Financially Constrained Arbitrage
and Cross-Market Contagion*

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Abstract

We propose a discrete time infinite horizon equilibrium model of financial markets in which arbitrageurs have valuable investment opportunities but face financial constraints. The investment opportunities, varying in horizon and volatility, are provided by pairs of similar assets trading at different prices in segmented markets. By exploiting these opportunities, arbitrageurs reduce the segmentation of markets, providing liquidity to other investors. The financial constraints arise from the arbitrageurs’ need to collateralize separately their positions in each asset. We characterize the optimal investment policy of arbitrageurs and derive implications for asset prices.

Keywords: Financial constraints, arbitrage, liquidity, contagion.

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1 Introduction

Financially Constrained Arbitrage

- Arbitrageurs:
  - Hedge funds, dealers, I-banks,...
  - Exploit price wedge between similar (portfolios of) assets.
  - Bring asset prices close to fundamentals.
  - Provide liquidity to other investors.

- Textbook:
  - Costless arbitrage.
  - ⇒ Absence of Arbitrage Opportunities.

- Reality:
  - Margin requirements + Access to external capital costly and limited.
  - If/when this is first order, there should be consequences for asset prices and liquidity.

The 1998 Financial Crisis

- Arbitrage.
  - Many hedge funds bet on the convergence of prices of similar-payoff assets.
  - During the crisis, prices diverged.
  - Hedge funds experienced heavy losses + distress ⇒ Liquidated positions deemed profitable.

- Asset prices and liquidity.
  - Prices were pushed away from fundamentals.
  - Liquidity dried up.
  - Contagion: Losses-liquidations chain propagated, affecting prices and liquidity in other markets.

- Public policy.
  - Liquidation of LTCM’s large positions could disrupt financial markets.
  - Concerns about systemic risk ⇒ US Fed coordinated LTCM’s rescue.
Some Questions

- Arbitrageurs’ strategy
  - What is the optimal investment strategy of an arbitrageur with financial constraints?
  - Financial constraints create a need for risk management.
  - How is this resolved when there are multiple arbitrage opportunities with different characteristics?
  - How does an arbitrageur’s optimal policy vary with his wealth? Asset characteristics?

- Asset prices and liquidity
  - Financial constraints ⇒ Wealth effects ⇒ Price and liquidity linkage across markets.
  - Which convergence spreads are more sensitive to changes in arbitrageurs’ capital?
  - How much of time-variation in convergence spreads is explained by contagion vs. fundamentals?
  - Is diversification of arbitrageurs effective despite contagion effects?

- Welfare and public policy
  - Do contagion and liquidity linkages across markets have welfare implications?
  - What are possible policy responses? Capital adequacy for arbitrageurs? Margin requirements?

Main Contributions (1)

- Model of financial markets in which some investors (arbitrageurs) face:
  - Better investment opportunities than others: Market segmentation.
  - Financial constraints: Margin constraints.

- Fairly tractable and flexible framework (e.g. closed for solutions):
  - Infinite horizon + Stationarity.
  - Model intertemporal aspects of arbitrage.
  - Multiple investment opportunities with different characteristics: volatility, horizon, market size...
  - Arbitrage: Riskfree or risky (fundamental risk and/or preference risk).
Main Contributions (2)

- Riskfree arbitrage case (for now).
- Equilibrium solved in closed form with intuitive properties:
  - Equalization of one-period excess returns per unit of collateral across arbitrage opportunities.
  - Risk premia are larger for arbitrage opportunities with longer time to convergence.
  - The effect of maturity is greater for arbitrage opportunities with more volatility.
- Effect of fundamental shocks:
  - Effect on arbitrageurs’ wealth and on risk premia.
  - Effect is stronger for arbitrage opportunities with longer time to convergence.
- Effect of integration of (arbitrage) markets:
  - Integration $\Rightarrow$ Contagion, i.e., correlation between otherwise unrelated assets.
  - Lowers the variance of (global) arbitrageurs’ portfolio.

Related Literature (Incomplete)

- Shleifer and Vishny (1997)
- Basak and Croitoru (2000,2006)
- Xiong (2001), Kyle and Xiong (2001)
- Gromb and Vayanos (2002)
- Pavlova and Rigobon (2005)
- Gorton and He (2006)
- Krishnamurthy and He (2007)
- Brunnermeier and Pedersen (2007)
- Kondor (2007a, 2007b)
- Anshuman and Viswanathan (2007)
- Hendershott, Moulton and Seasholes (2007)
2 The Model

The model has an infinite number of discrete time periods indexed by \( t \in \mathbb{Z} \). Section 2.1 describes the universe of assets which consists of a riskless asset and many risky assets. These assets can be traded by two classes of agents: outside investors and arbitrageurs, described in Section 2.2 and Section 2.3 respectively. All agents can invest in the riskless asset. However, only arbitrageurs can invest in all risky assets, while each outside investor can invest in only one specific risky asset.

2.1 Assets

The riskless asset has an exogenous return \( r \).

In each period \( t_0 \in \mathbb{Z} \), \( N \) pairs of zero net supply risky assets come into existence. We denote each such pair of assets as \((A_{n,t_0}, B_{n,t_0})\) with \( n = 1, ..., N \) indexing the pair, and \( M > 0 \) being the exogenous (time to) maturity common to all the assets.\(^2\) For \( i = A, B \), asset \( i_{n,h} \) pays a random stream of dividends in each period \( t \) from its initiation period \( t_0 \) to its maturity period \( h \equiv t_0 + M \):

\[
\delta_{i_{n,h},t}^i \equiv \bar{\delta} + \epsilon_{i_{n,h},t}^i,
\]

where \( \bar{\delta} \) is a constant mean, and \( \epsilon_{i_{n,h},t}^i \) is a zero-mean random variable revealed in period \( t \), and symmetrically distributed around zero over the finite support \([-\tau_n, +\tau_n]\).\(^3\) More specifically, we assume that the variables \( \left( \epsilon_{i_{n,h},t}^i / \tau_n \right) \) are identically and symmetrically distributed over \([-1, +1]\). Moreover, \( \epsilon_{n,h,t}^i \) and \( \epsilon_{n',h',t'}^i \) are independently distributed unless \( n = n', h = h' \) and \( t = t' \). Instead, \( \epsilon_{n,h,t}^A \) and \( \epsilon_{n,h,t}^B \) are assumed to be correlated. This specification nests the case in which the dividends paid by assets \( A_{n,h} \) and \( B_{n,h} \) are correlated but different (\( \epsilon_{n,h,t}^A \neq \epsilon_{n,h,t}^B \)) and that in which they are identical (\( \epsilon_{n,h,t}^A = \epsilon_{n,h,t}^B \)). These cases will correspond to arbitrage opportunities with and without fundamental risk respectively.

For \( i = A, B \), we denote by \( p_{i_{n,h},t} \) the ex-dividend price of asset \( i_{n,h} \) in period \( t \), and set

\[
\phi_{i_{n,h},t}^i \equiv \sum_{s=t+1}^{h} \frac{E_t \left( \delta_{i_{n,h,s}}^i \right)}{(1 + r)^{s-t}} - p_{i_{n,h},t}^i = \bar{\delta} \left( 1 - \frac{1}{(1 + r)^{h-t}} \right) - p_{i_{n,h},t}^i.
\]

The variable \( \phi_{i_{n,h},t}^i \) represents the expected excess return per share of asset \( i_{n,h} \) and, for simplicity, we refer to it as asset \( i_{n,h} \)'s risk premium in period \( t \). Since asset \( i_{n,h} \) pays no dividends after period \( h \), we have \( p_{i_{n,h},t}^i = 0 \), and therefore \( \phi_{i_{n,h},t}^i = 0 \) for all \( t \geq h \).

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\(^2\)The assumptions of exogenous riskless return, and zero net supply assets are for simplicity. In particular, the zero net supply assumption ensures that arbitrageurs hold opposite positions in the two risky assets. The assumption that all asset pairs come into existence with the same maturity \( M \) is only but a simple way of ensuring that in each period assets with a variety of maturities exist. It will also ensure the model’s stationarity.

\(^3\)The bounded support assumption plays a role for the financial constraint (see below).
2.2 Outside Investors

2.2.1 Market Segmentation

For the outside investors, the markets for all risky assets are segmented. Specifically, each outside investor can form portfolios of only two assets: the riskless asset and a single and specific risky asset. For $i = A, B$, the outside investors who can invest in asset $i_{n,h}$ form overlapping two-period generations, living in periods $t$ and $t + 1$ for $t \in \{h - M, ..., h - 1\}$. We refer to the generation living in periods $t$ and $t + 1$ as $(i_{n,h}, t)$-investors.$^{4}$

Market segmentation is taken as given, i.e., $(i_{n,h}, t)$-investors are assumed to face prohibitively large transaction costs for investing in any other risky asset than asset $i_{n,h}$. These costs can be due to physical factors (e.g., distance), information asymmetries or institutional constraints. Market segmentation is a realistic assumption in many contexts. In an international context, for example, it is well known that investors mainly hold domestic assets.

2.2.2 Endowment Shocks

We assume that $(i_{n,h}, t)$-investors are competitive and form a continuum with measure $\mu_n$. They have initial wealth $w_{n,h,t}^i$ in period $t$ and maximize the expected utility of wealth in period $t + 1$, $w_{n,h,t+1}^i$, which we assume to be exponential utility, i.e.,

$$-\exp(-\alpha w_{n,h,t+1}^i) \text{ with } \alpha > 0.$$  \hspace{1cm} (3)

In period $t$, each $(i_{n,h}, t)$-investor anticipates that in period $t + 1$ he will receive an endowment correlated with the dividend $\delta_{n,h,t+1}^i$ paid by asset $i_{n,h}$. Specifically, the endowment is equal to

$$u_{n,h,t}^i \cdot \epsilon_{n,h,t+1}^i.$$  \hspace{1cm} (4)

where $u_{n,h,t}^i$ is a coefficient revealed in period $t$.

The coefficient $u_{n,h,t}^i$, which can be positive or negative, measures the extent to which the endowment covaries with $\delta_{n,h,t+1}^i$. If $u_{n,h,t}^i$ is large and positive, the shock and the dividend are highly positively correlated, and thus the willingness of $(i_{n,h}, t)$-investors to hold asset $i_{n,h}$ in period $t$ is low. Conversely, if $u_{n,h,t}^i$ is large and negative, the shock and the dividend are highly negatively correlated, and thus the $(i_{n,h}, t)$-investors are keen to hold asset $i_{n,h}$ in period $t$ for insurance purposes. Therefore, we refer to $u_{n,h,t}^i$ as $(i_{n,h}, t)$-investors’ supply shock in period $t$, to emphasize that their demand for asset $i_{n,h}$ in period $t$ decreases with $u_{n,h,t}^i$.\cite{5}

$^{4}$The assumption that outside investors form overlapping generations is practical but not crucial.

$^{5}$To be consistent with the zero net supply assumption, the endowments can be interpreted as positions in a different but correlated asset, e.g. labor income. This specification of endowments is quite standard in the market microstructure literature (see O’Hara, 1995).
This specification nests the case in which \( u_{n,h,t} \) is effectively stochastic, i.e., unknown until period \( t \), and that in which it is deterministic. These cases correspond to arbitrage with and without preference risk respectively.

We assume that supply shocks are opposites for \((A_{n,h,t})-\) and \((B_{n,h,t})-\)investors, i.e.,

\[
u_{n,h,t}^A = -u_{n,h,t}^B \equiv u_{n,h,t}.
\]

(5)

Because \((A_{n,h,t})-\)investors and \((B_{n,h,t})-\)investors incur different shocks, they have different propensities to hold risky assets \(A_{n,h} \) and \(B_{n,h} \) respectively. However, they cannot realize the potential gains from trade due to market segmentation. This creates a role for arbitrageurs.

2.2.3 The Outside Investors’ Problem

Each \((i_{n,h,t})-\)investor chooses \(y_{i_{n,h,t}} \), her holding of asset \(i_{n,h} \) in period \(t \), to maximize their expected utility of period \(t + 1 \) wealth. Their optimization problem, \(P_{i_{n,h,t}} \), is

\[
\max_{y_{i_{n,h,t}}} -E_t \exp \left(-\alpha w_{i_{n,h,t+1}} \right),
\]

subject to the dynamic budget constraint. In period \(t \), each \((i_{n,h,t})-\)investor invests \(y_{i_{n,h,t}} \), in asset \(i_{n,h} \) and the rest of his wealth, \((w_{i_{n,h,t}} - y_{i_{n,h,t}}p_{i_{n,h,t}}) \), in the riskfree asset. By period \(t + 1 \), the first position is worth \(y_{i_{n,h,t}} (\delta_{i_{n,h,t+1}} + p_{i_{n,h,t+1}}) \) while the second has grown to \((1 + r)(w_{i_{n,h,t}} - y_{i_{n,h,t}}p_{i_{n,h,t}}) \). Moreover, the investor has received an endowment \(u_{i_{n,h,t}} \). Therefore, the \((i_{n,h,t})-\)investors’ dynamic budget constraint can be written as follows

\[
w_{i_{n,h,t+1}} = y_{i_{n,h,t}} (\delta_{i_{n,h,t+1}} + p_{i_{n,h,t+1}}) + (1 + r)(w_{i_{n,h,t}} - y_{i_{n,h,t}}p_{i_{n,h,t}}) + u_{i_{n,h,t}},
\]

(7)

\[
\quad = (1 + r)w_{i_{n,h,t}} + y_{i_{n,h,t}} [(1 + r)\delta_{i_{n,h,t}} - \delta_{i_{n,h,t+1}}] + (y_{i_{n,h,t}} + u_{i_{n,h,t}})\epsilon_{i_{n,h,t+1}}.
\]

(8)

The term \(y_{i_{n,h,t}} [(1 + r)\delta_{i_{n,h,t}} - \delta_{i_{n,h,t+1}}] \) is the capital gain realized by the \((i_{n,h,t})-\)investors between periods \(t \) and \(t + 1 \) over and above the riskfree return. This gain can be uncertain due to the possible uncertainty of price \(p_{i_{n,h,t+1}} \). The term \((y_{i_{n,h,t}} + u_{i_{n,h,t}})\epsilon_{i_{n,h,t+1}} \) represents the risk borne by the \((i_{n,h,t})-\)investors between periods \(t \) and \(t + 1 \) due to the uncertainty of the dividend \(\delta_{i_{n,h,t+1}} \). It is the sum of the uncertain component of the dividend and of the uncertain period \(t + 1 \) endowment.

2.3 Arbitrageurs

Compared to outside investors, arbitrageurs have better investment opportunities but face financial constraints.
2.3.1 Utility

Arbitrageurs are infinitely lived, i.e., they live in all periods \( t \in \mathbb{Z} \). They are competitive and form a continuum with measure 1. We denote their wealth in period \( t \) as \( W_t \).

We assume that in any given period \( t \), arbitrageurs consume a fraction \( g \) of their wealth, where \( g \) satisfies \((1 - g)(1 + r) < 1\) so that arbitrageurs’ total wealth remains bounded. The arbitrageurs’ utility is

\[
E_t \sum_{s=t}^{\infty} \beta^{s-t} U(g \cdot W_s),
\]

where \( U \) is a well-behaved utility function and \( \beta < 1 \) a discount factor. When utility is logarithmic, the arbitrageurs’ assumed linear consumption rule is optimal.

2.3.2 Financial Constraints

Unlike other investors, arbitrageurs can invest in all risky assets as well as in the riskless asset. Therefore, an arbitrageur’s dynamic budget constraint can be written as

\[
W_{t+1} = (1 - g) \left[ (1 + r) \left( W_t - \sum_{i=A,B} \sum_{n=1}^{N} \sum_{h=t+1}^{t+M} x_{n,h,t} p_{n,h,t}^i \right) + \sum_{i=A,B} \sum_{n=1}^{N} \sum_{h=t+1}^{t+M} x_{n,h,t} \left( \delta_{n,h,t+1}^i + p_{n,h,t+1}^i \right) \right],
\]

where the first term inside the bracket is the payoff from the riskless asset and the second term the payoff from the risky assets. This expression can be rewritten as

\[
W_{t+1} = (1 - g) \left[ (1 + r) W_t + \sum_{i=A,B} \sum_{n=1}^{N} \sum_{h=t+1}^{t+M} x_{n,h,t} \left( \epsilon_{n,h,t+1}^i + \left[ (1 + r) \phi_{n,h,t+1}^i - \phi_{n,h,t+1}^i \right] \right) \right].
\]

Arbitrageurs face financial constraints requiring them to hold separate and fully collateralized margin accounts for each risky asset they invest in. The margin account for a given asset consists of a position in that asset and of riskless collateral, but cross-margining is ruled out, i.e., positions in other accounts cannot be used as collateral. Suppose, for example, that an arbitrageur wants to borrow in order to long one share of risky asset \( i_{n,h} \). He must deposit as collateral in his margin \( i_{n,h}-\)account both the share of asset \( i_{n,h} \) he just bought and some additional collateral in the form of a position in the riskfree asset. Similarly, if an arbitrageur shorts asset \( i_{n,h} \), he must deposit as collateral in his \( i_{n,h}-\)account both the cash proceeds from selling asset \( i_{n,h} \) and

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6By fixing the measure of the arbitrageurs, we rule out entry into the arbitrage industry. This seems a reasonable assumption at least for understanding short-run market behavior. However, an alternative interpretation of the maturity \( M \) during which an asset pair pays dividends is the length of time it takes for enough new arbitrageurs or new arbitrage capital to enter and close the arbitrage opportunity.
some additional cash, to cover the cost of buying asset $i_{n,h}$ next period. In both cases however, he cannot deposit any other risky asset than asset $i_{n,h}$ in the $i_{n,h}$-account.\footnote{The no cross-margining assumption is related to that of market segmentation. Indeed, the same frictions that prevent $(i_{n,h}, t)$-investors from investing in any other risky asset can also prevent the custodians of arbitrageurs’ $i_{n,h}$-accounts from accepting any other risky asset as collateral. These custodians can be financial exchanges in the case of futures contracts, or brokers/dealers in that of stocks or bonds. The no cross-margining assumption is quite realistic in both cases. For example, futures exchanges generally accept as collateral only positions in contracts traded within the exchange, and dealers generally accept only positions in assets they are dealing in. In practice, arbitrageurs sometimes avoid cross-margining to avoid front running by single counterparties to whom their trades would reveal information (Ko, 2007).}

Second, the margin account must be fully collateralized, i.e., the arbitrageur must post enough collateral in the form of a position in the riskless asset to guarantee his counterparty the risk-free rate of return. The full collateralization assumption ensures that arbitrageurs never default. This allows us to avoid modeling the custodians of the arbitrageurs’ margin accounts. (For example, $(i_{n,h}, t)$-investors can be custodians.) Note that it is because of this assumption that we consider dividends with bounded supports. With an unbounded support, the maximum loss of a position would be infinite and borrowing any amount would require arbitrageurs to post an infinite amount of collateral. Therefore, arbitrageurs would not have access to external financing.\footnote{In one sense, our financial constraint is endogenous in that it depends on the properties of the price process. The notion that margin requirements are endogenously chosen to prevent default has appeared in the general equilibrium literature (see, e.g., Geanakoplos, 2003 and the references therein).}

Denoting by $x_{n,h,t}^i$ the arbitrageur’s position in asset $i_{n,h}$ in period $t$, the arbitrageur must add as extra collateral a position $z$ in the riskless asset that by period $t + 1$ will exceed the maximum shortfall relative to the riskfree rate of return:

$$\left((1 + r) x_{n,h,t}^i p_{n,h,t}^i \right) \leq \min_{p_{n,h,t+1}^i} \left\{ x_{n,h,t}^i \left(p_{n,h,t+1}^i + \delta_{n,h,t+1}^i\right) \right\} + (1 + r) z,$$

which can be rewritten as

$$\frac{1}{(1 + r)} \max_{i_{n,h,t+1}} \left\{ x_{n,h,t}^i \left(c_{n,h,t+1}^i - \left((1 + r) \phi_{n,h,t}^i - \phi_{n,h,t+1}^i\right)\right) \right\} \leq z.$$ \hfill (13)

Since the total collateral an arbitrageur can post in all positions is bounded by his wealth, the arbitrageur’s global financial constraint is written as:

$$\frac{1}{(1 + r)} \sum_{i=A,B} \sum_{n=1}^N \sum_{h=t+1}^{t+M} \max_{i_{n,h,t+1}} \left\{ x_{n,h,t}^i \left(c_{n,h,t+1}^i - \left((1 + r) \phi_{n,h,t}^i - \phi_{n,h,t+1}^i\right)\right) \right\} \leq W_t.$$ \hfill (14)

Hence, the positions an arbitrageur can take are restricted by his wealth.

\footnote{We do not consider financial constraints for outside investors. That is, we assume that either these investors do not face constraints or that their constraints are not binding. The latter situation will arise if the outside investors’ initial wealth is large enough. Indeed, with exponential utility, optimal holdings of the risky asset are independent of wealth, and so are capital gains. Moreover, since asset payoffs and supply shocks have bounded support, capital gains are also bounded. Therefore, for large enough initial wealth, capital losses are always smaller than wealth, and the financial constraint is not binding. Note that the initial wealth of the outside investors need not exceed that of the arbitrageurs. Indeed, if the measures $\mu_n$ of the outside investors are large enough, the arbitrageurs’ positions are much larger than those of the outside investors, and thus require more collateral.}
2.3.3 The Arbitrageurs’ Problem

The arbitrageurs’ optimization problem in period $t$, $\mathcal{P}_t$, is

$$\max_{x_{i,n,h,s}} E_t \sum_{s=t}^{\infty} \beta^{s-t} U(g \cdot W_s),$$

subject to the dynamic budget constraint (11) and the financial constraint (14).

2.4 Equilibrium

**Definition 1** A competitive equilibrium consists of prices $p_{i,n,h,t}$, asset holdings of the $(i_{n,h},t)$-investors, $y_{i,n,h,t}$, and of the arbitrageurs, $x_{i,n,h,t}$, such that

- given the prices, $y_{i,n,h,t}$ solves problem $\mathcal{P}_{i,n,h,t}$, and $\{x_{i,n,h,s}\}_{s\geq t}$ solve problem $\mathcal{P}_t$,
- and the markets for all risky assets clear:

$$\mu_i y_{i,n,h,t}^i + x_{i,n,h,t}^i = 0.$$  (16)

Because of our model’s symmetry, we can show the existence of a competitive equilibrium that is symmetric in the following sense.

**Definition 2** A competitive equilibrium is symmetric if

- the risk premia of two assets $A_{n,h}$ and $B_{n,h}$ in a given pair are opposites, i.e.,

$$\phi_{n,h,t}^A = -\phi_{n,h,t}^B \equiv \phi_{n,h,t},$$

(17)

- the arbitrageurs’ positions in the two assets are opposites, i.e.,

$$x_{n,h,t}^A = -x_{n,h,t}^B \equiv x_{n,h,t},$$

(18)

- and so are the positions of the outside investors, i.e.,

$$y_{n,h,t}^A = -y_{n,h,t}^B \equiv y_{n,h,t}.$$  (19)

Intuitively, risk premia are opposites because assets are in zero net supply and the supply shocks of the $A_{n,h}$- and $B_{n,h}$-investors are opposites. Note that symmetry implies that the risk premium $\phi_{n,h,t}$ is one-half of the price wedge between assets $A_{n,h}$ and $B_{n,h}$ since

$$2\phi_{n,h,t} = \phi_{n,h,t}^A - \phi_{n,h,t}^B,$$

(20)

$$= \left( \frac{3}{r} \left[ 1 - \frac{1}{(1 + r)^{h-t}} \right] - p_{n,h,t}^A \right) - \left( \frac{3}{r} \left[ 1 - \frac{1}{(1 + r)^{h-t}} \right] - p_{n,h,t}^B \right),$$

(21)

$$= p_{n,h,t}^B - p_{n,h,t}^A.$$  (22)
The arbitrageurs’ positions are opposites because the risk premia are opposites. Note that arbitrageurs act as intermediaries. Suppose, for example, that \((A_n,h,t)\)-investors receive a positive supply shock, in which case \((B_n,h,t)\)-investors receive a negative shock. Then arbitrageurs buy asset \(A_n,h\) from the \((A_n,h,t)\)-investors, who are willing to sell, and sell asset \(B_n,h\) to the \((B_n,h,t)\)-investors, who are willing to buy. Through this transaction arbitrageurs realize a profit, while at the same time providing liquidity to the other investors.

Finally, the positions of the outside investors are opposites because the arbitrageurs’ positions are opposites, and markets must clear. Market clearing in a symmetric equilibrium requires that

\[ \mu_n y_{n,h,t} + x_{n,h,t} = 0. \]  

(23)

### 3 Riskless Arbitrage: General Results

We start with the case in which there is no fundamental risk and no preference risk. The assumption of no fundamental risk means that the two assets \(A_n,h\) and \(B_n,h\) of any given pair pay identical dividends, i.e., \(\delta_{n,h,t}^A = \delta_{n,h,t}^B = \delta_{n,h,t}\) so that \(\epsilon_{n,h,t}^A = \epsilon_{n,h,t}^B \equiv \epsilon_{n,h,t}\). The assumption of no preference risk means that \(u_{n,h,t}\) is deterministic. For simplicity, we further assume that \(u_{n,h,t}\) can depend only on \(n\), i.e., there exists \(u_{n} > 0\) such that \(u_{n,h,t} \equiv u_{n}\).

In the riskless arbitrage case, we can show the existence of a symmetric equilibrium in which risk premia \(\phi_{n,h,t}\), outside investors’ positions \(y_{n,h,t}\), and arbitrageurs’ positions \(x_{n,h,t}\) and total wealth \(W_t\) are deterministic. The reason is as follows. Since there is no fundamental risk and the arbitrageurs’ positions in assets \(A_n,h\) and \(B_n,h\) are opposites, the arbitrageurs’ wealth \(W_t\) does not depend on the dividend \(\delta_{n,h,t}\). Therefore, \(\phi_{n,h,t}\) and \(x_{n,h,t}\) are also independent of the dividend. This means that \(\phi_{n,h,t}, x_{n,h,t}\) and \(W_t\) can be stochastic only because of the supply shocks, but these are deterministic in the riskless arbitrage case.

Sections 3.1-3.3 characterize properties of a symmetric equilibrium in which \(\phi_{n,h,t}, y_{n,h,t}, x_{n,h,t}\) and \(W_t\) are deterministic. A proof that such an equilibrium exists is in Section 3.3.

### 3.1 Outside Investors

Before studying the \((A_n,h,t)\)-investors’ problem, we begin with a technical result.

**Lemma 1** The function \(f\) defined by

\[ E_t \left[ \exp \left( -\alpha y \epsilon_{n,h,t+1} / \epsilon_n \right) \right] = \exp \left[ \alpha f(y) \right] \]

is positive and strictly convex. Moreover, it satisfies \(f(y) = f(-y), f'(y) \in (-1,1)\) and \(\lim_{y \to \infty} f'(y) = 1\).

Notice that \(f\) is indeed the same for all variables \(\epsilon_{n,h,t+1}\) because of our assumption that the variables \((\epsilon_{n,h,t+1}/\epsilon_n)\) are identically distributed.

\(10\) The assumption that \(u_{n,h,t}\) is the same for all asset pairs with the same index \(n\) will ensure the model’s stationarity.
We can now study the optimization problem $P_{A,n,h,t}$ of the $(A_{n,h}, t)$-investors. Equation (8), the budget constraint of the $(A_{n,h}, t)$-investors, can be rewritten as

$$w_{n,h,t+1}^A = (1 + r)w_{n,h,t}^A + y_{n,h,t} [(1 + r)\phi_{n,h,t} - \phi_{n,h,t+1}] + (y_{n,h,t} + u_n)\tau_n \left( \frac{\epsilon_{n,h,t+1}}{\tau_n} \right).$$  \hspace{1cm} (24)

The risk premia $\phi_{n,h,t}$ and $\phi_{n,h,t+1}$ being deterministic, using this expression and the definition of function $f$, problem $P_{A,n,h,t}$ boils down to

$$\max_{y_{n,h,t}} -E_t \exp \left[ -\alpha \left( (1 + r)w_{n,h,t}^A + y_{n,h,t} [(1 + r)\phi_{n,h,t} - \phi_{n,h,t+1}] - f \left( (y_{n,h,t} + u_n)\tau_n \right) \right) \right],$$  \hspace{1cm} (25)

or, since $w_{n,h,t}^A$ is a constant as of period $t$, to

$$\max_{y_{n,h,t}} y_{n,h,t} [(1 + r)\phi_{n,h,t} - \phi_{n,h,t+1}] - f \left( (y_{n,h,t} + u_n)\tau_n \right).$$  \hspace{1cm} (26)

The first term is the investor’s one-period excess return, i.e., the capital gain over and above the riskless rate of return realized between periods $t$ and $t+1$. This excess return is certain. The second term, $f \left( (y_{n,h,t} + u_n)\tau_n \right)$, is the cost of bearing risk between periods $t$ and $t+1$ due to the uncertainty of dividends and of the supply shock. This inventory cost increases with the position in the risky asset, $y_{n,h,t}$, with the magnitude of the supply shock, $u_n$, and with the volatility of dividends, $\tau_n$. At the optimum, the marginal inventory cost must equal the expected capital gain per unit of risky asset, i.e., the first-order condition is

$$\tau_n f' \left( (y_{n,h,t} + u_n)\tau_n \right) = (1 + r)\phi_{n,h,t} - \phi_{n,h,t+1}.$$  \hspace{1cm} (27)

By symmetry, the optimization problem $P_{B,n,h,t}$ of the $(B_{n,h}, t)$-investors yields the same first order condition. Indeed, the risk premia, the supply shocks, and investors’ positions for assets $A_{h,n}$ and $B_{h,n}$ are opposites, and $f'(y) = -f'(-y)$.

### 3.2 Arbitrageurs

Next we study the arbitrageurs’ optimization problem $P$. By symmetry, the arbitrageurs’ dynamic budget constraint in period $t$, equation (11), can be written as

$$W_{t+1} = (1 - g) \left[ (1 + r)W_t + 2 \sum_{n=1}^{N} \sum_{h=t+1}^{t+M} x_{n,h,t} [(1 + r)\phi_{n,h,t} - \phi_{n,h,t+1}] \right].$$  \hspace{1cm} (28)

The second term within the brackets is the sum of the arbitrageurs’ capital gains between periods $t$ and $t+1$ over and above the riskfree rate of return. Because the arbitrageurs’ positions in assets $A_{n,h}$ and $B_{n,h}$ are opposites, the capital gains are independent of the dividends, and thus riskless. Therefore, these gains are an excess return between periods $t$ and $t+1$.

Because $u_n > 0$, arbitrageurs long asset $A_{n,h}$, i.e., $x_{n,h,t} \geq 0$, and asset $A_{n,h}$ yields a positive one-period excess return, i.e., $(1 + r)\phi_{n,h,t} - \phi_{n,h,t+1} \geq 0$. The arbitrageurs’ financial constraint
(14) takes the form
\[ \frac{2}{(1 + r)} \sum_{n=1}^{N} \sum_{h=t}^{t+M} x_{n,h,t} (\tau_n - [(1 + r) \phi_{n,h,t} - \phi_{n,h,t+1}]) \leq W_t. \] (29)

All else equal, the constraint becomes more severe when the arbitrageurs’ wealth \( W_t \) decreases. It also becomes more severe when any of the bounds \( \tau_n \) increases. This is because the maximum loss a position in a given risky asset can experience increases with the volatility of that asset’s dividends. Finally, it becomes less severe when the one-period excess return \([(1 + r) \phi_{n,h,t} - \phi_{n,h,t+1}] \) increases. This is because the maximum loss of a position decreases.

From this we can derive each arbitrageur’s optimal investment policy.

**Proposition 1** In any given period \( t \), an arbitrageur’s optimal investment policy is as follows.

- If \( \forall (n, h), [(1 + r) \phi_{n,h,t} - \phi_{n,h,t+1}]) = 0 \), the arbitrageurs are indifferent between all investment policies.

- Otherwise, each arbitrageur’s optimal strategy is to invest only in arbitrage opportunities

\[ (A_{n,h}, B_{n,h}) \in \text{arg max} \left( \frac{(1 + r) \phi_{n,h,t} - \phi_{n,h,t+1}}{\tau_n} \right) \] (30)

up until his financial constraint (28) binds.

The intuition is as follows. Each arbitrageur’s optimization problem consists in maximizing the sum of one-period excess returns subject to the financial constraint. The solution to this problem is very simple. If these excess returns are zero for all asset pairs, the arbitrageur is indifferent between all investment policies within his feasible set. If however some pairs give strictly positive one-period excess returns, the arbitrageur should focus on those asset pairs giving the highest excess returns per unit of collateral, and invest up to the financial constraint in any subset of these pairs. Establishing the “return on collateral” per each asset pair is also simple. Indeed, the asset pair \((A_{n,h}, B_{n,h})\) yields a one-period excess return \([(1 + r) \phi_{n,h,t} - \phi_{n,h,t+1}] \) per leg of the spread trade but requires the arbitrageur to put up in period \( t \) some additional collateral in the form of a position in the riskfree asset. This position must grow in period \( t + 1 \) to cover the maximum loss per leg of the position. This maximum loss being \( [\tau_n - ((1 + r) \phi_{n,h,t} - \phi_{n,h,t+1})] \), he needs to post \( \tau_n - ((1 + r) \phi_{n,h,t} - \phi_{n,h,t+1}) \) / \((1 + r)\) of additional collateral per leg. In other words, investing one dollar of collateral in arbitrage opportunity \((A_{n,h}, B_{n,h})\) yield a “return on collateral” of

\[ (1 + r) + (1 + r) \times \frac{(1 + r) \phi_{n,h,t} - \phi_{n,h,t+1}}{\tau_n - ((1 + r) \phi_{n,h,t} - \phi_{n,h,t+1})}. \] (31)

This can be rewritten as

\[ \frac{(1 + r)}{1 - \left( \frac{(1 + r) \phi_{n,h,t} - \phi_{n,h,t+1}}{\tau_n} \right)}, \] (32)
which increases with the ratio \( \frac{(1 + r) \phi_{n,h,t} - \phi_{n,h,t+1}}{\epsilon_n} \). Finally, since at least some of the arbitrage opportunities offer arbitrageurs a certain return that strictly exceeds the riskfree rate, the arbitrageurs should invest as much as possible, i.e., they should “max out” their financial constraint.

As we will see, neither aspect of this optimal strategy remains optimal outside the case of riskfree arbitrage. That is, when arbitrage is risky (due to fundamental or preference risk), arbitrageurs do not necessarily find it optimal to “max out” their financial constraint nor do they necessarily only invest in the spread trades offering the highest one-period return on collateral.

3.3 Equilibrium

**Proposition 2** For any given period \( t \), there exists \( \Pi_t \in [0,1) \) such that arbitrageurs are only invested in opportunities satisfying \( f'(u_n \epsilon_n) > \Pi_t \). Moreover,

- all opportunities in which arbitrageurs are invested in offer the same one-period excess return per unit of volatility, equal to \( \Pi_t \), i.e.,
  \[
  \frac{(1 + r) \phi_{n,h,t} - \phi_{n,h,t+1}}{\epsilon_n} = \Pi_t, \tag{33}
  \]
  while opportunities in which arbitrageurs are not invested in offer lower returns.

- when arbitrageurs’ positions are non-zero, they are given by
  \[
  x_{n,h,t} \equiv x_{n,t} = \mu_n \left( u_n - \frac{(f')^{-1}(\Pi_t)}{\epsilon_n} \right). \tag{34}
  \]

That all opportunities in which arbitrageurs are invested in offer the same one-period excess return per unit of volatility is a consequence of equilibrium: if returns differ, then Proposition 1 implies that arbitrageurs would direct their investment to those opportunities with the highest returns. Arbitrageurs do not invest in opportunities with \( f'(u_n \epsilon_n) < \Pi_t \) since the returns on those opportunities in the absence of arbitrageurs are lower than \( \Pi_t \). For simplicity, from now on we consider equilibria in which arbitrageurs invest in all opportunities. This is without loss of generality because we can ignore the opportunities in which arbitrageurs do not invest in steady state (Section 4). For equilibria close to the steady state, those opportunities do not play a role.

Our assumption that pairs of assets with the same index \( n \) have the same characteristics (dividend volatility \( \epsilon_n \), supply shock \( u_n \), and investor measure \( \mu_n \)) implies that arbitrageurs as a group take the same position in all opportunities involving those asset pairs, i.e., \( x_{n,h,t} \equiv x_{n,t} \).

Proposition 2 has a number of implications for the cross-section of profitability and premia associated with different arbitrage opportunities.

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11This assumption is not important so far but will be needed to ensure the model’s stationarity in Section 4.
Corollary 1 In any given period $t$, all else equal, arbitrage opportunities with a longer maturity have higher risk premia, i.e.,

$$\phi_{n,h+1,t} > \phi_{n,h,t} > 0.$$  \hfill (35)

The intuition is as follows. From Proposition 2, in equilibrium, in any given period, arbitrage opportunities with the same dividend volatility must offer arbitrageurs the same one-period excess return, i.e.,

$$(1 + r) \phi_{n,h,t} - \phi_{n,h,t+1} = \tau_n \Pi_t.$$ \hfill (36)

Moreover, risk premia equal the present value of future one-period excess returns, i.e.,

$$\phi_{n,h,t} = \sum_{s=t}^{h-1} \frac{(1 + r) \phi_{n,h,s} - \phi_{n,h,s+1}}{(1 + r)^{s+1-t}} = \tau_n \sum_{s=t}^{h-1} \frac{\Pi_s}{(1 + r)^{s+1-t}}.$$ \hfill (37)

Therefore

$$\phi_{n,h+1,t} - \phi_{n,h,t} = \tau_n \frac{\Pi_h}{(1 + r)^{h+1-t}} > 0,$$

which implies that the risk premium $\phi_{n,h,t}$ increases with the maturity $h$.

Corollary 2 In any given period $t$, arbitrage opportunities with higher volatility $\tau_n$ offer higher one-period excess returns, i.e.,

$$\frac{\partial ((1 + r) \phi_{n,h,t} - \phi_{n,h,t+1})}{\partial \tau_n} > 0,$$ \hfill (38)

and have higher risk premia, i.e.,

$$\frac{\partial \phi_{n,h,t}}{\partial \tau_n} > 0.$$ \hfill (39)

Moreover, the effect of volatility is larger on the risk premia of opportunities with longer maturity, i.e.,

$$\frac{\partial \phi_{n,h+1,t}}{\partial \tau_n} > \frac{\partial \phi_{n,h,t}}{\partial \tau_n} > 0.$$ \hfill (40)

The intuition is that with higher dividend volatility, more collateral is needed for a given position in the arbitrage opportunity. Therefore, in equilibrium, arbitrage opportunities involving assets with more volatile dividends must offer a greater reward to arbitrageurs, i.e., a higher one-period excess return. Since risk premia are equal to the present value of future one-period excess returns, i.e.,

$$\phi_{n,h,t} = \tau_n \sum_{s=t}^{h-1} \frac{\Pi_s}{(1 + r)^{s+1-t}},$$ \hfill (41)

they are higher for opportunities with higher dividend volatility $\tau_n$. Moreover, since the present value increases with maturity, it is more sensitive to volatility for opportunities with longer maturity.
Given Proposition 2, we can rewrite the arbitrageurs’ dynamic budget constraint (28) as
\[ W_{t+1} = (1 - g) \left( (1 + r)W_t + 2M\Pi_t \sum_{n=1}^{N} \tau_n x_{n,t} \right), \] (42)
and their financial constraint (29) as
\[ \frac{2M (1 - \Pi_t)}{(1 + r)} \sum_{n=1}^{N} \tau_n x_{n,t} \leq W_t. \] (43)

Using equations (42) and (43), as well as the market-clearing equation (23) and the first-order condition (27) of outside investors, we can fully solve for equilibrium. Proposition 3 states that a solution exists.

**Proposition 3** There exists a symmetric equilibrium in which \((\phi_{n,h,t}, y_{n,t}, x_{n,t}, W_t)\) are deterministic.

### 4 Riskless Arbitrage: Steady-State Equilibrium

We now show that in the riskless arbitrage case there exists a steady-state equilibrium. This equilibrium has a simple structure and can be used as a basis for studying non-steady-state dynamics and risky arbitrage.

In a steady-state equilibrium, risk premia, risky asset positions, and arbitrageurs’ wealth are independent of time \(t\). Therefore risk premia and positions depend on the horizon of an arbitrage opportunity and on the index \(n \in \{1, \ldots, N\}\). We denote \(W_t \equiv W\), \(\phi_{n,h,t} \equiv \phi_{n,h-t}\), \(\Pi_t \equiv \Pi\), and \(x_{n,t} \equiv x_n\) for \(h = t, \ldots, t + M\) and \(n = 1, \ldots, N\). We also set \(Y \equiv (f')^{-1}(\Pi)\).

Equation (34) takes the form
\[ x_n = \mu_n \left( u_n - \frac{Y}{\tau_n} \right). \] (44)

The arbitrageurs’ dynamic budget constraint (42) takes the form
\[ W = (1 - g) \left( (1 + r)W + 2M\Pi \sum_{n=1}^{N} \tau_n x_n \right), \] (45)
and their financial constraint (43) takes the form
\[ \frac{2M (1 - \Pi)}{(1 + r)} \sum_{n=1}^{N} \tau_n x_n \leq W. \] (46)

**Proposition 4** In a steady state symmetric equilibrium of the riskless arbitrage case, the following hold.
The arbitrageurs’ financial constraint is binding and
\[ \Pi = 1 - (1 - g)(1 + r) > 0. \] (47)

Arbitrage opportunity \((A_{n,n+m}, B_{n,n+m})\) offers arbitrageurs an one-period excess return equal to:
\[ \varpi_n \cdot \Pi, \] (48)
and has a risk premium equal to:
\[ \phi_{n,m} = \frac{1}{r} \left( 1 - \frac{1}{(1 + r)^m} \right) \varpi_n \Pi. \] (49)

The arbitrageurs’ positions are:
\[ x_n = \mu_n \left( u_n - \frac{Y}{\varpi_n} \right). \] (50)

The arbitrageurs’ total wealth is:
\[ W = 2M (1 - g) \left[ \sum_{n=1}^{N} \mu_n u_n \varpi_n - \left( \sum_{n=1}^{N} \mu_n \right) Y \right]. \] (51)

To understand why the return on each spread trade must be strictly positive, i.e., \( \Pi > 0 \), suppose instead that \( \Pi = 0 \). Intuitively, if \( \Pi = 0 \) in equilibrium, there are no arbitrage opportunities. In that case, arbitrageurs have no means to increase their wealth at a rate exceeding the riskfree rate \( r \). Since the arbitrageurs’ wealth is depleted over time at a rate \( g \) exceeding \( r \), their wealth must be zero in a steady state. But if arbitrageurs have no wealth, they cannot invest in the spread trades and therefore cannot eliminate arbitrage opportunities. This contradicts the premise that \( \Pi = 0 \).

Since \( \Pi > 0 \), the arbitrageurs’ financial constraint must be binding (Proposition 1) and therefore equation (46) holds with equality. Solving for \( \Pi \) in equations (45) and (46), yields expression (47). The intuition for the expression of \( \Pi \) is simple. In the steady state, the return on collateral, given by expression (32), is \((1 + r) / (1 - \Pi)\). Therefore, one dollar of arbitrageur wealth in period \( t \) grows to \((1 - g) (1 + r) / (1 - \Pi)\) by period \( t+1 \), which in steady state must be equal to one. This gives the expression of \( \Pi \), from which the rest of the proposition’s expressions are derived.

This result has several implications for the cross-section of profitability and premia associated with different arbitrage opportunities.

**Corollary 3** An increase in the dividend volatility of one asset pair (i.e., a tightening of the financial constraint for these assets) has the following effects.
For these assets, the one-period excess returns and the risk premia increase, i.e.,
\[
\forall (n,m), \quad \frac{\partial ((1 + r) \phi_{n,m} - \phi_{n,m-1})}{\partial \epsilon_n} > 0, \quad \text{and} \quad \frac{\partial \phi_{n,m}}{\partial \epsilon_n} > 0,
\]  
(52)

while the arbitrageurs’ positions increase, i.e.,
\[
\forall n, \quad \frac{\partial x_n}{\partial \epsilon_n} > 0.
\]  
(53)

The other assets are unaffected, i.e.,
\[
\forall n \neq n', \forall m, \quad \frac{\partial x_{n'}}{\partial \epsilon_n} = \frac{\partial \phi_{n',m}}{\partial \epsilon_n} = 0.
\]  
(54)

The intuition is as follows. First, arbitrage opportunities with higher dividend volatility must provide greater compensation to arbitrageurs in the form of higher excess returns. These also translate into higher risk premia. Second, greater dividend volatility implies that the outside investors’ demand is more inelastic. This in turn means that arbitrageurs can take larger positions without narrowing the price wedge between the assets too much.

**Corollary 4** An increase in the withdrawal rate $g$:

- increases the risk premia (i.e., $\frac{\partial \phi_{n,m}}{\partial g} > 0$),
- reduces the asset positions (i.e., $\frac{\partial x_n}{\partial g} < 0$),
- and reduces the steady-state fund size (i.e., $\frac{\partial W}{\partial g} < 0$).

The intuition is as follows. As $g$ increases, the arbitrage trades must be more profitable in equilibrium for arbitrageurs to maintain a constant wealth level. Therefore, $\Pi$ must increase. This translates directly into an increase of the risk premia. For this to be the case, however, the equilibrium positions taken by the arbitrageurs must decrease ($f$ being convex, $(f')^{-1}$ is increasing). Indeed these are given by $\Pi$ and the outside investors’ demand functions which are unaffected by a change in $g$.

**Corollary 5** An increase in the number $NM$ of opportunities increases the steady-state arbitrageurs’ wealth $W$ but has no effect on risk premia $\phi_{n,m}$.

Intuitively, when arbitrageurs have access to more opportunities, they become wealthier in steady state. At the same time, their investment in any given opportunity does not increase because while wealth is larger, it is used to finance more positions.
5 Riskless Arbitrage: Unanticipated One-Off Shocks

We next consider the impact of shocks to arbitrageurs’ wealth, arising because of either fundamental or preference risk. To remain within the confines of the riskless arbitrage case, we assume that these shocks are unanticipated and are not followed by future shocks. Studying the impact of these unanticipated one-off shocks provides insights into the mechanisms operating in the risky arbitrage case.

5.1 Fundamental Shocks

Suppose that we are in the steady state of the riskless arbitrage case, and in a given period \( t \) the two assets in the pair \( (A_{n_0,h_0}, B_{n_0,h_0}) \) do not pay exactly the same dividend. That is,

\[
\epsilon_{n_0,h_0,t}^A = \epsilon_{n_0,h_0,t} + \eta_{n_0,h_0,t},
\]

while

\[
\epsilon_{n_0,h_0,t}^B = \epsilon_{n_0,h_0,t} - \eta_{n_0,h_0,t},
\]

where \( \eta_{n_0,h_0,t} \neq 0 \) is a small shock. Our goal is to understand how this shock affects the wealth of arbitrageurs and the risk premia.

The arbitrageurs’ budget constraint between periods \( t \) and \( t + 1 \) is

\[
W_{t+1} = (1 - g) \left[ (1 + r)W + 2 \sum_{n=1}^N \sum_{h=t+1}^{t+M} x_n [(1 + r)\phi_{n,h,t} - \phi_{n,h,t+1}] + 2x_{n_0}\eta_{n_0,h_0,t} \right].
\]

Note that the shock \( \eta_{n_0,h_0,t} \) does not affect the arbitrageurs’ pre-shock wealth \( W \), their holdings of the risky assets \( x_n \), and the risk-premia \( \phi_{n,h,t} \), which are all at their steady state values. However, the shock has both a direct and an indirect effect on the arbitrageurs’ post-shock wealth. The direct effect is the change in value of arbitrageurs’ positions, holding prices constant: it corresponds to the term \( 2x_{n_0}\eta_{n_0,h_0,t} \). The indirect effect is a feedback effect: the change in arbitrageurs’ wealth affects prices and this feeds back into wealth. The feedback amplifies the direct effect of the shock. For example, if the direct effect reduces arbitrageur wealth, then arbitrageurs reduce their positions, risk premia increase, and this further reduces wealth. The amplification effect works both through the change in risk premia of the assets \( (A_{n_0,h_0}, B_{n_0,h_0}) \) generating the shock, and through the change in premia of the other assets. The latter channel represents cross-market contagion.

Differentiating (57) with respect to \( \eta_{n_0,h_0,t} \), we find

\[
\frac{\partial W_{t+1}}{\partial \eta_{n_0,h_0,t}} = 2(1 - g) \left[ \sum_{n=1}^N \sum_{h=t+1}^{t+M} x_n \left( -\frac{\partial \phi_{n,h,t+1}}{\partial \eta_{n_0,h_0,t}} \right) + x_{n_0} \right],
\]

\[
= 2(1 - g) \left[ \sum_{n=1}^N \sum_{h=t+1}^{t+M} x_n \left( -\frac{\partial \phi_{n,h,t+1}}{\partial W_{t+1}} \right) \left( \frac{\partial W_{t+1}}{\partial \eta_{n_0,h_0,t}} \right) + x_{n_0} \right].
\]
where the first term inside the square bracket is the indirect effect and the second term is the direct effect. Therefore, the effect of the shock $\eta_{n_0,h_0,t}$ on arbitrageur wealth is

$$\frac{\partial W_{t+1}}{\partial \eta_{n_0,h_0,t}} = \frac{2(1 - g)x_{n_0}}{1 + 2(1 - g)\sum_{n=1}^{N} x_n \sum_{h=t+1}^{t+M} \left( \frac{\partial \phi_{n,h,t+1}}{\partial W_{t+1}} \right)}.$$  \hspace{1cm} (60)

To fully compute this expression, we need to evaluate the term in the denominator, i.e., how a change in wealth in period $t+1$ affects risk premia in that period. Since there are no shocks after period $t+1$, we can compute the effect of wealth on risk premia using the equations of the riskless arbitrage case, i.e., the market-clearing equation (23), the first-order condition (27) of outside investors, the arbitrageurs’ budget constraint (42), and financial constraint (43). These equations determine the effect of wealth on risk premia in closed form. Closed-form solutions can also be derived for the effect of wealth on one-period excess returns and arbitrageurs’ positions. Proposition 5 reports the closed-form solutions, using the notation

$$Z \equiv \frac{1 - g}{\left[ \sum_{n'=1}^{N} \mu_{n'} x_{n'} - Y \right] f''(Y) + (1 - g)(1 + r)}.$$ \hspace{1cm} (61)

**Proposition 5** An increase in arbitrageur wealth in period $t+1$, relative to its steady-state value, has the following effects

- The one-period excess return per unit of volatility decreases
  $$\frac{\partial \Pi_{t+1}}{\partial W_{t+1}} = -\frac{(1 + r)f''(Y)Z}{2M(1 - g)\sum_{n'=1}^{N} \mu_{n'}} < 0.$$ \hspace{1cm} (62)

- Risk premia decrease
  $$\frac{\partial \phi_{n,h,t+1}}{\partial W_{t+1}} = -\frac{\tau_n f''(Y)}{2M(1 - g)\sum_{n'=1}^{N} \mu_{n'}} \frac{Z - Z^{h-t}}{1 - Z} < 0.$$ \hspace{1cm} (63)

- Arbitrageurs’ positions increase
  $$\frac{\partial x_{n,t+1}}{\partial W_{t+1}} = \frac{(1 + r)\mu_n Z}{2M(1 - g)\tau_n \sum_{n'=1}^{N} \mu_{n'}} > 0.$$ \hspace{1cm} (64)

- Arbitrageurs’ wealth in future periods decreases
  $$\forall s \geq t + 1, \quad \frac{\partial W_s}{\partial W_{t+1}} = [(1 + r)Z]^{s-t-1}.$$ \hspace{1cm} (65)

Intuitively, an increase in wealth relaxes the arbitrageurs’ financial constraint, allowing them to invest more aggressively in all arbitrage opportunities. Therefore, arbitrageurs’ positions increase, and this reduces one-period excess returns. It also reduces risk premia, since these are equal the present value of future one-period excess returns. Since this present value increases with maturity, the effect of wealth is larger on the risk premia of opportunities with longer maturity. Likewise, since one-period excess returns increase with volatility, the effect of wealth is larger on the risk premia of more volatile opportunities.
Corollary 6 An increase in arbitrageur wealth in period $t+1$, relative to its steady-state value,

- has a stronger (more negative) effect on the risk premia of arbitrage opportunities with longer maturity, i.e.,
  \[
  \frac{\partial \phi_{n,h,t+1}}{\partial W_{t+1}} < \frac{\partial \phi_{n,h,t+1}}{\partial W_{t+1}} < 0. \quad (66)
  \]

- has a stronger (more negative) effect on the risk premia of arbitrage opportunities with higher volatility, i.e.,
  \[
  \frac{\partial^2 \phi_{n,h,t+1}}{\partial \epsilon_n \partial W_{t+1}} < 0. \quad (67)
  \]

Using the closed-form solutions of Proposition 5, we can return to equation (60) and compute the effect of a fundamental shock on arbitrageur wealth and risk premia.

Proposition 6 A small unanticipated one-off shock fundamental shock in period $t$ has the following effects.

- The arbitrageurs’ wealth changes by:
  \[
  \frac{\partial W_{t+1}}{\partial \eta_{n_0,h_0,t}} = \frac{2 (1 - g) \mu_{n_0} \left( u_{n_0} - \frac{Y}{s_{n_0}} \right)}{1 - \frac{1 - g}{M} \left( 1 - \frac{rZ}{1-Z} \right) \left( M - \frac{1-Z^M}{1-Z} \right)}. \quad (68)
  \]

- The risk premium of asset $A_{n,h}$ changes by:
  \[
  \frac{\partial \phi_{n,h,t+1}}{\partial \eta_{n_0,h_0,t}} = -\frac{\tau_n f''(Y) \mu_{n_0} \left( u_{n_0} - \frac{Y}{s_{n_0}} \right)}{M \sum_{n'=1}^N \mu_{n'} \left[ 1 - \frac{1 - g}{M} \left( 1 - \frac{rZ}{1-Z} \right) \left( M - \frac{1-Z^M}{1-Z} \right) \right] \left( 1 - Z - Z^{h-t} \right)} \cdot (69)
  \]

Proposition 6 implies that a negative fundamental shock to one asset pair leads to an increase in the risk premia of all assets. The intuition for this contagion effect is similar to that for the effect of financial constraints in a conglomerate involved in multiple unrelated lines of business. A negative shock to one business tightens the conglomerate’s financial constraint and restricts its investment capacity in all lines of business. In the case at hand, investment has a feedback effect on asset prices, and therefore, financial constraints create a link between the prices of otherwise unrelated assets.

5.1.1 Integration versus Segmentation

We next use our model to study the effects of arbitrageur diversification. We compare two polar cases: integration, where all arbitrageurs hold the same portfolio, thus being invested in all opportunities, versus segmentation, where each arbitrageur is invested only in opportunities indexed by a specific $n$. 


When arbitrage is riskless, the integration and segmentation cases produce the same outcome: risk premia are identical, and so are arbitrageurs’ total wealth and aggregate position in each opportunity. This can be seen from Corollary 5: the steady-state risk premia corresponding to a set of opportunities are independent of whether the arbitrageurs accessing those opportunities can also access other opportunities. Differences between integration and segmentation arise when arbitrage is risky. We consider risky arbitrage in Section 6, but analyzing the impact of unanticipated one-off shocks gives a flavor of the results.

When arbitrage is risky, an individual arbitrageur can benefit from investing in all opportunities since this increases diversification. At the same time, diversification by arbitrageurs as a whole is limited because of contagion: when all arbitrageurs are invested in all opportunities, shocks to one asset are transmitted to all assets. Our goal is to examine whether diversification is effective despite contagion. A first step in answering this question is to compare the effects of unanticipated one-off fundamental shocks in the cases of integration and segmentation.

Suppose that in period $t$, unanticipated one-off fundamental shocks can occur to all asset pairs $(A_{n,h}, B_{n,h})$. Suppose also that these shocks, $\eta_{n,h,t}$, are normal with mean zero, variance $\lambda \bar{\epsilon}_n^2$ for $\lambda$ small, and are independent across opportunities. We can then define the variance of arbitrageurs’ total wealth induced by these shocks as

$$\text{Var}_\eta(W) \equiv \lambda \sum_{n' = 1}^{N} \sum_{h' = t+1}^{t+M} \left[ \frac{\partial W_{t+1}}{\partial \eta_{n',h',t}} \right]^2 \bar{\epsilon}_{n'}^2, \quad (70)$$

and the variance of risk premia as

$$\text{Var}_\eta(\phi_{n,h}) \equiv \lambda \sum_{n' = 1}^{N} \sum_{h' = t+1}^{t+M} \left[ \frac{\partial \phi_{n,h,t+1}}{\partial \eta_{n',h',t}} \right]^2 \bar{\epsilon}_{n'}^2. \quad (71)$$

In the case of integration, the variances $\text{Var}_\eta(W)$ and $\text{Var}_\eta(\phi_{n,h})$ can be computed using Proposition 6. The formulas in Proposition 6 can also be extended to cover the case of segmentation. Proposition 7 compares integration and segmentation when asset pairs are symmetric, i.e., $(\mu_n, u_n, \bar{\epsilon}_n)$ is independent of $n$, and Proposition 8 considers the asymmetric case.

**Proposition 7** If $(\mu_n, u_n, \bar{\epsilon}_n)$ is independent of $n$, then

- The variance of arbitrageurs’ total wealth is identical under integration and segmentation.
- The variance of risk premia is lower under integration.

Since risk premia have lower variance under integration, the outside investors benefit from arbitrageur diversification. On the other hand, arbitrageur diversification does not reduce the variance of arbitrageurs’ total wealth. Indeed, while each arbitrage opportunity has lower variance, opportunities also become correlated. As a result, the variance of arbitrageurs’ total wealth, i.e., of a diversified investment across all opportunities, stays constant.
Note that while the variance of arbitrageurs’ total capital is the same under integration and segmentation, integration benefits each individual arbitrageur. This is because variations in arbitrageurs’ total capital are better shared among arbitrageurs.

**Proposition 8** If \((\mu_n, u_n, \epsilon_n)\) depends on \(n\), then

- The variance of arbitrageurs’ total wealth is lower under integration.
- The average variance of risk premia over all asset pairs is lower under integration. However, for asset pairs with low \((u_n, \epsilon_n)\), the variance of risk premia can be higher under integration.

The intuition is as follows. Under segmentation, fundamental shocks have large effects for opportunities where arbitrageurs’ positions are large \(((u_n, \epsilon_n) \text{ large})\), because amplification effects are strong. Integration dampens these effects because arbitrageurs can transfer capital from other opportunities (e.g., with small \((u_n, \epsilon_n)\)) to smooth the shocks. At the same time, the contagion generated by these transfers can raise the variance of risk premia of opportunities with small \((u_n, \epsilon_n)\). The average variance of risk premia over all asset pairs decreases, however, consistent with the symmetric case.

The variance of arbitrageurs’ total wealth decreases under integration because the reduction in variance for opportunities with large \((u_n, \epsilon_n)\) dominates the increase for opportunities with small \((u_n, \epsilon_n)\). Intuitively, amplification effects exhibit convexity: their strength increases with arbitrageurs’ positions at an increasing rate.

### 5.2 Preference Shocks (To be written)

Suppose that we are in the steady state of the riskless arbitrage case, and in a given period \(t\) the supply shock for the two assets in the pair \((A_{n_0, h_0}, B_{n_0, h_0})\) becomes \(u_{n_0, h_0, t_0} = u\).

### 6 Risky Arbitrage (To be written)

We next consider the case where there is fundamental risk, i.e., \(\epsilon^A_{n, h, t} \neq \epsilon^B_{n, h, t}\), and preference risk, i.e., \(u_{n, h, t}\) is stochastic.

More precisely, we assume that

\[
\begin{align*}
\epsilon^A_{n, h, t} &= \epsilon_{n, h, t} + \eta_{n, h, t}, \\
\epsilon^B_{n, h, t} &= \epsilon_{n, h, t} - \eta_{n, h, t}, \\
u_{t+1, s, n} &= u + \rho(u_{n, h, t} - u) + \omega_{n, h, t},
\end{align*}
\]
where the three sequences \( \{ \epsilon_{n,h,t}, \eta_{n,h,t}, \omega_{n,h,t} \} \) are zero-mean, i.i.d., and independent of each other. We denote the finite support of \( \epsilon_{n,h,t} \) by \([-\overline{\epsilon}, \overline{\epsilon}] \), \( \eta_{n,h,t} \) by \([-\overline{\eta}, \overline{\eta}] \), and \( \omega_{n,h,t} \) by \([-\overline{\omega}, \overline{\omega}] \). The shocks \( \eta_{n,h,t} \) represent the fundamental risk of the arbitrage, and the shocks \( \omega_{n,h,t} \) represent the preference risk.

To derive closed-form solutions, we consider the case where arbitrageurs face small uncertainty. Since the arbitrageurs’ uncertainty is described by the shocks \( \eta_{n,h,t} \) and \( \omega_{n,h,t} \), we take \( \eta \) to zero, holding \( \Omega \equiv \omega/\eta \) constant. For \( \eta = 0 \) we are back in the riskless arbitrage case.

We consider Taylor expansions of the risk premia and the arbitrageurs’ positions in order \( \eta \).

We set
\[
\phi_{n,h,t} = \phi_{n,h,t}^0 + \eta \phi_{n,h,t}^1 + o(\eta),
\]
\[
x_{n,h,t} = x_{n,h,t}^0 + \eta x_{n,h,t}^1 + o(\eta),
\]
where \( o(\eta) \) denotes terms of order smaller than \( \eta \). The terms with superscript zero are the values in the riskless arbitrage case (\( \eta = 0 \)). The terms with superscript one are new terms introduced by uncertainty. We are interested in the following questions:

- What is the volatility of the price wedge (i.e., of the risk premium \( \phi_{n,h,t} \)) that is generated by fundamental shocks? What is the volatility generated by preference shocks? How is volatility influenced by the presence of arbitrageurs and by changes in their wealth? To answer these questions, we need to know how \( \phi_{n,h,t} \) depends on the arbitrageurs’ wealth \( W_t \) and the preference shock \( u_t \). For small uncertainty, we can focus on the term of order zero, \( \phi_{n,h,t}^0 \).
- How are the arbitrageurs’ positions influenced by uncertainty? How do positions differ across opportunities, depending on the risk of each opportunity and on the arbitrageurs’ wealth? To answer these questions, we need to consider the terms of order one in the function \( X_{n,h,t} \). Indeed, the terms of order zero correspond to the riskless arbitrage case, and are equal across all opportunities. The terms of order one can differ across opportunities, depending on the opportunities’ risks. Moreover, the difference can depend on the arbitrageurs’ wealth.

7 Conclusion

- Financial market model where some investors (arbitrageurs)
  - have better investment opportunities than others.
  - face financial constraints.
- Tractable equilibrium model with explicit theory of financial constraints.
• Work in progress:
  – Uncertainty.
    * Preference risk (random supply shocks) and/or fundamental risk ($\epsilon_{t,A_{h,n}} \neq \epsilon_{t,B_{h,n}}$).
    * Role for diversification ⇒ Arbitrageurs may not max out the constraint.
    * Contagion vs. diversification.
    * Arbitrageurs may exacerbate price volatility.
  – Welfare analysis.
  – Partial collateralization, VAR, etc.
  – Partial cross-margining.
References (List under construction. Suggestions welcome)


APPENDIX

Proof of Lemma 1: By assumption, the variables \( \left( \frac{\epsilon_n h t}{\epsilon_n} \right) \) are identically and symmetrically distributed over \([-1, +1]\). Therefore \( f \) is indeed identical for all these variables. Let \( \epsilon \) denote one such variable.

To show that \( f \) is positive, we use Jensen’s inequality:

\[
\exp(\alpha f(y)) = E \exp(-\alpha y e) > \exp[E(-\alpha y e)] = \exp[-\alpha y E(e)] = 1. \tag{77}
\]

Note that Jensen’s inequality is strict since \( \epsilon \) is stochastic. The last equality holds because \( E(\epsilon) = 0 \).

To show that \( f \) is strictly convex, we compute its second derivative. We have

\[
f(y) = \frac{1}{\alpha} \log \left[ E \exp(-\alpha y e) \right]. \tag{78}
\]

Therefore,

\[
f'(y) = -\frac{E(\epsilon \exp(-\alpha y e))}{E \exp(-\alpha y e)}, \tag{79}
\]

and

\[
f''(y) = \alpha \frac{E(\epsilon^2 \exp(-\alpha y e)) E \exp(-\alpha y e) - [E(\epsilon \exp(-\alpha y e))]^2}{[E \exp(-\alpha y e)]^2}. \tag{80}
\]

That \( f''(y) > 0 \) follows from the Cauchy-Schwarz inequality

\[
E(GH)^2 \leq E(G^2)E(H^2), \tag{81}
\]

for the functions \( G = \epsilon \exp(-\alpha y e/2) \) and \( H = \exp(-\alpha y e/2) \). The Cauchy-Schwarz inequality is strict since \( \epsilon \) is stochastic, and thus \( G \) and \( H \) are not proportional.

To show that \( f(y) = f(-y) \), we use the symmetry of the probability distribution of \( \epsilon \) around zero:

\[
\exp(\alpha f(y)) = E \exp(-\alpha y e) = E \exp(\alpha y e) = \exp(\alpha f(-y)). \tag{82}
\]

Finally, to show that \( \lim_{y \to \infty} f'(y) = 1 \), we note that

\[
|f'(y) - 1| = -\frac{E(\epsilon \exp(-\alpha y e))}{E \exp(-\alpha y e)} - 1 = \frac{E[(\epsilon + 1) \exp(-\alpha y e)]}{E \exp(-\alpha y e)}. \tag{83}
\]

To show that the last term goes to zero when \( y \) goes to \( \infty \), we fix \( \eta > 0 \). We have

\[
\left| \frac{E[(\epsilon + 1) \exp(-\alpha y e) 1_{\epsilon \leq -1+\eta}]}{E \exp(-\alpha y e)} \right| \leq \eta \left| \frac{E[\exp(-\alpha y e) 1_{\epsilon \leq -1+\eta}]}{E \exp(-\alpha y e)} \right| \leq \eta. \tag{84}
\]

Moreover, for \( y \) large enough,

\[
\left| \frac{E[(\epsilon + 1) \exp(-\alpha y e) 1_{\epsilon > -1+\eta}]}{E \exp(-\alpha y e)} \right| \leq \eta. \tag{85}
\]
becomes smaller than $\eta$.

**Proof of Proposition 6:** Denote:

$$Y \equiv (f')^{-1} (\Pi) = (f')^{-1} (1 - (1 - g) (1 + r))$$  \hfill (86)

$$\tau \equiv \frac{1}{N} \sum_{n=1}^{N} \tau_n$$  \hfill (87)

$$B \equiv \frac{(1 - g)}{[u\tau - Y] f'' (Y) + (1 - g) (1 + r)}$$  \hfill (88)

**Lemma 2** For all $s \geq t + 1$:

$$\frac{\partial \Pi_s}{\partial W_s} = -\frac{(1 + r) \mu f'' (Y)}{2NM (1 - g)} B$$  \hfill (89)

$$\tau_n \frac{\partial x_{n,s}}{\partial W_s} = \frac{(1 + r)}{2NM (1 - g)} B$$  \hfill (90)

$$\frac{\partial W_{s+1}}{\partial W_s} = (1 + r) B$$  \hfill (91)

Differentiating

$$x_{n,s} = \frac{u - (f')^{-1}(\Pi_s)}{\mu}$$  \hfill (92)

with respect to $W_s$ for $\Pi_s$ at its steady state value yields

$$\tau_n \frac{\partial x_{n,s}}{\partial W_s} = -\frac{1}{\mu f'' (Y)} \frac{\partial \Pi_s}{\partial W_s}.$$  \hfill (93)

Differentiating the arbitrageurs’ financial constraint (assumed to be binding since we are close to the steady state)

$$\frac{2M}{(1 + r)} (1 - \Pi_s) \sum_{n=1}^{N} \tau_n x_{n,s} = W_s$$  \hfill (94)

with respect to $W_s$ for $\Pi_s$ at its steady state value yields

$$\frac{2M}{(1 + r)} \left[ -\frac{\partial \Pi_s}{\partial W_s} \sum_{n=1}^{N} \tau_n x_{n,s} + (1 - \Pi) \sum_{n=1}^{N} \tau_n \frac{\partial x_{n,s}}{\partial W_s} \right] = 1,$$  \hfill (95)

which can be rewritten as

$$-\frac{2M}{(1 + r)} \left[ \frac{\partial \Pi_s}{\partial W_s} \sum_{n=1}^{N} \tau_n x_{n,s} + (1 - \Pi) \sum_{n=1}^{N} \frac{1}{\mu f'' (Y)} \frac{\partial \Pi_s}{\partial W_s} \right] = 1.$$  \hfill (96)

This implies that

$$\frac{\partial \Pi_s}{\partial W_s} = \frac{(1 + r)}{2M} \frac{\mu f'' (Y)}{\left[ f'' (Y) \sum_{n=1}^{N} \mu \tau_n x_{n,s} + (1 - g) (1 + r) \right]}.$$  \hfill (97)
At the steady state values we have
\[
\sum_{n=1}^{N} \mu \varepsilon_{n}x_{n} = \sum_{n=1}^{N} \left( u \varepsilon_{n} - (f')^{-1}(\Pi) \right) = N (u \varepsilon - Y) \tag{98}
\]
Hence
\[
\frac{\partial \Pi_{s}}{\partial W_{s}} = - \frac{(1 + r) \mu f''(Y)}{2M} \left[ f''(Y) N (u \varepsilon - Y) + (1 - g)(1 + r) \right] \tag{99}
\]
This implies directly that at the steady state values
\[
\frac{\partial x_{n,s}}{\partial W_{s}} = \frac{(1 + r)}{2MN} B. \tag{100}
\]
Assuming that the arbitrageurs’s financial constraints (43) is binding in period s and plugging
the expression of \( W_{s} \) into that of the arbitrageurs dynamic budget constraint (42) yields.
\[
W_{s+1} = 2M(1 - g) \sum_{n=1}^{N} \varepsilon_{n}x_{n,s}. \tag{102}
\]
Differentiating this expression with respect to \( W_{s} \) yields
\[
\frac{\partial W_{s+1}}{\partial W_{s}} = 2M(1 - g) \sum_{n=1}^{N} \varepsilon_{n} \frac{\partial x_{n,s}}{\partial W_{s}} = 2M(1 - g) \sum_{n=1}^{N} \frac{(1 + r) B}{2MN (1 - g)} = (1 + r) B. \tag{103}
\]
For \( m \geq 2 \), we have
\[
\phi_{n,t_{0} + m,t_{0} + 1} = \varepsilon_{n} \sum_{\theta=1}^{m-1} \frac{\Pi_{t_{0} + \theta}}{(1 + r)^{\theta}}. \tag{104}
\]
Therefore
\[
\frac{\partial \phi_{n,t_{0} + m,t_{0} + 1}}{\partial W_{t_{0} + 1}} = \varepsilon_{n} \sum_{\theta=1}^{m-1} \frac{1}{(1 + r)^{\theta}} \frac{\partial \Pi_{t_{0} + \theta}}{\partial W_{t_{0} + 1}}. \tag{105}
\]
Note that for \( \theta \geq 2 \)
\[
\frac{\partial \Pi_{t_{0} + \theta}}{\partial W_{t_{0} + 1}} = \frac{\partial \Pi_{t_{0} + \theta}}{\partial W_{t_{0} + \theta}} \prod_{\alpha=1}^{\theta-1} \frac{\partial W_{t_{0} + \alpha}}{\partial W_{t_{0} + \alpha}} = \frac{\partial \Pi_{t_{0} + \theta}}{\partial W_{t_{0} + \theta}} \prod_{\alpha=1}^{\theta-1} (1 + r) B = \frac{\partial \Pi_{t_{0} + \theta}}{\partial W_{t_{0} + \theta}} (1 + r)^{\theta-1} B^{\theta-1} \tag{106}
\]
Clearly this expression holds true for \( \theta = 1 \) as well.
Therefore we have
\[
\frac{\partial \phi_{n,t_{0} + m,t_{0} + 1}}{\partial W_{t_{0} + 1}} = \varepsilon_{n} \sum_{\theta=1}^{m-1} \frac{1}{(1 + r)^{\theta}} \frac{\partial \Pi_{t_{0} + \theta}}{\partial W_{t_{0} + \theta}} B^{\theta-1} = \varepsilon_{n} \sum_{\theta=1}^{m-1} \frac{\partial \Pi_{t_{0} + \theta}}{\partial W_{t_{0} + \theta}} B^{\theta-1}. \tag{107}
\]
Moreover, for $\theta \geq 1$,
\[
\frac{\partial \Pi_{t_0+\theta}}{\partial W_{t_0+\theta}} = -\frac{(1 + r) \mu f''(Y)}{2MN (1 - g)} B.
\] (108)

This implies
\[
\frac{\partial \phi_{n,t_0+m,t_0+1}}{\partial W_{t_0+1}} = -\frac{\tau_n \mu f''(Y)}{2MN (1 - g)} \sum_{\theta=1}^{m-1} B\theta = -\frac{\tau_n \mu f''(Y)}{2MN (1 - g)} \frac{B (1 - B^{m-1})}{(1 - B)}.
\] (109)

From this we have:
\[
\sum_{m=2}^{M} \left( \frac{\partial \phi_{n,t_0+m,t_0+1}}{\partial W_{t_0+1}} \right) = -\sum_{m=2}^{M} \frac{\tau_n \mu f''(Y)}{2MN (1 - g)} \frac{B (1 - B^{m-1})}{(1 - B)}
\] (110)

\[
= -\frac{\tau_n \mu f''(Y)}{2MN (1 - g)} \frac{B}{(1 - B)} \sum_{m=2}^{M} (1 - B^{m-1})
\] (111)

\[
= -\frac{\tau_n \mu f''(Y)}{2MN (1 - g)} \frac{B}{(1 - B)} \left( M - 1 - \frac{1 - B^M}{1 - B} \right)
\] (112)

\[
= -\frac{\tau_n \mu f''(Y)}{2MN (1 - g)} \frac{B}{(1 - B)} \left( M - 1 - \frac{1 - B^M}{1 - B} \right)
\] (113)

We can now write
\[
\frac{\partial W_{t_0+1}}{\partial \eta_{n_0,h_0,t_0}} = \frac{2 (1 - g) x_{n_0}}{1 + 2 (1 - g) \sum_{n=1}^{N} x_n \sum_{m=2}^{M} \left( \frac{\partial \phi_{n,t_0+m,t_0+1}}{\partial W_{t_0+1}} \right)}
\] (114)

\[
= \frac{2 (1 - g) x_{n_0}}{1 - 2 (1 - g) \sum_{n=1}^{N} x_n \tau_n \mu f''(Y) \frac{B}{2MN (1 - g) (1 - B)} \left( M - 1 - \frac{1 - B^M}{1 - B} \right)}
\] (115)

\[
= \frac{2 (1 - g) x_{n_0}}{1 - \frac{f''(Y)}{M} \frac{B}{(1 - B)} \left( M - 1 - \frac{1 - B^M}{1 - B} \right) \frac{1}{N} \sum_{n=1}^{N} \mu \tau_n x_n}
\] (116)

\[
= \frac{2 (1 - g) x_{n_0}}{1 - \frac{f''(Y)}{M} \frac{B}{(1 - B)} \left( M - 1 - \frac{1 - B^M}{1 - B} \right) \left( u\overline{c} - Y \right)}
\] (117)

Using this expression we get
\[
\frac{\partial \phi_{n,t_0+m,t_0+1}}{\partial \eta_{n_0,h_0,t_0}} = \left( \frac{\partial \phi_{n,t_0+m,t_0+1}}{\partial W_{t_0+1}} \right) \left( \frac{\partial W_{t_0+1}}{\partial \eta_{n_0,h_0,t_0}} \right)
\] (118)

\[
= -\frac{\tau_n \mu f''(Y)}{2MN (1 - g)} \frac{B (1 - B^{m-1})}{(1 - B)} \frac{2 (1 - g) x_{n_0}}{1 - \frac{f''(Y)}{M} \frac{B}{(1 - B)} \left( M - 1 - \frac{1 - B^M}{1 - B} \right) (u\overline{c} - Y)}
\] (119)

\[
= -\left[ \frac{x_{n_0} (1 - B^{m-1})}{\frac{M}{\mu f''(Y)} \frac{1}{(1 - B)} - \left( M - 1 - \frac{1 - B^M}{1 - B} \right) (u\overline{c} - Y)} \right] \left( \frac{\tau_n}{N} \right).
\] (120)