regressions.

	R^2-Values				
	(A)	(B)	(C)	(D)	(E)
$R^e(0,1,2)$	0.2469	0.3700	0.4663	0.4677	0.2662
$R^e(0,1,3)$	0.2834	0.4150	0.5189	0.5214	0.3071
$R^e(0,1,4)$	0.2881	0.4248	0.5278	0.5310	0.3147
$R^e(0,1,5)$	0.2845	0.4267	0.5264	0.5302	0.3139

Table V
Beta coefficients

The table reports the beta coefficients of different regression equations for excess returns (regressions (A) through (E)). The estimations are based on monthly data spanning a time period of 20 years. T-statistics are shown in parentheses and are based on Newey-West standard errors with a lag truncation parameter of 11.

Regression (A)

	$R^e(0,1,2)$		$R^e(0,1,3)$		$R^e(0,1,4)$		$R^e(0,1,5)$	
β_0	-0.0614	(-2.90)	-0.0635	(-2.84)	-0.0635	(-2.75)	-0.0632	(-2.69)
β_1	9.3517	(1.99)	9.9435	(2.20)	10.0847	(2.25)	10.0983	(2.26)
β_2	-17.4608	(-2.43)	-18.7929	(-2.65)	-19.1366	(-2.69)	-19.1669	(-2.68)
β_3	10.6014	(2.66)	11.4519	(2.88)	11.6716	(2.90)	11.6852	(2.88)

Regression (B)

	$R^e(0,1,2)$		$R^e(0,1,3)$		$R^e(0,1,4)$		$R^e(0,1,5)$	
β_0	-0.0274	(-1.26)	-0.0283	(-1.18)	-0.0274	(-1.10)	-0.0263	(-1.04)
β_1	8.2304	(1.70)	8.7808	(1.90)	8.8943	(1.95)	8.8830	(1.96)
β_2	-15.4556	(-2.15)	-16.7138	(-2.38)	-17.0081	(-2.42)	-16.9938	(-2.42)
β_3	9.3568	(2.40)	10.1614	(2.63)	10.3504	(2.66)	10.3363	(2.64)
β_4	-61.8879	(-2.93)	-64.1701	(-2.85)	-65.6952	(-2.85)	-67.0706	(-2.90)

Regression (C)

	$R^e(0,1,2)$		$R^e(0,1,3)$		$R^e(0,1,4)$		$R^e(0,1,5)$	
β_0	-0.0378	(-1.99)	-0.0391	(-1.89)	-0.0383	(-1.77)	-0.0370	(-1.67)
β_1	8.1996	(2.05)	8.7488	(2.30)	8.8622	(2.34)	8.8515	(2.33)
β_2	-14.5328	(-2.39)	-15.7527	(-2.63)	-16.0461	(-2.65)	-16.0471	(-2.62)
β_3	8.5880	(2.57)	9.3607	(2.78)	9.5490	(2.78)	9.5477	(2.73)
β_4	-106.3084	(-7.49)	-110.4347	(-6.57)	-111.9999	(-6.05)	-112.6382	(-5.81)
β_5	61.3822	(4.01)	63.9305	(4.12)	63.9859	(4.00)	62.9674	(3.88)

Regression (D)

	$R^e(0,1,2)$		$R^e(0,1,3)$		$R^e(0,1,4)$		$R^e(0,1,5)$	
β_0	-0.0380	(-2.00)	-0.0393	(-1.91)	-0.0385	(-1.79)	-0.0372	(-1.70)
β_1	7.9813	(2.04)	8.4492	(2.32)	8.5225	(2.37)	8.4834	(2.37)
β_2	-14.1453	(-2.41)	-15.2210	(-2.71)	-15.4432	(-2.74)	-15.3940	(-2.71)
β_3	8.3703	(2.61)	9.0620	(2.89)	9.2102	(2.89)	9.1806	(2.85)
β_4	-103.9258	(-7.88)	-107.1660	(-6.71)	-108.2929	(-6.05)	-108.6222	(-5.73)
β_5	62.9070	(3.63)	66.0224	(3.74)	66.3583	(3.67)	65.5376	(3.57)
β_6	-0.1525	(-0.47)	-0.2092	(-0.66)	-0.2373	(-0.72)	-0.2571	(-0.76)

Regression (E)

	$R^e(0,1,2)$		$R^e(0,1,3)$		$R^e(0,1,4)$		$R^e(0,1,5)$	
β_0	-0.0548	(-2.72)	-0.0561	(-2.62)	-0.0556	(-2.50)	-0.0549	(-2.42)
β_1	8.4322	(1.77)	8.9212	(2.00)	8.9973	(2.07)	8.9534	(2.08)
β_2	-15.8968	(-2.20)	-17.0541	(-2.48)	-17.2872	(-2.55)	-17.2196	(-2.55)
β_3	9.7212	(2.45)	10.4733	(2.74)	10.6307	(2.79)	10.5892	(2.78)
β_6	-0.4995	(-1.37)	-0.5553	(-1.44)	-0.5907	(-1.47)	-0.6219	(-1.51)

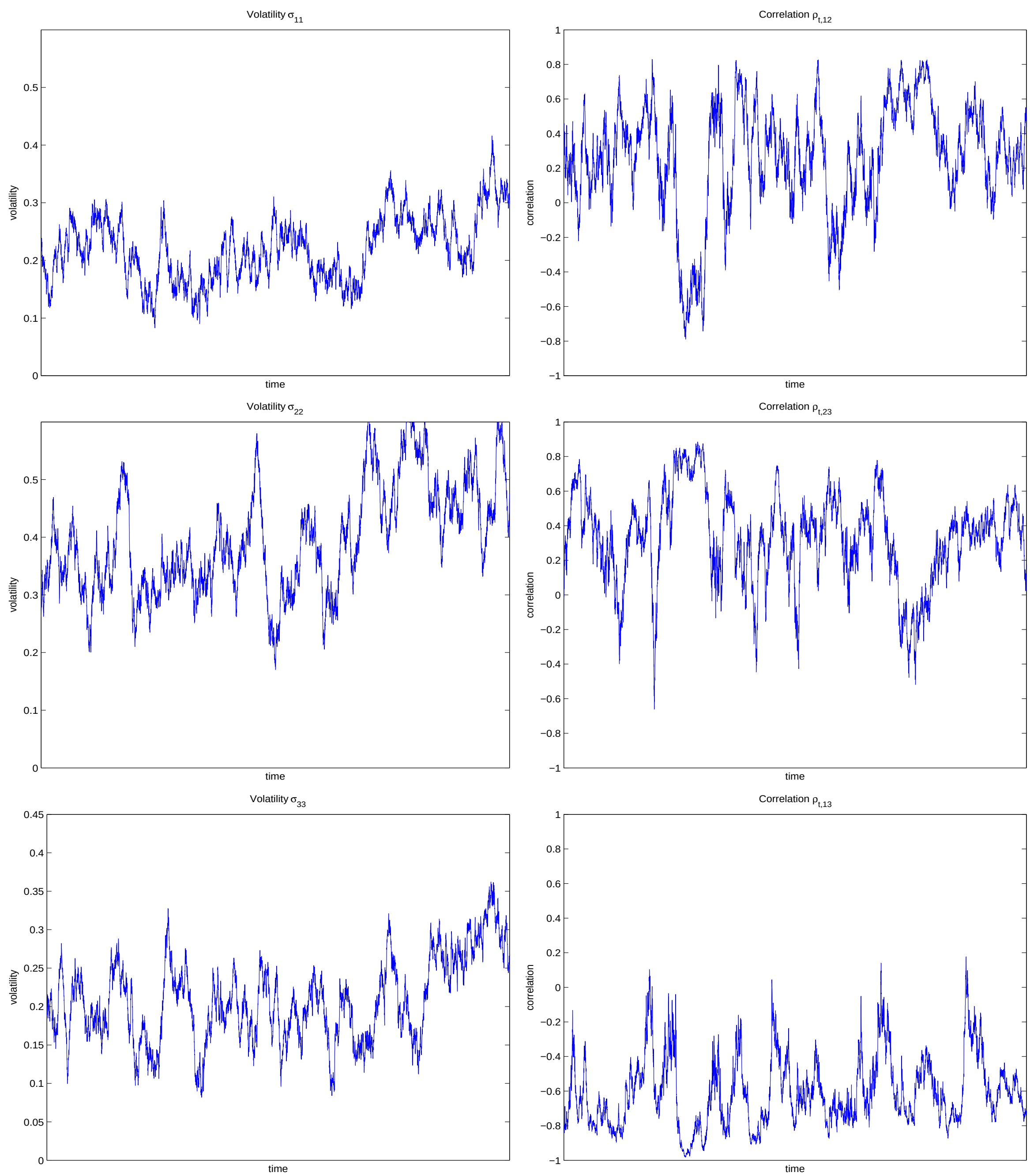

Figure 1. Volatility and correlation process.
Panels on the left present realized trajectories of volatilities $\sqrt{X_{11}}$, $\sqrt{X_{22}}$, $\sqrt{X_{33}}$ (from the top to the bottom) simulated under the matrix AJD process in Assumption 1 for the parameters reported in Panel A of Table III. Panels on the right present realized trajectories of correlations ρ_{12}, ρ_{23}, ρ_{13} (from the top to the bottom) simulated under the same matrix AJD process.

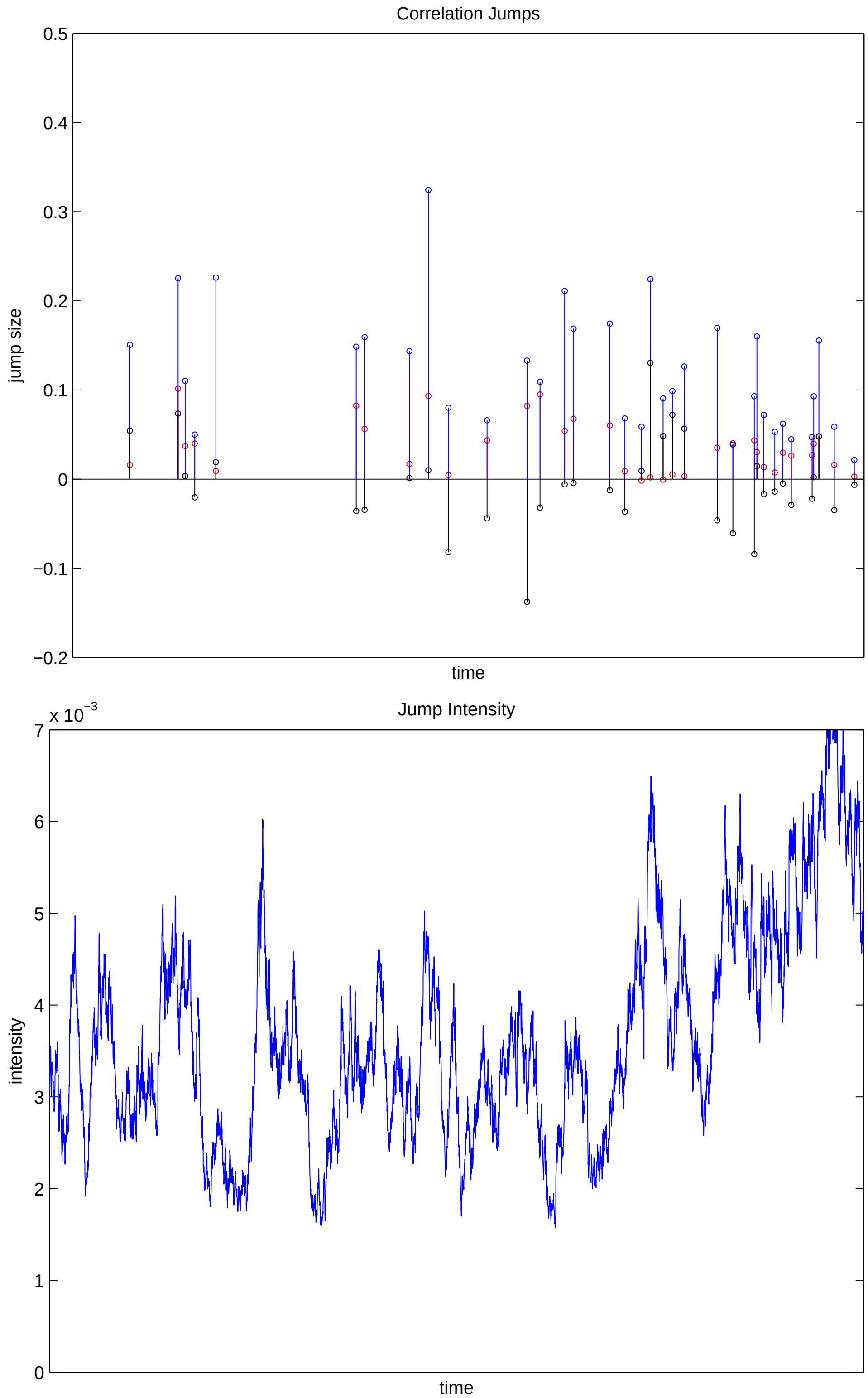

Figure 2. Jump sizes and jump intensity for correlation processes.
The top panel presents realized correlation jump sizes for ρ_{12} (red), ρ_{23} (green), ρ_{13} (blue) simulated under the matrix AJD process in Assumption 1 for the parameters reported in Panel A of Table III. The bottom panel presents realized intensities $\lambda(X_t)$ simulated under the same matrix AJD process.

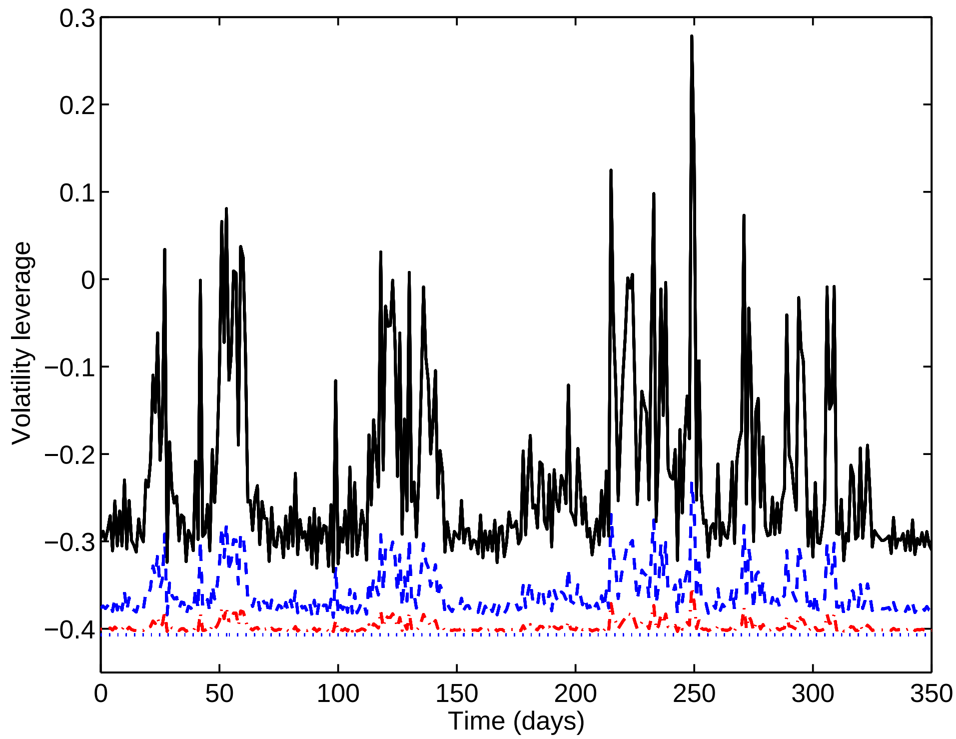

Figure 3. Volatility leverage with and jump specification.
The figure displays the volatility leverage $Corr_t(dY_{it}, dV_{iit})$ for an asset i within a multivariate return setting. The dotted line represents the case when there are no jumps, neither in the covariances nor in the return process. The dash-dotted line correspond to the case when there are jumps the return process only and the dashed line to the case when there are jumps in the covariance process only. The solid line represents the volatility leverage, when there are jumps both in the covariances and in the return process.

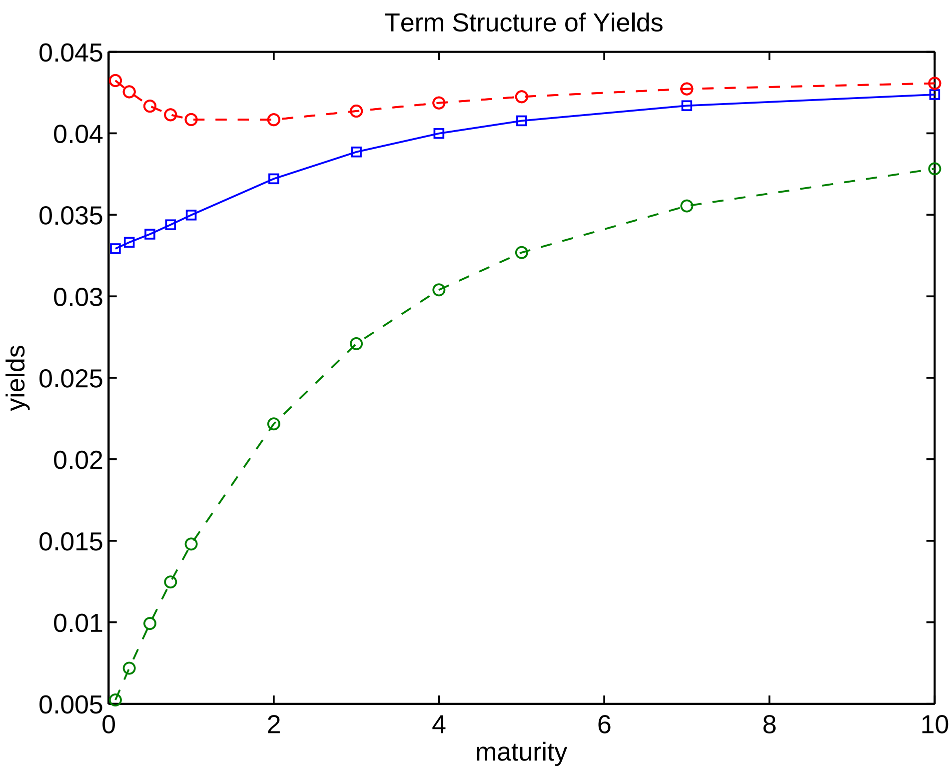

Figure 4. Term structure shapes.
The figure shows different shapes of the term structure of bond yields for different maturities, ranging from one month to ten years. The solid line represent the mean term structure for the simulated twenty-year time series using the parameter values .

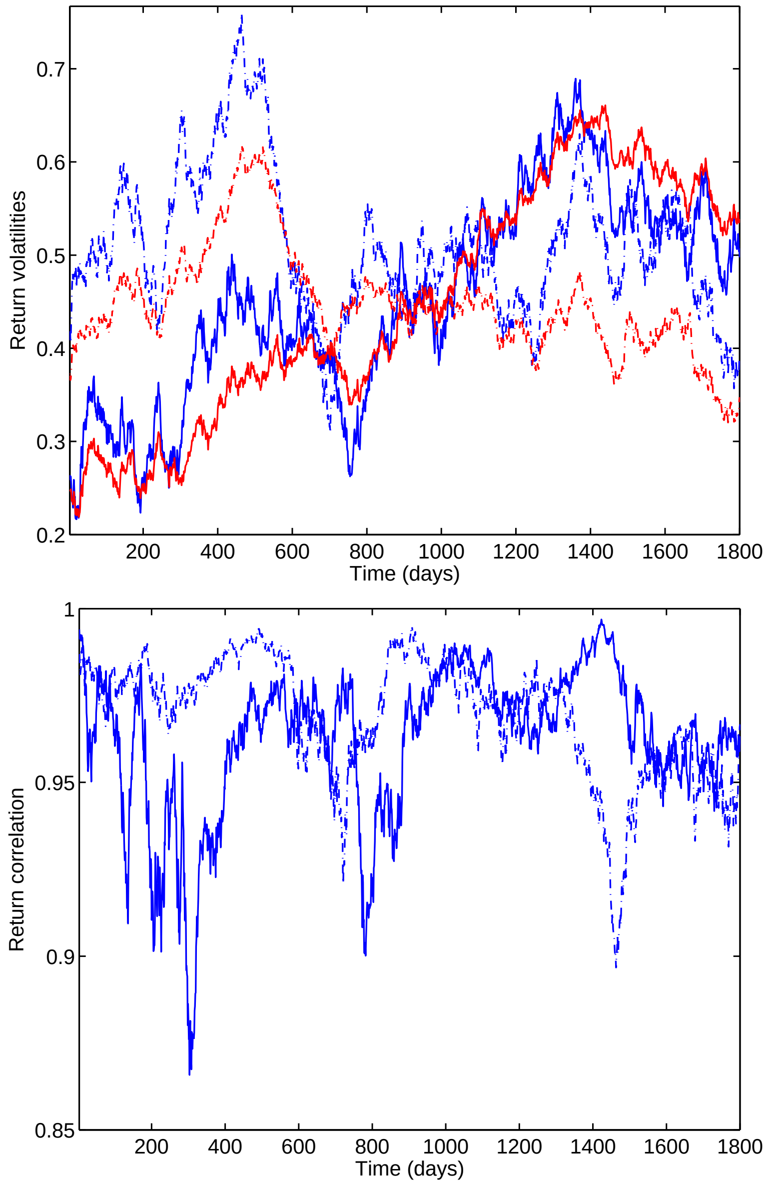

Figure 5. Simulation of return volatilities and return correlations.
The upper panel of the figure displays the simulated return variances using the parameter values in Panel C of Table III, together with $\Omega\Omega' = kQQ'$ with $k = 10$. We perform a simulation for daily data over a time horizon of five years. For the constant part of the jump intensity, we set $\lambda_0 = 0$. For the matrix λ_1 we choose a value such that the covariance matrix exhibits jumps at each time step with an average probability of approximately 1%.

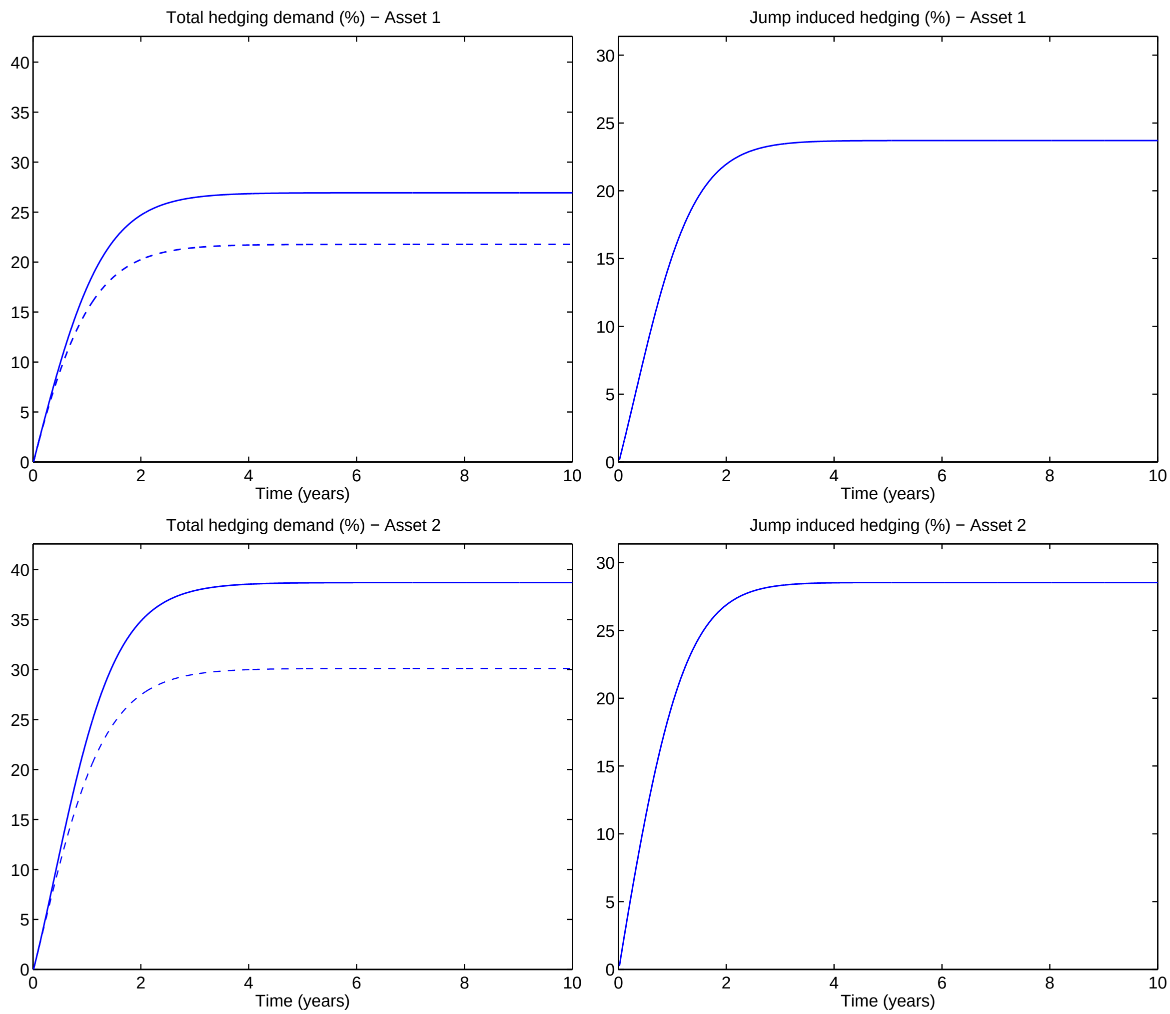

Figure 6. Intertemporal hedging demand and jumps in covariances.
The figure displays the intertemporal hedging demand in an optimal portfolio allocation for a CRRA utility investor with relative risk aversion coefficient $\gamma = 6$. In the left panels, we calculate the hedging demand as a fraction of the myopic portfolio for different time horizons, ranging from zero to ten years. The dotted lines represent the hedging demand when there is no jump risk in covariances and the solid lines when there are jumps in covariances. The left panels plot the fraction of the hedging demand induced by the presence of jumps in covariances.

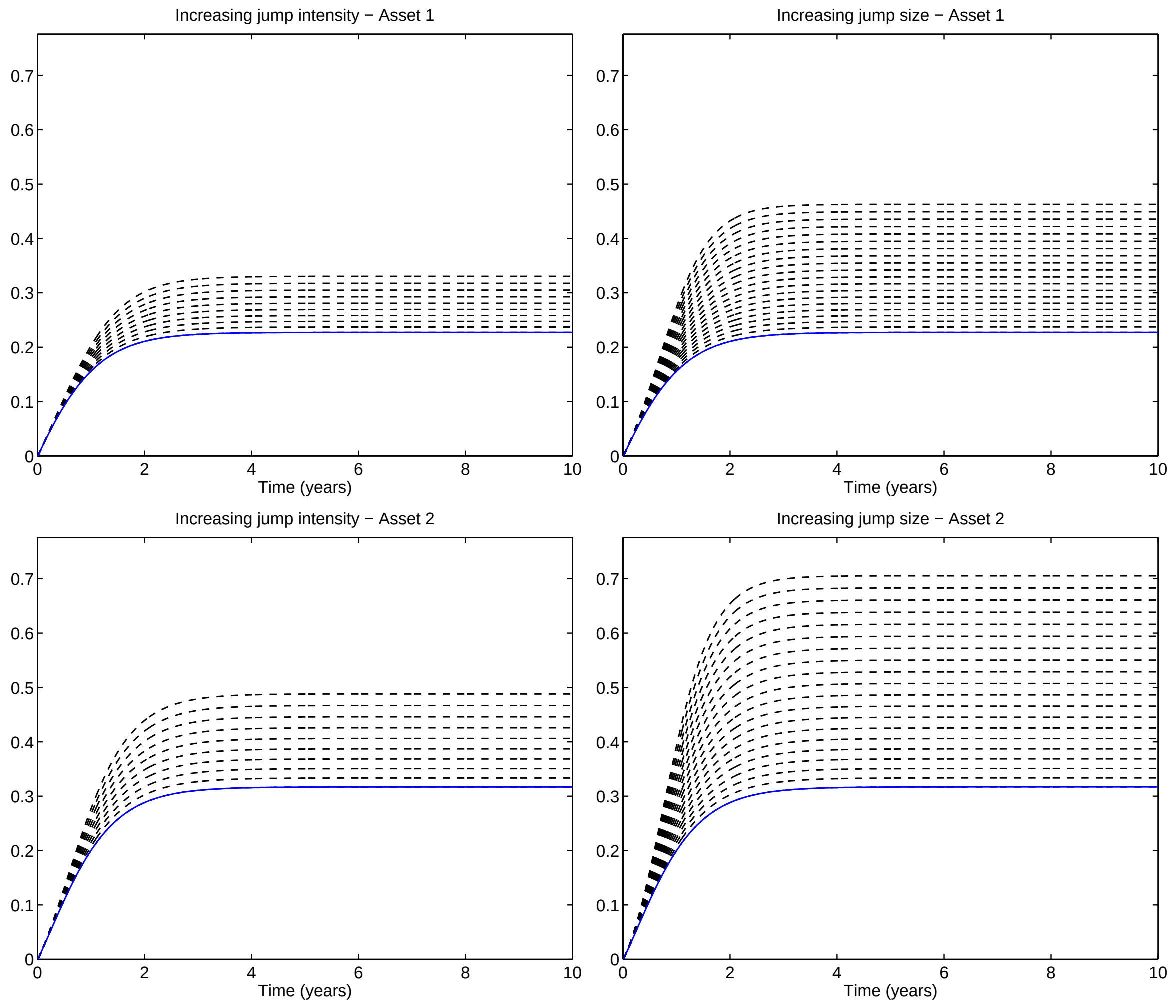

Figure 7. Intertemporal hedging with different jump intensities and sizes. The figure displays the intertemporal hedging demand in an optimal portfolio allocation for a CRRA utility investor with relative risk aversion coefficient $\gamma = 6$. We calculate the hedging demand as a fraction of the myopic portfolio for different time horizons, ranging from zero to ten years. The solid line represents the hedging demand when there is no covariance jump risk present. For the left panels, we start with an average daily jump probability of around 0.3% per day and we increase this number to 10%, while keeping ξ^X fixed. For the right panels, we fix the average daily jump probability to 1% and we vary the jump sizes from $\frac{1}{5}\xi^X$ to $4\xi^X$.