A Theoretical Model of Cross-market Arbitrage†

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Abstract

I develop a preferred-habitat term structure model of interest rates across two parallel sovereign bond markets, in which I study the interactions among preferred-habitat investors, within-market arbitrageurs, and cross-market arbitrageurs. First, I discuss the ingredients of my model and introduce three forms of market segmentation. Second, I show that my model generates a rich set of theoretical implications, under certain limits to arbitrage, for the impact of short rate shocks, supply shocks and demand shocks on the two term structures. In particular, I analyze the propagation of these shocks within each market and their internalization and transmission from one market to the other. Third, I perform an empirical implementation of my model on China's parallel sovereign bond markets with heterogeneous market participants, a perfect setting to test its performance. I focus on analyzing the impact of the cross-market arbitrageur's trading behaviors on the comovements of the two yield curves. Last, in terms of policy implications, I discuss how regulators can use my model to assess the impact of realistic policy options on the two term structures.

Keywords: Term structure of interest rates, limits to arbitrage, market segmentation, clientele effects, sovereign bonds

JEL classification: E04, E05, F15, G01, G11, G12, G15, G18

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1 Introduction

As the rapid development of the global financial system over the past few decades has made segmented financial markets over the world more interconnected than ever, more securities have become tradable across two segmented markets operating in tandem. However, securities traded across two segmented markets, despite their identical cash flows, may exhibit drastically different prices. One nature question to ask is what the underlying reasons that contribute to such price differentials are. In this paper, I examine the difference in prices of riskless bonds that are traded across two parallel markets. Some of the typical examples can be (1) sovereign bonds traded in the interbank OTC market and the exchange market in China, (2) sovereign bonds traded in the Islamic and the non-Islamic financial systems in Malaysia, and (3) on-the-run and off-the-run sovereign bonds traded in the U.S. bond markets. To achieve the goal, I develop a preferred-habitat term structure model of cross-market arbitrage under certain market frictions and limits to arbitrage. I examine the extent to which short rate shocks, supply shocks and demand shocks generated from one market propagate within that particular market and transmit to the other market through trading activities of a cross-market arbitrageur. I also investigate some important policy implications in financial markets generated from the model.

The preferred-habitat view of term structure of interest rates refers to the notion that investors may have a specific preference of one maturity clientele over the others (Culbertson, 1957; Modigliani and Sutch, 1966). This preferred-habitat hypothesis of interest rate determination has not only received wide acceptance and been incorporated into several large econometric models, it is also popular among industry practitioners, as it is supported by numerous real-world investment practices. For instance, pension funds have specific demand for bonds with fixed maturities to guarantee scheduled payments. Quantitative easing (QE), a type of unconventional monetary policy, also refers to central banks' large-scale purchase of bonds with long maturities, in order to inject liquidity into the market and to stimulate corporate investment. Therefore, in the *extreme-form* of market segmentation, where no investor invests in bonds across two or more distinct maturities, the interest rate at any given maturity is driven *only* by the supply and demand shocks affecting that particular maturity. However, this form of extreme maturity segmentation hardly exists in reality, as one of its fundamental assumptions is that *any* type of investor in the market is required to not switch to any nearby maturity clientele that offers a higher risk-to-return tradeoff. Intuitively, once the yield gap between two maturity clienteles is sufficiently large to outweigh transaction costs, arbitrageurs have the incentive to profit from the yield differential, closing the gap between the yields for these two maturity clienteles and rendering a smoother term structure of interest rates.

As developed in Modigliani and Sutch (hereafter the M&S model, 1966), the preferred-habitat model incorporates three logically independent hypotheses: (1) market participants' tendency to match the term structure of their assets and liabilities, (2) the dependence of long-term rates on expected future short-term rates, and (3) that market expectations about future short-term rates contain both regressive and extrapolative elements. Despite the prevalence of preferred-habitat investments in real-world practices, the economic notion of the preferred-habitat concept has not received much attention in academic literature. One of the reasons has to do with the Markovian nature of interest rates, which refers to the fact that movement in a given period is independent of movements in other periods. This contradicts the second hypothesis in the M&S model. If both long-term and short-term interest rates follow a random walk, long-term rates are not determined by a distributed lag of short-term rates, rendering several important econometric models potentially mispecified (Phillips and Pippenger, 1974). Another reason is that, from the perspective of standard economic theory, interest rate at a specific maturity τ is determined by the willingness of investors to substitute consumption between now and time τ, contrasting sharply to the preferred-habitat view of maturity clienteles and partly to the first hypothesis in the M&S model (Vayanos and Vila, 2009 & 2019).

Vayanos and Vila (hereafter the V&V model, 2019) also attribute the little attention paid to the preferred-habitat view to the lack of a formal model and the impression that it may conflict with the logic of no-arbitrage. They are among the first to formally model the term structure of interest rates that results from the interaction between preferred-habitat investors and risk-averse arbitrageurs in a *one-market* setting. The V&V model can explain the effects of clientele demands on the term structure and the effects of large-scale bond purchases by central banks, and generate implications for the transmission of monetary policy from short to long rates. Although it embeds the notion of preferred-habitat into a modern no-arbitrage term structure framework, the V&V model is only applicable to a *one-market* setting but not to the more complex market structures in which securities are traded across two or more parallel markets, segmented yet connected *to-some-extent* by the arbitrageurs across these markets. Guibaud, Nosbusch and Vayanos (2008) also propose a clientele-based theory of the optimal maturity structure of government debt. They find that the optimal fraction of long-term debt increases in the weight of the long-horizon clientele, given that agents are more risk-averse than log, and show that lengthening the maturity structure raises the slope of the yield curve. Greenwood and Vayanos (2014) investigate around a term-structure model where risk-averse arbitrageurs absorb shocks to the demand and supply for bonds of different maturities and examine empirically how the supply and maturity structure of government debt affect yields and expected returns. Moreover, Gourinchas, Ray and Vayanos (2020) build an integrated preferred-habitat model of term premia and exchange rates, based on the V&V model. Their model generates deviations from the uncovered interest parity conidition.

They also explore the transmission of monetary policy to domestic and currency markets and the spillovers to the foreign term premia. On the empirical side, Chen *et al.* (2019) find that clientele effects in Malaysian bonds cause Islamic sovereign bond investors to enjoy 4.8 bps higher in yield than do conventional bond investors, after controlling for bond characteristics and liquidity. Liu *et al.* (2019) find that enterprise bonds in the Chinese exchange market, which has greater demand from retail investors, are more expensive than those traded by institutional investors in the interbank market. However, they ignore the fact that the exchange market is not homogenous in terms of retail investors, due to the presence of institutional investors, who also can access the interbank market and arbitrage across the two markets.

In this paper, similar to the setup in the V&V model, I develop a cross-market preferred-habitat model that is applicable to a much larger set of empirical settings. There have been an extensive literature on the modelling of term structure of interest rates (Cox, Ingersoll and Ross, 1985; Frachot and Lesne, 1993; Duffie and Kan, 1996; Dai and Singleton, 2000; Duffee, 2002). To model the term structure within each market, I follow Duffie and Kan (1996) and assume that bond prices are exponentially affine in the factors that can potentially affect bond pricing. The short rate r_t, being the limit of the spot rate as maturity approaches zero, is exogenously given and is uniform across the two markets. I further assume that r_t follows the *Ornstein–Uhlenbeck process*. I retain the assumption that the term structure of interest rates is determined through the interaction between the preferred-habitat investors and the risk-averse arbitrageurs, while I expand the preferred-habitat view to a two-market structure that is widely seen in many financial markets in the real world. In specific, to link the two originally segmented markets I introduce a cross-market arbitrageur, who trades across the two markets, arbitrageurs profit away from the discrepancy in the two yield curves, and, more importantly, functions as a channel through which shocks to the short rate, demand, and supply in one market transmits to the other market. In addition to generating similar theoretical implications to the V&V model, this cross-market model is able to quantify the change in the yield curve in one market due to one unit of the three types of shocks from the other market. To capture the demand differential across the two markets, I include a net demand factor β_t. Under the introduction of the net demand factor, the term structure in one market can be modelled using only one risk factor, the short rate, which the term structure in the other market is dependent on the short rate factor and the net demand factor. In Section 2, I describe in detail the model setup and its components.

The introduction of within-market arbitrageurs and the cross-market arbitrageurs allow me to discuss the other two forms of market segmentation, in addition to the in the *extreme form* of market segmentation that I introduced at the beginning. The *semi form* of market segmentation characterizes the situation in which only the within-market arbitrageurs are present. The absence of the cross-market arbitrageur forces the two parallel bond markets to remain segmented from each other, while the two term structures are smoothened due to the arbitraging behavior from the within-market arbitrageurs. The *weak form* of market segmentation characterizes the situation in which both the within-market arbitrageurs and the cross-market arbitrageurs are present. In this case, the two parallel bond markets are connected through the trading behavior of the cross-market arbitrageur, while the two term structures remain smoothened due to the arbitraging behavior from the within-market arbitrageurs.

The existence of two parallel trading venues is not rare in practice. One typical example can be the well-known segmented bond markets in China. The interbank OTC market and the exchange market, regulated by the People's Bank of China (PBOC) and the China Securities Regulatory Commission (CSRC), respectively, serve as the two primary trading venues for a variety of bonds in China (Amstad and He, 2019; Hu, Pan and Wang, 2018; Mo and Subrahmanyam, 2020). In particular, sovereign bonds are traded in both markets. Besides, market participants of each market also differ: while banks and retail investors can trade only in the interbank market and the exchange market, respectively, institutional investors have access to both markets. The heterogeneity in market participants has caused the same maturity clientele to be traded by investors with drastically different investment objectives. I explain in detail how my model is able to capture the demand differential at the same maturity across the two markets in Section 2.

In addition, my model is applicable to the Malaysian bond markets, where there exist an Islamic financial system and a conventional (*non-Islamic*) financial system. Islamic bonds are traded only in the Islamic financial system, and conventional bonds are traded only in the conventional financial system.[1] These two markets are parallel to each other and contain distinct bonds traded by distinct investor groups: the group of Islamic investors and the group of non-Islamic investors. Islamic investors are restricted to only trade bonds in the Islamic market, while non-Islamic investors have the freedom to trade bonds in both markets (Chen *et al.*, 2019). In other words, non-Islamic investors have the opportunity to profit from the pricing differential, if there is any, across the two markets.

Although the structures of market participation in the above two examples appear to be similar to each other, in the sense that a particular subset of investors may trade in one market but not in its parallel counterpart, the nature of market participants differ. In the Chinese bond markets, it should be noted that household investors speculate heavily in the exchange market, while large banks only trade in the interbank market whenever they have to. Not only does this feature render the trading volume of the interbank market much larger than that of the exchange market,

[1] Islamic bonds refer to bonds that do not pay coupons.

but it has also made the interbank market much deeper, yet lacking immediacy, than the exchange market. In addition to the more complex market structure my model accounts for, it also allows arbitrageurs to smooth the term structure in each market by arbitraging across different maturity clienteles to profit from unequal risk-to-return tradeoffs till the entire term structure of interest rates is rendered arbitrage-free. Moreover, I further classify the risk-averse arbitrageurs into two categories, the *within-market* arbitrageurs, who only trade in one of the two markets and do not switch between the two markets, and the *cross-market* arbitrageurs, who trade in both markets and profit from the pricing discrepancies between the two markets.

One significant contribution of my model to the literature is that it formally captures the limits to arbitrage under partially segmented markets. Though some papers investigate the role of limits to arbitrage in financial markets, no paper has ever done this formally with partially segmented markets. The theoretical developments in the literature on the limits of arbitrage examines how various costs faced by arbitrageurs can prevent them from eliminating mispricings and providing liquidity to other investors (Gromb and Vayanos, 2010). Shleifer and Vishny (2012) discuss a number of interesting implications of professional risky arbitrage for security pricing and the reasons that arbitrage fails to eliminate anomalies in financial markets. Acharya, Lochstoer and Ramadorai (2013) build an equilibrium model of commodity markets with capital-constrained speculators and commodity producers with hedging demands on futures and find that limits to financial arbitrage cause limits to hedging by producers and affect equilibrium prices in commodity markets. Hanson and Sunderam (2013) develop a novel methodology that infers the capital allocated to value and momentum arbitrage strategies and exploit time-variation in the cross-section of short interest. He and Krishnamurthy (2013) model the dynamics of risk premia during crises in asset markets where financial intermediaries face an equity capital constraint, which reflects the capital scarcity. Other papers study empirically the role of limits to arbitrage in financial markets include Gabaix, Krishnamurthy and Vigneron (2007), Lam and Wei (2011), Lewellen (2011), and Ljungqvist and Qian (2016). In my model, I consider various types of limits to arbitrage, including the cross-market arbitrageur's risk aversion level, settlement period differences, uncertainty in settlement periods, sequencing of the arbitrage trade (buy before you can sell), etc. Specifically, my model quantifies the pricing effects of limits to arbitrage under partially segmented markets, identifies the economic meaning of variable coefficients, allows for an empirical implementation to the model, and permits policy experimentation to study the impact of realist policy options with closed-form optimal solutions.

With regard to model implications, my model also contributes to the design of macroeconomic strategies. First, I analyze the impact of shocks to short rate expectations on the term structure. In the V&V model, through carry trades arbitrageurs earn rents from transmitting the shocks from short rate to long rates. However, my model is able to quantify not only the impact of shocks to short rate in one market on its own term structure, but also on the term structure of the other market. Second, I address the effects of supply shocks and demand shocks at specific maturity clienteles on the term structure within each market. In the V&V model, demand shocks have the same relative effect on the term structure regardless of the maturities they originate when the short rate is the only risk factor, while they become more localized under stochastic demand, when demand becomes another risk source. In this paper, however, I analyze the impact of shocks to the short rate and to the supply on the term structures in the two markets, under the latter two types of market segmentation. Consider first the *semi form* of market segmentation where the two parallel markets remain segmented. In the market where the short rate is the only risk factor, a positive exogenous supply shock will generate the same global effect at all maturity clienteles and lead to an upward shift in the term structure. Meanwhile, a positive change in preferred habitat investors' demand will generate the same global effect at all maturity clienteles and lead to a downward shift in the term structure. The supply effect and demand effect are both global as the relative effect across maturities is independent of where the shocks originate. In the parallel counterpart where both the short rate and the net demand factor are risk factors, a positive exogenous supply shock will generate the same global effect at all maturity clienteles, while a change in preferred habitat investors' demand will generate localized effect across different maturity clienteles. The supply effect remains global, while the demand effect is now more localized relative to the one-factor situation, and the relative effect across maturities depends on the origin of the demand shock. Moreover, I show that yields of longer-maturity bonds are more sensitive to expected jumps in the short rate. Equilibrium allocations of more risk-averse within-market arbitrageurs are less sensitive to uncertainty due to expected jump size of the short rate. On the other hand, a positive supply shock to the market with stochastic demand will either increase or decrease its term structure, depending on the relative importance of the short rate and the demand factor as priced risk factors.

One key feature of my model is that it allows me to study how supply shocks and demand shocks from one market impact the term structures in both markets as the cross-market arbitrageurs transmit the shocks from one to the other. This corresponds to the *weak-form* of market segmentation. In particular, I show that any supply shock or demand shock will cause the two term structures to co-move with each other under either constant demand or stochastic demand, though the relative effects may differ. The global feature and localization of the impact of such shocks on the two term structures inherits from the *semi-form* of market segmentation. Moreover, I investigate the induced demand shocks thanks to the cross-market arbitrageur's trading behavior under different risk-aversion levels and examine their propagation from short maturities to long maturities. Last, I examine the reactions of the term structures to supply shocks under different values of the cross-market arbitrageur's risk aversion and short-selling constraints from two dimensions. I first consider the internalization and transmission of exogenous supply shocks under the presence of the cross-market arbitrageur under the *weak-form* of market segmentation. I then study the reactions of the two term structures to the cross-market arbitrageur's intervention as the market segmentation type evolves from the *semi-form* to the *weak-form*. In particular, I examine the extent to which the gap between the two term structures can be fully closed or partially reduced.

The fact that changes in the term structures of the two markets, due to different types of shocks, are now quantifiable allows me to examine the real consequences of the policy implications from my model through performing a calibration exercise on the Chinese sovereign bond markets. The central bank can now use my model to analyze the impact of its monetary policies on the two sovereign bond markets. The Chinese Ministry of Finance can study the impact of debt issuance on the two yield curves. Taking for example the establishment of the *Bond Connect*, a trading platform that allows offshore investors and mainland China investors to invest in each other's bond markets on July 3, 2017. The *Southbound* of the platform was opened immediately after the initialization, while the *Northbound* has not been opened so far.[2] In particular, my model can be used to forecast the term structure comovements once the Southbound of the Bond Connect platform is opened. The government can use my model to determine to what extent they should let domestic investors participate in the foreign bond markets. The government may control the degree of integration by allowing particular groups of institutional investors to trade in the *Southbound*. In the calibration exercise, I explain how this can be done through controlling for the cross-market arbitrageur's risk-aversion and setting up limits on short-selling constraints. I show that the extent to which the gap between the two term structures can be reduced depends on the degree to which the cross-market arbitrageur is allowed to short-sell.

The rest of the paper is organized as follows. Section 2 gives a detailed description of the model and the economy. Section 3 discusses the three forms of segmentation and the equilibrium solutions in each case. Section 4 and section 5 list propositions that capture the theoretical insights generated from the model. Section 6 reports the calibration results to China's parallel sovereign bond markets. Section 7 reports the calibration results on the transmission of supply and demand shocks from one market to the other and the response of the two yield curves to these shocks

[2] The *Southbound* refers to the channel that allows foreign investors to invest in the Chinese interbank bond markets. The *Northbound* refers to the channel that allows mainland China investors to invest in the overseas bond markets. See Cherian, Mo and Subrahmanyam (2020) for detail discussions of *Bond Connect*.

under different cross-market arbitrageur's risk-aversion levels. Section 8 discusses the policy implications on improvement in market efficiency. Section 9 concludes the paper.

2 The Model

In this section, I present the constituents, setup, assumptions, and demand functions in the economic framework of the theoretical model, which models the term structure of interest rates due to the interaction among the preferred-habitat investors and the two types of arbitrageurs, the within-market arbitrageurs and cross-market arbitrageurs. Figure 1 provides an overview of the parallel sovereign bond market structure in China and shows the general framework of my model. Within each market, my model admits a similar framework to the V&V model in Vayanos and Vila (2019) to model the term structure of interest rates. At the same maturity clientele, my model adopts a modified framework to Goldstein, Li and Yang (2014). Household investors can only invest in market E, which represents the exchange market in China. Banks can only invest in market C, which represents the interbank OTC market in China.

[Insert Figure 1 and Table 1 Here]

2.1 *Markets, assets, and market participants*

The economy consists of two parallel markets, market E and market C, and contains two types of assets, the market-E sovereign bonds and the market-C sovereign bonds.[3] Each market contains sovereign bonds with maturities ranging from 1 year to T years, with supplies of $S_{t,\tau}^{E}$ for market-E sovereign bonds and supplies of $S_{t,\tau}^{C}$ for market-C sovereign bonds with maturity τ at time t. The supplies of both types of bonds are exogenous and are taken as given.

The preferred-habitat investors in the two markets constitute the $2T$ maturity clienteles: a total number of T types of preferred-habitat investors at maturities 1, 2, …, T in market C and a total number of T types of preferred-habitat investors at maturities 1, 2, …, T in market E. In total, there exist $2T + 3$ types of investors in the economy. I also adopt a representative agent framework and assume that each type of investors' aggregate trading behavior can be delineated under a *representative agent* framework, so that I arrive at $2T + 3$ *representative* agents in the economy: T preferred-habitat investors in market E, T preferred-habitat investors in each market C, one within-market arbitrageur in market E, one within-market arbitrageur in market C, and one cross-

[3] In the setup of the Chinese bond markets, "market C" denotes the interbank OTC market, and "market E" denotes the exchange market.

market arbitrageur across the two markets. As a result, the economy is populated with *five* categories of market-wise *representative* investors:

(1) *PHI.E* – the *T representative* preferred-habitat investors who invest in bonds with specific maturity clienteles *only* in market *E*, but not in market *C*.

(2) *PHI.C* – the *T representative* preferred-habitat investors who invest in bonds with specific maturity clienteles *only* in market *C*, but not in market *E*.

(3) *WMA.E* – the *representative* within-market arbitrageur who invests in bonds with maturity clienteles in market *E* and smoothens the term structure in market *E*.

(4) *WMA.C* – the *representative* within-market arbitrageur who invests in bonds with maturity clienteles in market *C* and smoothens the term structure in market *C*.

(5) *CMA* – the *representative* cross-market arbitrageur who invests in bonds with maturity clienteles in both market *E* and market *C*, with the amount of investment in each market depending on the yield level in each market.

For notational purpose, I further use *PHI.E*τ to denote the *representative* preferred-habitat investor who invests only in maturity- τ bonds in market *E*, but not in market *C*. Similarly, I use *PHI.C*τ to denote the *representative* preferred-habitat investor who invests only in maturity-τ bonds in market *C*, but not in market *E*.

2.2 *Term structures*

Time is assumed to be continuous in this economy and can go to infinity. The maturities of the bonds in the two markets fall in the interval $[1, T]$, with a time-t bond that matures in τ years paying \$1 at time $t + \tau$. Let $P_{t,\tau}^M$ denote the time-t price of the market-*M* bond with maturity τ, where $M \in \{E, C\}$, and let $y_{t,T}^M$ be the spot rate for that maturity, so that

$$y_{t,\tau}^M = -\tau^{-1} \log P_{t,\tau}^M$$

The short rate r_t is the limit of $y_{t,\tau}^M$ when τ goes to zero. Let r_t be exogenously given and assume that it is uniform across the two markets, such that $y_{t,\tau}^C = y_{t,\tau}^M \equiv y_{t,\tau}$. Assume further that r_t follows the *Ornstein–Uhlenbeck process*, such that

$$dr_t = \kappa_r(\bar{r} - r_t)d_t + \sigma_r dW_{r,t}$$

where $\bar{r}$, κ_r, and σ_r are positive constants and represent the long-run mean, the mean reversion rate, and the volatility of the short rate, respectively. The term $W_{r,t}$ denotes the Wiener process that governs the evolution of the short rate. I take the short rate r_t as given and focus on investigating the impact of shocks for the short rate to the term structures of interest rate in the two markets. I assume that the term structure in market C is fully specified by the short rate r_t, while the term structure in market E is fully specified by the short rate r_t and, in addition, a net demand factor β_t. The net demand factor captures the difference in demand across the two markets at each maturity clientele, resulting from the heterogeneity of market participants, and also follows the *Ornstein–Uhlenbeck process*, such that

$$d\beta_t = -\kappa_\beta \beta_t dt + \sigma_\beta dW_{\beta,t}$$

where κ_β, and σ_β are positive constants and the mean reversion rate and the volatility of the short rate, respectively. I assume that the long-run mean of this net demand factor is zero. The term $W_{\beta,t}$ denotes the Wiener process that governs the evolution of the net demand factor.

2.3 *The net demand factor*

I now interpret the net demand factor in the exchange market in the case of the parallel sovereign bond markets in China. To reiterate, while banks can only trade in the interbank market and household investors can only trade in the exchange market, financial institutions can trade across the two markets. Thus, financial institutions that trade across the two markets function as the cross-market arbitrageurs.[4] It should also be noted that distinct sovereign bonds are traded in each market, so that the short rates are different across the two markets. If I adopt the actual scenario where short rates are different across the two markets, such that r_t^C and r_t^E are the short rates in market C and market E, respectively, the net demand factor captures the difference in demand between market participants at each maturity clientele and comprises three components. The first component illustrates the difference in demand between household investors in the exchange market (*WMA.E*) and banks in the interbank market (*WMA.C*) at each maturity clientele. The second component captures the difference in demand of institutional investors across the two markets (*CMA*) at each maturity clientele. The third component captures any potential discrepancies in trading frictions (*e.g.* transaction costs) across the two markets.

The calibration exercise of my model can follow two approaches. First, I adopt the actual scenario where each market shares a distinct short rate process. In this case, the net demand factor captures three components, delineated previously. Second, for analytical tractability, I assume that the short rate process in market E is characterized by the same short rate process in market C, with *perturbed* mean-reversion parameter and long-run mean. The term structures of sovereign bond

[4] Due to the drastic difference in aggregate trading volume across the interbank and the exchange markets, large institutional investors usually find it difficult to find counterparties entering into comparable large trades in the much smaller exchange market. Thus, financial institutions that trade across the interbank and the exchange markets are usually medium- to small-sized financial institutions.

yields across the two markets highly co-move with each other. Empirically, Figure 2 shows that the term structure in the exchange market lies mostly above that in the interbank market. This spread between the two short rates on a trading day t is given by $r_t^C - r_t^E$. The net demand factor now captures three components. Besides the three components of demand differential and trading frictions mentioned in the case of using distinct short rates, the net demand factor further captures the spread between the two term structures of sovereign yield curves across the two markets, $s_t^{CE} \equiv r_t^C - r_t^E$. One way to incorporate this spread in the *Ornstein–Uhlenbeck process* of the net demand factor is to introduce a *modified* long-run mean of β_t, which was originally zero and has now become $\bar{\beta} > 0$. The interpretation of $\bar{\beta}$ is the long run mean of the time series s_t^{CE}. The *Ornstein–Uhlenbeck process* of the net demand factor now becomes

$$d\beta_t = \kappa_\beta(\bar{\beta} - \beta_t)dt + \sigma_\beta dW_{\beta,t}$$

2.4 *Demand functions*

I assume that preferred habitat investors have linear demand functions for maturity-τ bonds and are increase in the bond yield $R_{t,\tau}$. The demand functions of *PHI.E* and *PHI.C* investors, $Y_{t,\tau}^E$ and $Y_{t,\tau}^C$, are given by

$$Y_{t,\tau}^E = \alpha^E(\tau)\tau\left(y_{t,\tau}^E - d^E - \theta(\tau)\beta_t\right) = -\alpha^E(\tau)\log\left(P_{t,\tau}^E\right) - \beta_{t,\tau}^E$$
$$Y_{t,\tau}^C = \alpha^C(\tau)\tau\left(y_{t,\tau}^C - d^C\right) = -\alpha^C(\tau)\log\left(P_{t,\tau}^C\right) - \beta_{t,\tau}^C$$

where $\beta_{t,\tau}^C$ and $\beta_{t,\tau}^E$ are intercept terms and $\alpha^C(\tau)$ and $\alpha^E(\tau)$ are some functions of maturity τ which characterize the slopes of demand. Since the term structure in market C is fully specified by the short rate, I assume that $\beta_{t,\tau}^C = d^C(\tau)$, where $d^C(\tau)$ is a constant over time but can depend on τ. However, since the term structure in market E is specified by both the short rate r_t and the net demand factor β_t, I assume $\beta_{t,\tau}^E$ to be a function linear in β_t such that $\beta_{t,\tau}^E = d^E(\tau) + \theta(\tau)\beta_t$, where $d^E(\tau)$ and $\theta(\tau)$ are constants over time but can depend on τ if necessary.

In addition to bonds with respective maturity clienteles that preferred habitat investors buy, they are accessible to another *private investment vehicle* which allows them to substitute consumption freely between this investment vehicle and their respective maturity clientele. This assumption is crucial in solving this cross-market preferred-habitat model; I later discuss in the solution process.

On the other hand, I assume that the three representative arbitrageurs (*WMA.C*, *WMA.E*, and *CMA*) are mean-variance utility optimizers and choose their portfolios based on the trade-off between instantaneous risk and return (variance and mean). The market-*C*, market-*E*, and cross-market arbitrageurs have risk versions of γ^C, γ^E, and γ^A, respectively. I assume that they have CARA utility functions $U(W_t) = \exp(-\gamma W_t)$, where W_t is the representative arbitrageurs' endowment at time t. They choose their portfolio allocation at time t and maximize their expected utility at time $t+1$. The three representative arbitrageurs have different investment opportunity sets, though they all observe bond prices from both market C and market E.

I further assume that the three representative arbitrageurs can all borrow an infinite amount, i.e. no borrowing constraint. I later show that the restriction on the issuance amount of market-C and market-E bonds that they are allowed to borrow cannot exceed a certain fraction λ of their time t endowment is tantamount to assuming that γ^C, γ^E, and γ^A are all positive.

2.5 *Asset payoffs*

Denote the log-prices of the market-C bonds and of the market-E bonds by $p_{t,\tau}^C$ and $p_{t,\tau}^E$. Denote also the demand shocks, as a result of the intervention from the cross-market arbitrageur, to market-C and market-E as β_t^C and β_t^E, respectively. The preferred-habitat investors and the within-market arbitrageurs use the information sets (r_t, β_t^C) and $(r_t, \beta_t, \beta_t^E)$ to forecast yields $y_{t,\tau}^C$ and $y_{t,\tau}^E$ in the two markets. The actual demand from the cross-market arbitrageur are functions of the demand shocks, $X_{t,\tau}^C \equiv F^C(\tau, \gamma^A)\beta_t^C$ and $X_{t,\tau}^E \equiv F^E(\tau, \gamma^A)\beta_t^E$, where $F^C(\tau)$ and $F^E(\tau)$ are finite integrable functions of τ to be solved for in the equilibrium. If one of $X_{t,\tau}^C$ and $X_{t,\tau}^E$ is zero, the corresponding demand shock is zero. From Duffie and Kan (1996), bond prices are exponentially affine in the factors

$$P_{t,\tau}^C = \exp\left[-\left(A_r^C(\tau)r_t + A_\beta^C(\tau)\beta_t^C + C^C(\tau)\right)\right]$$
$$P_{t,\tau}^E = \exp\left[-\left(A_r^E(\tau)r_t + A_\beta^E(\tau)(\beta_t + \beta_t^E) + C^E(\tau)\right)\right]$$

where yields to maturity are

$$y_{t,\tau}^C = \tau^{-1}\left[A_r^C(\tau)r_t + A_\beta^C(\tau)\beta_t^C + C^C\right]$$
$$y_{t,\tau}^E = \tau^{-1}\left[A_r^E(\tau)r_t + A_\beta^E(\tau)(\beta_t + \beta_t^E) + C^E(\tau)\right]$$

so that each investment unit of $\exp(-\tau y_{t,\tau})$ at time t yields 1 unit of payoff at maturity τ.

2.6 *Poisson jumps in the short rate*

The diffusion processes of the short rate and the demand factor evolve continuously over time. However, policy announcements and macroeconomic news may trigger large movements in bond prices within a short period, and these announcements may arrive at either stochastic or deterministic times. In other words, the times at which price jumps occur may be either stochastic or deterministic. The value of the state vector, the short rate r_t, right before a jump at time t is the left limit $r_{t-} = \lim_{s\to t} r_s$. The jump in r at t is $\Delta r_t \equiv r_t - r_{t-}$. In addition, the conditional probability

$\lambda_t dt$ of a jump during the interval $[t, t + dt]$ and the distribution of the corresponding jump size may or may not be state dependent. As pointed out in Piazzesi (2010), the jump size and the jump timing cannot be specified to be state dependent at the same time, and one of them has to be state independent and loses tractability. In this case, since the dates of policy announcements are determined by the government, it is more appropriate to assume that the distribution of jump size is state dependent. Duffie and Kan (1996) models jumps at stochastic jump times by

$$dr_t = \mu_r(r_{t-})dt + \sigma_r(r_{t-})dW_{r,t} + \Delta r_t dJ_t$$
$$\mu_r(r_{t-}) = \mu_r + \lambda_r E[\Delta r_t]$$

The jump process can be activated in two possible ways, while in this paper I assume that it is caused by a Poisson process N^P with stochastic intensity λ_t. For small enough intervals, we can intuitively think of this Poisson process as having probability $\lambda_t dt$ of observing a price jump at time $t + dt$ and having probability $(1 - \lambda_t dt)$ of not observing a price jump at time $t + dt$.

2.7 *Model summary*

Market C – one of the two parallel markets in the economy that contains exogenous supply $S^C_{t,\tau}$ of maturity-τ bonds at time *t*, with $S^C_{t,\tau} \in [0, \infty)$.

Market E – one of the two parallel markets in the economy that contains exogenous supply $S^E_{t,\tau}$ of maturity-τ bonds at time *t*, with $S^E_{t,\tau} \in [0, \infty)$.

PHI.C and *PHI.E* – the two types of representative preferred-habitat investors in market *C* and market *E*.

PHI.Cτ and *PHI.Eτ* – the two representative preferred-habitat investors in market *C* and market *E*, with their respective demand $\mathrm{Y}^{\mathrm{C}}_{\mathrm{t},\tau}$ and $\mathrm{Y}^{\mathrm{E}}_{\mathrm{t},\tau}$ for maturity-τ bonds in time-*t* dollars.

WMA.C and *WMA.E* – the two representative within-market arbitrageurs, with $\mathrm{Z}^{\mathrm{C}}_{\mathrm{t},\tau}$ and $Z^E_{t,\tau}$ being their respective demand for maturity-τ bonds in time-*t* dollars.

CMA – the one representative cross-market arbitrageur, with $\mathrm{X}^{\mathrm{C}}_{\mathrm{t},\tau}$ and $\mathrm{X}^{\mathrm{E}}_{\mathrm{t},\tau}$ being her demand for maturity-τ bonds in time-*t* dollars in market *C* and market *E*, respectively.

CARA utility preferences – *WMA.C*, *WMA.E* and *CMA* have constant absolute risk aversion utility functions with risk-aversion parameters γ^{C}, γ^E and γ^A, respectively.

Borrowing constraint – infinite short-selling is allowed.

One risky investment vehicle – *PHI.C* and *PHI.E* save consumption through either this investment vehicle or their respective maturity clienteles.

Market clearing conditions –maturity clientele τ bonds at time *t* in the two markets:

$$S^C_{t,\tau} = X^C_{t,\tau} + Y^C_{t,\tau} + Z^C_{t,\tau} \quad \text{and} \quad S^E_{t,\tau} = X^E_{t,\tau} + Y^E_{t,\tau} + Z^E_{t,\tau}$$

3 The Three Forms of Market Segmentation

In this section, I solve this cross-market preferred-habitat term structure model under three different forms of market segmentation. I start with the simplest case of the market in the absence of three representative arbitrageurs, which contains only the $2T$ preferred-habitat investors, who solely clear the markets. In this extreme case, the two parallel markets, market C and market E, remain segmented; beyond that, the T maturity clienteles in each market remain segmented. I call this case the *extreme-form* of market segmentation. The second case contains the $2T$ preferred-habitat investors and the two within-market arbitrageurs, *WMA.C* and *WMA.E*. In this case, market C and market E remain segmented, while the term structures in market C and market E are smoothened and rendered arbitrage-free by *WMA.C* and *WMA.E*. I call this case the *semi-form* of market segmentation. However, in the absence of the representative cross-market arbitrageur, arbitrage opportunities exist in the *semi-form* segmentation due to the yield differential across the two term structures. The third case, being the most intricate one to solve, contains the last market participant in the game, and perhaps the most vital one, the cross-market arbitrageur, *CMA*. The cross-market arbitrageur exploits the yield differential across the two term structures as fully as possible and closes the gap between the two markets to some extent. In specific, the arbitrageur's ability to close the yield spread gap is contingent on a group of factors, which I discuss later.

3.1 *The extreme-form of market segmentation*

Consider a simple situation with the absence of all three representative arbitrageurs. In this case, demand for bonds with specific maturities are only demanded by preferred-habitat investors, *PHI.C* and *PHI.E*, so that the term structure in each market displays extreme segmentation. In this extreme case, the equilibrium allocations satisfy the market clearing conditions

$$Y_{t,\tau}^{E} = \alpha^{E}(\tau)\tau\left(y_{t,\tau}^{E} - \beta_{t,\tau}^{E}\right) = \alpha^{E}(\tau)\tau y_{t,\tau}^{E} - \beta_{t,\tau}^{E} = S_{t,\tau}^{E}$$
$$Y_{t,\tau}^{C} = \alpha^{C}(\tau)\tau\left(y_{t,\tau}^{C} - d^{C}\right) = \alpha^{C}(\tau)\tau y_{t,\tau}^{C} - \beta_{t,\tau}^{C} = S_{t,\tau}^{C}$$

so that the equilibrium yields to maturity and allocations are

$$\left(y_{t,\tau}^{E\,*}, y_{t,\tau}^{C\,*}\right) = \left(d^{E} + \theta(\tau)\beta_t + \frac{S_{t,\tau}^{E}}{\alpha^{E}(\tau)\tau}, d^{C} + \frac{S_{t,\tau}^{C}}{\alpha^{C}(\tau)\tau}\right)$$
$$\left(Y_{t,\tau}^{E\,*}, Y_{t,\tau}^{C\,*}\right) = \left(S_{t,\tau}^{E}, S_{t,\tau}^{C}\right)$$

Such segmentation, however, is not of great interest and does not exist in reality, as the within-market arbitrageurs exploit the unequal risk-to-return trade-offs across nearby maturities and eventually render the term structure arbitrage-free in equilibrium.

3.2 *The semi-form of market segmentation*

Consider a more complicated situation under the absence of only the cross-market arbitrageur. In this case, the two parallel bond markets remain fully segmented, while the presence of the two within-market arbitrageurs, *WMA.C* and *WMA.E*, render the two term structures smooth and arbitrage-free. Without the intervention of the cross-market arbitrageur, *CMA*, the within-market arbitrageur in market *C*, *WMA.C*, solves the optimization problem:

$$\max_{Z_{t,\tau}^C} E[-\exp(-\gamma^C dW_t)\,|F_t]$$

$$\text{s.t.}\quad dW_t = W_t r_t dt - r_t \int_0^T Z_{t,\tau}^C d\tau dt + \int_0^T Z_{t,\tau}^C \left[\frac{dP^C(r_t,\tau)}{P^C(r_{t-},\tau)}\right] d\tau$$

The solution of the *WMA.C*'s optimization problem leads to lemma 1.

***Lemma 1.* (*Semi-form, WMA.C*)**

The within-market arbitrageur WMA.C's first order condition leads to the following two equations, with one variable κ_r^C to solve for:

$$A_r^{C\prime} + \kappa_r A_r^C - 1 = -\gamma^C[\sigma_r^2 + \mathrm{Var}(\Delta r)]A_r^C \int_0^T \alpha^C {A_r^C}^2 d\tau \tag{3.2.1}$$

$$C^{C\prime} - A_r^C[\kappa_r \bar{r} + \lambda_r E(\Delta r_t)] + \frac{1}{2}[\sigma_r^2 + \mathrm{Var}(\Delta r)]{A_r^C}^2 = \gamma^C[\sigma_r^2 + \mathrm{Var}(\Delta r)]A_r^C \int_0^T A_r^C\left[S_{t,\tau}^C - \alpha^C C^C + d^C\right] d\tau \tag{3.2.2}$$

Under the one-factor setting, the no-arbitrage condition turns into requiring the price of the short rate factor to be equal to the ratio of any asset's expected excess return to its factor sensitivity, which leads to the first order condition in the proof of Lemma 1 in the appendix. The two first-order linear ordinary differential equations (3.2.1) and (3.2.2) are characterized by the functions $A_r^C(\tau)$ and $C^C(\tau)$ and are obtained from extracting the coefficient of r_t in the first order condition with respect to $Z_{t,\tau}^C$, *WMA.C*'s allocation. The price of the short rate is $-\gamma^C[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^T Z_{t,\tau}^C A_r^C d\tau$, where $Z_{t,\tau}^C = S_{t,\tau}^C - \alpha^C(A_r^C r_t + C^C) + d^C$ is imposed under the market clearing condition. The price of the short rate increases in *WMA.C*'s risk-aversion γ^C, the variance of the short rate $\sigma_r^2 + \mathrm{Var}(\Delta r)$, and the sensitivity of portfolio's return $-\int_0^T Z_{t,\tau}^C A_r^C d\tau$. The next step is to solve for the functions $A_r^C(\tau)$ and $C^C(\tau)$ under the initial conditions $A_r^C(0) = C^C(0) = 0$, as a bond with zero maturity is priced at its face value. However, both ordinary differential equations cannot be solved using the standard approach, as the coefficient of $A_r^C(\tau)$ in (3.2.1) involves $A_r^C(\tau)$, and the constant term in (3.2.2) involves $C^C(\tau)$. This leads to Proposition 1.

Proposition 1. (Semi-form, WMA.C)

The function $A_r^C(\tau)$ is given by

$$A_r^C(\tau) = \frac{1 - e^{-\kappa_r^C \tau}}{\kappa_r^C}$$

where κ_r^C is the perturbed risk-neutral mean-reversion parameter and is given by

$$\kappa_r^C = \kappa_r + \gamma^C[\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T \alpha^C {A_r^C}^2 d\tau$$

For some perturbed risk-neutral long-run mean $\bar{r}^C(\{S_{t,\tau}^C\})$, which is a function of the sequence of market-C bond supplies at each maturity τ, the constant term in the yield $C^C(\tau)$ satisfies

$$C^C(\tau) = \kappa_r^C \bar{r}^C \int_0^\tau A_r^C(s)ds - \frac{1}{2}[\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^\tau A_r^C(s)^2 ds$$

where $\bar{r}^C$ is the risk-neutral long-run mean of the short rate factor and is given by

$$\bar{r}^C = \bar{r} + \frac{\lambda_r E(\Delta r_t) + \gamma^C[\sigma_r^2 + \mathrm{Var}(\Delta r)]\left\{\int_0^T \left(S_{t,\tau}^C + d^C - \tau\alpha^C\bar{r}\right)A_r^C d\tau + \frac{1}{2}\int_0^T \alpha^C\left[\int_0^\tau A_r^C(s)^2 ds\right]A_r^C d\tau\right\}}{\kappa_r^C\left\{1 + \gamma^C[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^T \alpha^C\left[\int_0^\tau A_r^C(s)ds\right]A_r^C d\tau\right\}}$$

Having analyzed the one-factor situation of market *C* in which the term structure is fully specified by the short rate factor, I now generalize the analysis to the less restrictive situation in which demand is non-stochastic and is captured by the net demand factor, β_t. This corresponds to the term structure in market *E*, which is specified by both the short rate factor and the net demand factor. Without the intervention of the cross-market arbitrageur, *CMA*, the within-market arbitrageur in market *E*, *WMA.E*, solves the optimization problem:

$$\max_{Z_{t,\tau}^E} E[-\exp(-\gamma^E dW_t)\,|F_t]$$

$$\text{s.t.} \quad dW_t = W_t r_t dt - (r_t + \beta_t)\int_0^T Z_{t,\tau}^E d\tau dt + \int_0^T Z_{t,\tau}^E \left[\frac{dP^E(r_t,\tau)}{P^E(r_{t-},\tau)}\right] d\tau$$

Note that the optimization problem of the *WMA.E* is slightly different from that of the *WMA.C*, as the coefficient of $\int_0^T Z_{t,\tau}^E d\tau dt$ is now $r_t + \beta_t$, rather than r_t in the *WMA.C*'s problem. First, the risk-aversion of the *WMA.E* should be strictly positive to get her involved in trading across different maturity clienteles and profit from the pricing differentials, so that the net demand factor can affect yields. Second, the coefficient of $\int_0^T Z_{t,\tau}^E d\tau dt$ should be r_t^E, which is the short rate in market *E* and is different from r_t, the short rate in market *C*. As explained in Section 2.3, not only does the net demand factor β_t capture the spread between the two short rates, but it also captures the difference in demand across the two markets at different maturities. The assumption that requires the net demand factor β_t to capture the difference in demand across the two markets is not necessary, but it more closely represents what happens in reality due to the heterogeneity in market participants across the two markets. Suppose the two markets are homogeneous in investor types, The solution of the *WMA.E*'s optimization problem leads to lemma 2.

***Lemma 2.* (*Semi-form, WMA.E*)**

The within-market arbitrageur WMA.E's first order condition leads to the following three equations, with two variables $(\kappa_r^E, \kappa_\beta^E)$ *to solve for:*

$$A_r^{E'} + \kappa_r A_r^E - 1 = -\gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)]A_r^E \int_0^T \alpha^E {A_r^E}^2 d\tau - \gamma^E \sigma_\beta^2 A_\beta^E \int_0^T \alpha^E A_\beta^E A_r^E d\tau \tag{3.2.3}$$

$$A_\beta^{E'} + \kappa_\beta A_\beta^E - 1 = -\gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)]A_r^E \int_0^T A_r^E\left(\alpha^E A_\beta^E - \theta\right) d\tau - \gamma^E \sigma_\beta^2 A_\beta^E \int_0^T A_\beta^E\left(\alpha^E A_\beta^E - \theta\right) d\tau \tag{3.2.4}$$

$$\begin{aligned} &C^{E'} - A_r^E[\kappa_r \bar{r} + \lambda_r E(\Delta r_t)] + \frac{1}{2}[\sigma_r^2 + \mathrm{Var}(\Delta r)]{A_r^E}^2 + \frac{1}{2}\sigma_\beta^2 {A_\beta^E}^2 \\ &\qquad = \gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)]A_r^E \int_0^T \left[S_{t,\tau}^E - \alpha^E C^E + d^E\right] A_r^E d\tau \\ &\qquad + \gamma^E \sigma_\beta^2 A_\beta^E \int_0^T \left[S_{t,\tau}^E - \alpha^E C^E + d^E\right] A_\beta^E d\tau \end{aligned} \tag{3.2.5}$$

Under the two-factor setting, the no-arbitrage condition turns into requiring the prices of the short rate factor and of the net demand factor to be equal to the ratios of any asset's expected excess return to these two factors' sensitivity, respectively, leading to the first order condition in the proof of Lemma 2 in the appendix. The functions $A_r^E(\tau)$, $A_\beta^E(\tau)$ and $C^E(\tau)$ characterize the three first-order linear ordinary differential equations in (3.2.3), (3.2.4) and (3.2.5), which are obtained after imposing the market clearing condition and from extracting the coefficients of r_t and β_t in the first order condition with respect to the *WMA.E*'s allocation $Z_{t,\tau}^E$. The price of the short rate and the price of the net demand factor are $-\gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)]A_r^E \int_0^T Z_{t,\tau}^E A_r^E d\tau$ and $-\gamma^E \sigma_\beta^2 A_\beta^E \int_0^T Z_{t,\tau}^E A_\beta^E d\tau$, respectively, where $Z_{t,\tau}^E = S_{t,\tau}^E - \alpha^E\left(A_r^E r_t + A_\beta^E \beta_t + C^E\right) + d^E + \theta\beta_t$. The price of the short rate increases in the *WMA.E*'s risk-aversion γ^E, the variance of the short rate $\sigma_r^2 + \mathrm{Var}(\Delta r)$, and the sensitivity of portfolio's return to the short rate $-\int_0^T Z_{t,\tau}^E A_r^E d\tau$. The price of the net demand factor increases in the *WMA.E*'s risk-aversion γ^E, the variance of the net demand process σ_β^2, and the sensitivity of portfolio's return to the demand process $-\int_0^T Z_{t,\tau}^E A_\beta^E d\tau$. The next step is to solve for the functions $A_r^E(\tau)$, $A_\beta^E(\tau)$ and $C^E(\tau)$, under the initial conditions $A_r^E(0) = A_\beta^E(0) = C^E(0) = 0$, as a bond with zero maturity is priced at its face value. However, all three ordinary differential equations can neither be solved using the standard approach, nor can the system be solved using the approach in the one-factor case as in Lemma 1. This is because the ordinary differential equation (3.2.3) of A_r^E involves A_β^E, while A_r^E also appears in the ordinary differential equation (3.2.4) of A_β^E. The inter-dependency of the parameters that govern the short rate process and those that model the net demand process leads to the ODE system in Proposition 2.

Proposition 2. **(*Semi-form, WMA.E*)**

The functions $A_r^E(\tau)$ and $A_\beta^E(\tau)$ are solutions of the first order ODE system $\boldsymbol{X}' = \boldsymbol{A}\boldsymbol{X} + \boldsymbol{g}$:

$$\underbrace{\begin{bmatrix} A_r^{E\prime} \\ A_\beta^{E\prime} \end{bmatrix}}_{\boldsymbol{X}'} = -\underbrace{\begin{bmatrix} k_r + \gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^T \alpha^E A_r^{E^2} d\tau & \gamma^E \sigma_\beta^2 \int_0^T \alpha^E A_\beta^E A_r^E d\tau \\ \gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^T A_r^E\left(\alpha^E A_\beta^E - \theta\right) d\tau & \kappa_\beta + \gamma^E \sigma_\beta^2 \int_0^T A_\beta^E\left(\alpha^E A_\beta^E - \theta\right) d\tau \end{bmatrix}}_{\boldsymbol{A}} \underbrace{\begin{bmatrix} A_r^E \\ A_\beta^E \end{bmatrix}}_{\boldsymbol{X}} + \underbrace{\begin{bmatrix} 1 \\ 1 \end{bmatrix}}_{\boldsymbol{g}}$$

for some perturbed long-run means $\bar{r}^E(\{S_{t,\tau}^E\})$ and $\bar{\beta}^E(\{S_{t,\tau}^E\})$, which are functions of the sequence of market-E bond supplies at each maturity τ. Suppose that the matrix $\boldsymbol{A}$ has two distinct eigenvalues $(\kappa_r^E, \kappa_\beta^E)$, the functions $A_r^E(\tau)$ and $A_\beta^E(\tau)$ are given by

$$A_r^E(\tau) = \psi_r\left(\frac{1 - e^{-\kappa_\beta^E \tau}}{\kappa_\beta^E} - y_1 \frac{1 - e^{-\kappa_r^E \tau}}{\kappa_r^E}\right)$$
$$A_\beta^E(\tau) = \psi_\beta\left(\frac{1 - e^{-\kappa_\beta^E \tau}}{\kappa_\beta^E} - y_2 \frac{1 - e^{-\kappa_r^E \tau}}{\kappa_r^E}\right)$$

for some constants ψ_r, ψ_β, y_1 and y_2, where $\psi_r(1 - y_1) = 1$ and $\psi_\beta(1 - y_2) = 1$. The constant term in the yield, $C^E(\tau)$, is given by

$$C^E(\tau) = \kappa_r^E \bar{r}^E \int_0^\tau A_r^E(s)ds + \kappa_\beta^E \bar{\beta}^E \int_0^\tau A_\beta^E(s)ds - \frac{1}{2}[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^\tau A_r^E(s)^2 ds - \frac{1}{2}\sigma_\beta^2 \int_0^\tau A_\beta^E(s)^2 ds$$

where $\bar{r}^E$ and $\bar{\beta}^E$ are the risk-neutral long-run means of the short rate factor and the demand factor, respectively, and are given by

$$\bar{r}^E = \bar{r} + \frac{\lambda_r E(\Delta r_t) + \gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)]\left\{\int_0^T \left(S_{t,\tau}^E + d^E - \tau \alpha^E \bar{r}\right) A_r^E d\tau + \frac{1}{2}\int_0^T \alpha^E \left[\int_0^\tau A_r^{E^2} ds\right] A_r^E d\tau\right\}}{\kappa_r^E \left\{1 + \gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^T \alpha^E \left[\int_0^\tau A_r^E ds\right] A_r^E d\tau\right\}}$$
$$\bar{\beta}^E = \frac{\gamma^E \sigma_\beta^2 \left\{\int_0^T \left(S_{t,\tau}^E + d^E\right) A_\beta^E d\tau + \frac{1}{2}\int_0^T \alpha^E \left[\int_0^\tau A_\beta^{E^2} ds\right] A_\beta^E d\tau\right\}}{\kappa_\beta^E \left\{1 + \gamma^E \sigma_\beta^2 \int_0^T \alpha^E \left[\int_0^\tau A_\beta^{E^2} ds\right] A_\beta^E d\tau\right\}}$$

Solution algorithm – Semi-form of market segmentation

① Solve for κ_r^C from (3.2.1).

② Substitute κ_r^C into equation (3.2.2) to solve for $C^C(\tau)$.

③ Solve for κ_r^E and κ_β^E jointly from equations (3.2.3) and (3.2.4).

④ Substitute κ_r^E and κ_β^E into equation (3.2.5) to solve for $C^E(\tau)$.

3.3 *The weak-form of market segmentation*

Once the cross-market arbitrageur comes into play, the five types of representative agent co-determine their equilibrium allocations. I take the estimated net demand factor series β_t as given from the case of *semi-form* market segmentation. I simultaneously solve the first order conditions of the three representative arbitrageurs. Given the presence of a yield differential across the two markets, the *CMA* would like to short an infinite amount of the more expensive bonds in one market and use the proceeds to buy an infinite amount of the cheaper bonds in the other market. However, she could not find a counterparty to enter the trade. This explains why imposing short-selling constraints is unnecessary, as long as the cross-market arbitrageur remains risk-averse.

First, I consider the cross-market arbitrageur's optimization problem. To reiterate, I conjecture that the solutions can be written as $X_{t,\tau}^C \equiv F^C(\tau)\beta_t^C$ and $X_{t,\tau}^E \equiv F^E(\tau)\beta_t^E$ for some finite integrable functions $F^C(\tau), F^E(\tau) \le 1$ of maturity τ. The functional forms of $F^C(\tau)$ and $F^E(\tau)$ are discussed in the calibration exercise. The cross-market arbitrageur, *CMA*, solves the optimization problem:

$$\max_{X_{t,\tau}^C, X_{t,\tau}^E} E[-\exp(-\gamma^A dW_t)\,|F_t]$$

$$\begin{aligned}
\text{s.t.}\quad dW_t &= W_t r_t dt - r_t \int_0^T X_{t,\tau}^C d\tau dt - (r_t + \beta_t)\int_0^T X_{t,\tau}^E d\tau dt \\
&\quad + \int_0^T X_{t,\tau}^C \left[\frac{dP^{C,A}(r_t,\tau)}{P^{C,A}(r_{t-},\tau)}\right] d\tau + \int_0^T X_{t,\tau}^E \left[\frac{dP^{E,A}(r_t,\tau)}{P^{E,A}(r_{t-},\tau)}\right] d\tau \\
S_{t,\tau}^C &= X_{t,\tau}^C + Y_{t,\tau}^C + Z_{t,\tau}^C \\
S_{t,\tau}^E &= X_{t,\tau}^E + Y_{t,\tau}^E + Z_{t,\tau}^E
\end{aligned}$$

***Lemma 3.* (*Weak-form, CMA*)**

The cross-market arbitrageur's first order conditions lead to the following two equations, with five variables $\{\kappa_r^{C,A}, \kappa_r^{E,A}, \kappa_\beta^{E,A}, \beta_t^C, \beta_t^E\}$ *to solve for, together with the two within-market arbitrageurs' first order conditions:*

$$\begin{aligned}
&A_r^{C,A'} r_t + C^{C,A'} - [\kappa_r(\bar{r} - r_t) + \lambda_r E(\Delta r_t)]A_r^{C,A} + \frac{1}{2}[\sigma_r^2 + \mathrm{Var}(\Delta r)]A_r^{C,A^2} - r_t \\
&\qquad = \gamma^A[\sigma_r^2 + \mathrm{Var}(\Delta r)]A_r^{C,A}\left(\beta_t^C \int_0^T F^C A_r^{C,A} d\tau + \beta_t^E \int_0^T F^E A_r^{E,A} d\tau\right)
\end{aligned} \tag{3.3.1}$$

$$\begin{aligned}
&A_r^{E,A'} r_t + A_\beta^{E,A'} \beta_t + C^{E,A'} - [\kappa_r(\bar{r} - r_t) + \lambda_r E(\Delta r_t)]A_r^{E,A} + A_\beta^{E,A}\kappa_\beta\beta_t + \frac{1}{2}[\sigma_r^2 + \mathrm{Var}(\Delta r)]A_r^{E,A^2} \\
&\qquad + \frac{1}{2}\sigma_\beta^2 A_\beta^{E,A^2} - r_t - \beta_t \\
&\qquad = \gamma^A[\sigma_r^2 + \mathrm{Var}(\Delta r)]A_r^{E,A}\left(\beta_t^E \int_0^T F^E A_r^{E,A} d\tau + \beta_t^C \int_0^T F^C A_r^{C,A} d\tau\right) \\
&\qquad + \gamma^A \sigma_\beta^2 A_\beta^{E,A}\left(\beta_t^E \int_0^T F^E A_\beta^{E,A} d\tau\right)
\end{aligned} \tag{3.3.2}$$

The equilibrium allocations are co-determined among the $2T+3$ representative agents. Since lemma 3 leads to two equations with five variables to solve for, they have to be solved along with the first order conditions of the two within-market arbitrageurs. Specifically, the mean-reversion parameters $\{\kappa_r^{C,A}, \kappa_r^{E,A}, \kappa_\beta^{E,A}\}$ can be solved directly from the first order conditions of the *WMA.C* and the *WMA.E* through equating the coefficients of the corresponding pricing factors to zero.

With the intervention of the cross-market arbitrageur, *CMA*, the *within-market* arbitrageur in market *C*, *WMA.C*, solves the optimization problem:

$$\max_{Z_{t,\tau}^C} E[-\exp(-\gamma^C dW_t)\,|F_t]$$

$$\text{s.t.}\quad dW_t = W_t r_t dt - r_t \int_0^T Z_{t,\tau}^C d\tau dt + \int_0^T Z_{t,\tau}^C \left[\frac{dP^{C,A}(r_t,\tau)}{P^{C,A}(r_{t-},\tau)}\right] d\tau$$

***Lemma 4.* (*Weak-form, WMA.C*)**

Under the CMA's intervention, the within-market arbitrageur WMA.C's first order condition leads to the following two equations, with two variables $\{\kappa_r^{C,A}, \beta_t^C\}$ and one constant term $\{C^{C,A}\}$ to solve for:

$$A_r^{C,A\prime} + \kappa_r A_r^{C,A} - 1 = -\gamma^C[\sigma_r^2 + \mathrm{Var}(\Delta r)]A_r^{C,A} \int_0^T \alpha^C {A_r^{C,A}}^2 d\tau \tag{3.3.3}$$

$$C^{C,A\prime} - A_r^{C,A}[\kappa_r \bar{r} + \lambda_r E(\Delta r_t)] + \frac{1}{2}[\sigma_r^2 + \mathrm{Var}(\Delta r)]{A_r^{C,A}}^2$$
$$= \gamma^C[\sigma_r^2 + \mathrm{Var}(\Delta r)]A_r^{C,A} \int_0^T A_r^{C,A}\left[S_{t,\tau}^C - F^C\beta_t^C - \alpha^C C^{C,A} + d^C\right] d\tau \tag{3.3.4}$$

***Proposition 4.* (*Weak-form, WMA.C*)**

$A_r^{C,A}(\tau)$ is a function of the perturbed mean-reversion parameter $\kappa_r^{C,A}$

$$A_r^{C,A}(\tau) = \frac{1 - e^{-\kappa_r^{C,A}\tau}}{\kappa_r^{C,A}}$$

where $\kappa_r^{C,A}$ is the perturbed risk-neutral mean-reversion parameter and is given by

$$\kappa_r^{C,A} = \kappa_r + \gamma^C[\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T \alpha^C {A_r^{C,A}}^2 d\tau$$

For some perturbed long-run mean parameter $\bar{r}^{C,A}(S_{t,\tau}^C, S_{t,\tau}^E)$, which is a function of market-C and market-E bond supplies at each maturity τ, $C^{C,A}(\tau)$, the constant term in the yield, is solved from (3.3.4) and is specified in the Appendix.

With the intervention of the cross-market arbitrageur *CMA*, the *within-market* arbitrageur in market *E*, *WMA.E*, solves the optimization problem:

$$\max_{Z_{t,\tau}^{E}} E[-\exp(-\gamma^E dW_t)\,|F_t]$$

$$\text{s.t.} \quad dW_t = W_t r_t dt - (r_t + \beta_t)\int_0^T Z_{t,\tau}^E d\tau dt + \int_0^T Z_{t,\tau}^E \left[\frac{dP^{E,A}(r_t,\tau)}{P^{E,A}(r_{t-},\tau)}\right] d\tau$$

Lemma 5. (Weak-form, WMA.E)

Under the CMA's intervention, the within-market arbitrageur WMA.E's first order condition leads to the following three equations, with three variables $\{\kappa_r^{E,A}, \kappa_\beta^{E,A}, \beta_t^E\}$ and one constant term $\{C^{E,A}\}$ to solve for:

$$A_r^{E,A\prime} + \kappa_r A_r^{E,A} - 1 = -\gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)]A_r^{E,A}\int_0^T \alpha^E A_r^{E,A^2} d\tau - \gamma^E \sigma_\beta^2 A_\beta^{E,A}\int_0^T \alpha^E A_\beta^{E,A} A_r^{E,A} d\tau \tag{3.3.5}$$

$$\begin{aligned} A_\beta^{E,A\prime} + \kappa_\beta A_\beta^{E,A} - 1 = &-\gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)]A_r^{E,A}\int_0^T A_r^{E,A}\left(\alpha^E A_\beta^{E,A} - \theta\right) d\tau \\ &-\gamma^E \sigma_\beta^2 A_\beta^{E,A}\int_0^T A_\beta^{E,A}\left(\alpha^E A_\beta^{E,A} - \theta\right) d\tau \end{aligned} \tag{3.3.6}$$

$$\begin{aligned} C^{E,A\prime} - A_r^{E,A}[\kappa_r \bar{r} + \lambda_r E(\Delta r_t)] + \frac{1}{2}[\sigma_r^2 + \mathrm{Var}(\Delta r)]A_r^{E,A^2} + \frac{1}{2}\sigma_\beta^2 A_\beta^{E,A^2} \\ = \gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)]A_r^{E,A}\int_0^T A_r^{E,A}\left[S_{t,\tau}^E - F^E\beta_t^E - \alpha^E C^{E,A} + d^E\right] d\tau \\ + \gamma^E \sigma_\beta^2 A_\beta^{E,A}\int_0^T A_\beta^{E,A}\left[S_{t,\tau}^E - F^E\beta_t^E - \alpha^E C^{E,A} + d^E\right] d\tau \end{aligned} \tag{3.3.7}$$

Proposition 5. (Weak-form, WMA.E)

$A_r^{E,A}(\tau)$ and $A_\beta^{E,A}(\tau)$ are solutions of the first order ODE system

$$\boldsymbol{X}' = \boldsymbol{A}\boldsymbol{X} + \boldsymbol{g}$$

where

$$\boldsymbol{X} = \begin{bmatrix} A_r^{E,A} & A_\beta^{E,A} \end{bmatrix}^T \qquad \boldsymbol{g} = [1 \quad 1]^T$$

$$\boldsymbol{A} = -\begin{bmatrix} \kappa_r + \gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^T \alpha^E A_r^{E,A^2} d\tau & \gamma^E\sigma_\beta^2\int_0^T \alpha^E A_\beta^{E,A} A_r^{E,A} d\tau \\ \gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^T A_r^{E,A}\left(\alpha^E A_\beta^{E,A} - \theta\right) d\tau & \kappa_\beta + \gamma^E\sigma_\beta^2\int_0^T A_\beta^{E,A}\left(\alpha^E A_\beta^{E,A} - \theta\right) d\tau \end{bmatrix}$$

For some perturbed long-run mean parameter $\bar{r}^E\left(\{S_{t,\tau}^C\},\{S_{t,\tau}^E\}\right)$, which is a function of the sequence of both market-C and market-E bond supplies at each maturity τ, $C^{E,A}(\tau)$, the constant term in the yield, is solved from (3.3.7) and is specified in the Appendix.

Solution algorithm – Weak-form of market segmentation

I take as given the demand factor β_t and the volatility term σ_β obtained from the segmented case in 3.B. I solve for the five variables $\{\kappa_r^{C,A}, \kappa_r^{E,A}, \kappa_\beta^{E,A}, \beta_t^C, \beta_t^E\}$ and the two constant terms $\{C^{C,A}, C^{E,A}\}$ from the system of seven equations (3.3.1) ~ (3.3.7).

① Solve for $\kappa_r^{C,A}$ from (3.3.3).

② Solve for $\kappa_r^{E,A}$ and $\kappa_\beta^{E,A}$ from (3.3.5) and (3.3.6).

③ Rewrite $C^{C,A}$ as a function of β_t^C from (3.3.4).

④ Rewrite $C^{E,A}$ as a function of β_t^E from (3.3.7).

⑤ Substitute $C^{C,A}$ and $C^{E,A}$ into (3.3.1) and (3.3.2) so that they are functions of β_t^C and β_t^E alone.

⑥ Substitute $\kappa_r^{C,A}$, $\kappa_r^{E,A}$ and $\kappa_\beta^{E,A}$ into (3.3.1) and (3.3.2) to solve for β_t^C and β_t^E.

4 Theoretical Implications I

In this section, I discuss some of the important theoretical implications from the previous section for the *semi-form* of market segmentation. I study how exogenous supply and demand shocks affect the term structures in both markets. The first three corollaries correspond to market *C*, with the short rate being the only risk factor and the net demand factor being non-stochastic. The next two corollaries correspond to market *E*, with the short rate and the net demand factor both functioning as the risk factors.

Suppose market *C* experiences an exogenous supply shock $\Delta S_{t,\tau^*}^C$ at maturity τ^*. Suppose $\Delta S_{t,\tau^*}^C > 0$ (< 0), this will lead to an increase (a decrease) in the *perturbed* risk-neutral long-run mean of the short rate $\bar{r}^C$ by $\Delta\bar{r}^C(\Delta S_{t,\tau^*}^C)$. The intercept term $C^C(\tau)$ will subsequently increase (decrease) by $\kappa_r^C \Delta\bar{r}^C(\Delta S_{t,\tau^*}^C)\int_0^\tau A_r^C(s)ds$, leading to an increase (decrease) in the yield at maturity τ, $y_{t,\tau}^C$, by $\Delta y_{t,\tau}^C \equiv \tau^{-1}\kappa_r^C \Delta\bar{r}^C(\Delta S_{t,\tau^*}^C)\int_0^\tau A_r^C(s)ds$. The dependence of $\Delta y_{t,\tau}^C$ on τ indicates that that exogenous supply shock affects the term structure unevenly at different maturities.[5] Moreover, the supply change is a local shock at maturity τ^* and is thus independent of other maturities, causing the preferred habitat investors and the within-market arbitrageurs to adjust their bond allocations accordingly. This leads to Proposition 6.

Proposition 6. (Semi-form, market C, global supply effects)

In the semi-form of market segmentation, an exogenous positive shock that increases the supply from S_{t,τ^*}^C to $S_{t,\tau^*}^C + \Delta S_{t,\tau^*}^C$ at maturity τ^* in market C affects yields when the within-market arbitrageur WMA.C is risk-averse. The yield $y_{t,\tau}^C$ increases at any maturity τ as the entire term structure in market C is pushed up. The supply effect is global as the relative effect across maturities is independent of the origin of the supply shock. The yields at longer maturities are more affected. The supply shock induces a higher demand of the

[5] The term structure moves unevenly in the sense that the upward or downward shift is not parallel.

PHI.C at any maturity τ. For the WMA.C, the supply shock induces a lower demand at maturity $\tau \neq \tau^*$ and a higher demand at maturity $\tau = \tau^*$.

Proof. See Appendix.

One key takeaway from Proposition 6 is its delineation of the redistribution of bond investments in the new equilibrium due to an exogenous positive supply shock. The unanticipated supply of $\Delta S_{t,\tau^*}^C$ at maturity τ^* has to be shared between the clientele investor at maturity τ^*, *PHI.C*τ^*, and the within-market arbitrageur in market *C*, *WMA.C*. Corollary 1 states that each agent shares a positive proportion of $\Delta S_{t,\tau^*}^C$, so that it will induce a decline in *WMA.C*'s allocation in maturities other than τ^* due to the global effect of the supply shock to the entire term structure. As a result, the preferred-habitat investors at each maturity clientele will increase their positions by resorting to the outside private technology.

Suppose further that the demand intercept function of the preferred habitat investors in market *C* changes from $d^C(\tau)$ to $d^C(\tau) + \Delta d^C(\tau)$, which leads to a change in the *perturbed* risk-neutral long-run mean of the short rate by $\Delta\bar{r}^C\big(\Delta d^C(\tau)\big)$. Suppose $\int_0^T \Delta d^C(\tau) A_r^C(\tau) d\tau > 0$ (< 0), this will lead to an increase (a decrease) in the intercept $C^C(\tau)$ by $\kappa_r^C \Delta\bar{r}^C\big(\Delta d^C(\tau)\big) \int_0^\tau A_r^C(s) ds$. The yield $y_{t,\tau}^C$ at maturity τ increases (decreases) by $\Delta y_{t,\tau}^C \equiv \tau^{-1} \kappa_r^C \Delta\bar{r}^C\big(\Delta d^C(\tau)\big) \int_0^\tau A_r^C(s) ds$, and the entire term structure is pushed up (down) unevenly in market *C*. Note also that the sign of $\Delta d^C(\tau)$ in the preferred-habitat investors' demand function $Y_{t,\tau}^C$ is negative, indicating that a positive (negative) $\Delta d^C(\tau^*)$ leads to a decrease (an increase) in demand. If $\Delta d^C(\tau) > 0$ (< 0) at any maturity τ, it is always true that $\int_0^T \Delta d^C(\tau) A_r^C(\tau) d\tau > 0$ (< 0). If $\Delta d^C(\tau)$ is not uniformly positive or negative, $\int_0^T \Delta d^C(\tau) A_r^C(\tau) d\tau$ can have either positive or negative signs, leading to either an increase or a decrease in the yield $y_{t,\tau}^C$. This implies that a uniform increase (decrease) in the preferred-habitat investors' demand pushes down (up) the entire term structure in market *C*. However, changes in the preferred-habitat investors' demand that differ in signs can either push down or push up the term structure in market *C*. This implication can be very useful for the government to assess the impact on buy-back programs of longer-maturity bonds. Moreover, the demand shock is global and represents a shift in the *functional* form of the demand intercept at all maturities, causing the clientele investors and the within-market arbitrageur to re-allocate their positions accordingly. This leads to Proposition 7.

Proposition 7. (Semi-form, market C, global demand effects)

In the semi-form of market segmentation, a change of the demand intercept $\Delta d^C(\tau)$ of the preferred habitat investors in market C affects yields when the within-market arbitrageur WMA.C is risk-averse. The sign *of $\int_0^T \Delta d^C(\tau) A_r^C(\tau) d\tau$ matters. For a uniformly positive (negative) demand shock, i.e. $\forall\tau$, $\Delta d^C(\tau) < 0$ ($\forall\tau$, $\Delta d^C(\tau) > 0$), the yield $y_{t,\tau}^C$ increases (decreases) at maturity τ as the entire term structure is pushed up (down). For a demand shock with unequal signs of $\Delta d^C(\tau)$ at two maturities, $\int_0^T \Delta d^C(\tau) A_r^C(\tau) d\tau$ can be either positive or negative, and the yield $y_{t,\tau}^C$ can increase or decrease at any maturity as the entire term structure is pushed up or down. The demand effect is global as the relative effect across maturities is independent of where the demand shock originates. It also induces either a higher or lower demand of the PHI.C at any maturity τ and a corresponding lower or higher demand of the WMA.C at any maturity τ.*

Proof. See Appendix.

The V&V model measures the extent to which arbitrageurs transmit short rate shocks to bond yields by examining their impact on expected short rate $f_{t,\tau}$ and on forward rates $E(r_{t+\tau})$. They find that one unit shock to the short rate raises $f_{t,\tau}$ by $e^{-k_r^C \tau}$ and raises $E(r_{t+\tau})$ by $e^{-\kappa_r \tau} > e^{-k_r^C \tau}$. Hence, arbitrageurs transmit short rate shocks to bond yields only partially as forward rates under-react and do not move one-to-one with expected future short rates. The difference $k_r^C - \kappa_r$ quantifies the extent of under-reaction of forward rates to expected short rates. As policy announcements at any time could potentially trigger unexpected jumps in the short rate, I further incorporate jumps in the short rate with deterministic jump size that follow a normal distribution of $N(E(\Delta r_t), \sigma_r^2)$. A large expected jump size of $E(\Delta r_t)$ indicates more uncertainty, and risk-averse within-market arbitrageurs would undercut their demand under more uncertainty. On the other hand, the clientele investors raise their holdings, as the exogenous supply remains unchanged. In addition, as the risk-averse within-market arbitrageurs maximize CARA utilities, equilibrium allocations of more risk-averse arbitrageurs should be less sensitive to uncertainty due to expected jumps of the short rate. This leads to Proposition 8.

Proposition 8. (Semi-form, market C, jumps in the short rate)

In the semi-form of market segmentation, an expected jump of the short rate $E(\Delta r_t)$ in market C affects yields when the within-market arbitrageur is risk-averse. Her demand $Z_{t,\tau}^C$ decreases at any maturity τ as uncertainty in the expected jump size of the short rate increases, and the preferred habitat investors increase their demand $Y_{t,\tau}^C$ accordingly. The yields of longer-maturity bonds are more sensitive to expected jumps in the short rate. The equilibrium allocations of the WMA.C, being more risk-averse, are less sensitive to uncertainty due to expected jump size of the short rate.

Proof. See Appendix.

Suppose that market E experiences an unanticipated supply shock $\Delta S_{t,\tau^*}^E$ at maturity τ^*. Suppose $\Delta S_{t,\tau^*}^E > 0$ (< 0), this will lead to an increase (a decrease) in the *perturbed* risk-neutral long-run means of the short rate $\bar{r}^E$ by $\Delta\bar{r}^E(\Delta S_{t,\tau^*}^E)$ and of the net demand factor $\bar{\beta}^E$ by $\Delta\bar{\beta}^E(\Delta S_{t,\tau^*}^E)$. The intercept term $C^E(\tau)$ will increase (decrease) by $\kappa_r^E \Delta\bar{r}^E(\Delta S_{t,\tau^*}^E)\int_0^\tau A_r^E(s)ds + \kappa_\beta^E \Delta\bar{\beta}^E(\Delta S_{t,\tau^*}^E)\int_0^\tau A_\beta^E(s)ds$. Since the change in yield $\Delta y_{t,\tau}^E(\Delta S_{t,\tau^*}^E)$ is $\tau^{-1}\Delta C^E(\tau)$, which is affine in both $\int_0^\tau A_r^E(s)ds$ and $\int_0^\tau A_\beta^E(s)ds$, its sign is unclear without detailed calculation. I show that $\Delta y_{t,\tau}^E(\Delta S_{t,\tau^*}^E) > 0$ for all $\tau \geq t$. Hence, the yield $y_{t,\tau}^C$ at maturity τ increases (decreases) as the entire term structure is pushed up (down) unevenly. Moreover, the supply shock has a global effect on all maturities, causing the clientele investors and the within-market arbitrageur to adjust their bond allocations accordingly. This leads to Proposition 9.

Proposition 9. (Semi-form, market E, global supply effects)

In the semi-form of market segmentation, an exogenous positive shock that increases supply from S_{t,τ^*}^E to $S_{t,\tau^*}^E + \Delta S_{t,\tau^*}^E$ at maturity τ^* in market E affects yields when the within-market arbitrageur is risk-averse. The yield $y_{t,\tau}^E$ increases at any maturity τ as the entire term structure in market E is pushed up. The supply effect is global as the relative effect across maturities is independent of the origin of the supply shock. This leads to a higher demand of the PHI.E at any maturity τ, yet an either higher or a lower demand of the WMA.C at any maturity τ.

Proof. See Appendix.

Unlike in market *C*, where the short rate is the only risk factor, in market *E* the net demand factor is the second risk factor and generates more *localized* demand effects relative to the one-factor situation. Suppose market *E* experiences a shock to the demand intercept $\Delta d^E(\tau)$. A positive shock, *i.e.* $\forall\tau$, $\Delta d^E(\tau) > 0$ (< 0), will increase (decrease) the *perturbed* risk-neutral long-run means of the short rate by $\Delta\bar{r}^E(\Delta d^E(\tau))$ and of the net demand factor by $\Delta\bar{\beta}^E(\Delta d^E(\tau))$, which leads to an increase (a decrease) in $C^E(\tau)$ by an amount of $\kappa_r^E \Delta\bar{r}^E(\Delta d^E(\tau))\int_0^T A_r^E(\tau)d\tau + \kappa_\beta^E \Delta\bar{\beta}^E(\Delta d^E(\tau))\int_0^T A_\beta^E(\tau)d\tau$. Unlike in Corollary 2, I show that the sign of $\Delta y_{t,\tau}^E(\Delta d^E(\tau))$ cannot be precisely pinned down at any maturity $\tau \geq T$ and depends on the relative contribution of $\int_0^\tau A_r^E(s)ds$ and $\int_0^\tau A_\beta^E(s)ds$, the coefficients of which further depend on $\int_0^T \Delta d^E(\tau)A_r^E(\tau)d\tau$ and $\int_0^T \Delta d^E(\tau)A_r^E(\tau)d\tau$, respectively. Suppose $\Delta d^E(\tau_1) > 0$ and $\Delta d^E(\tau_2) < 0$ for some $0 < \tau_1, \tau_2 < T$, both $\int_0^T \Delta d^E(\tau)A_r^E(\tau)d\tau$ and $\int_0^T \Delta d^E(\tau)A_\beta^E(\tau)d\tau$ can take either positive or negative values. If both terms are positive (negative), the yield $y_{t,\tau}^E$ increases (decreases) at all maturities, even for maturity clienteles with $\Delta d^E(\tau) < 0$ (< 0). Since both terms $\int_0^T \Delta d^E(\tau)A_r^E(\tau)d\tau$ and $\int_0^T \Delta d^E(\tau)A_\beta^E(\tau)d\tau$ come into the expression of the coefficients of $\int_0^\tau A_r^E(s)ds$ and $\int_0^\tau A_\beta^E(s)ds$, respectively, the sign of $\Delta y_{t,\tau}^E(\Delta d^E(\tau))$ is not pinned down. Thus, the demand shock is now *localized*, while yields of longer-maturity bonds can be either more or less affected. It also causes the preferred-habitat investors and the within-market arbitrageurs to adjust their bond allocations accordingly. This leads to Proposition 10.

Proposition 10. (Semi-form, market E, localized demand effects)

In the semi-form of market segmentation, a change of the demand intercept $\Delta d^E(\tau)$ of the preferred habitat investors in market E affects yields when the within-market arbitrageur is risk-averse. The sign of $\int_0^T \Delta d^E(\tau) A_r^E(\tau) d\tau$ matters. For a uniformly positive (negative) demand shock, i.e. $\forall\tau$, $\Delta d^E(\tau) < 0$ ($\forall\tau$, $\Delta d^E(\tau) > 0$), the yield $y_{t,\tau}^E$ increases (decreases) at maturity τ as the entire term structure is pushed up (down). For a demand shock with unequal signs of $\Delta d^E(\tau)$ at two distinct maturities, both $\int_0^T \Delta d^E(\tau) A_r^E(\tau) d\tau$ and $\int_0^T \Delta d^E(\tau) A_\beta^E(\tau) d\tau$ could take either positive or negative values, and the yield $y_{t,\tau}^C$ can either increase or decrease at all maturities as the entire term structure is pushed up or down. The demand effect is more localized relative to the one-factor situation in market C, as the relative effect across maturities now depends on the origin of the demand shock. This leads to either a higher or lower demand of the PHI.Eτ at any maturity τ, and either a higher or lower demand of the WMA.E at any maturity τ.

Proof. See Appendix.

5 Theoretical Implications II

In this section, I discuss some of the important theoretical implications on shock transmissions for the *weak-form* of market segmentation. I investigate how exogenous supply and demand shocks from one market affect the term structures in both markets. Again, the short rate is the only risk factor in market *C*, while both the short rate and the net demand factor are risk factors in market *E*. In Section 4, I have illustrated how exogenous supply shocks, shocks to the demand intercept and shocks to the short rate can affect the term structure, and how the preferred habitat investors and the within-market arbitrageurs internalize these shocks through adjusting bond allocations upon observing shift in the term structure. Under a parallel market structure of market *C* and market *E*, shocks to the short-rate and to the demand intercept in market *C* affect not only the term structure in market *C*, but they also affect the term structure in market *E* through the intervention of the cross-market arbitrageur, or vice versa. Meanwhile, the preferred habitat investors and the within-market arbitrageurs adjust their bond allocations accordingly in response to the changes in yield curves. Specifically, my model is able to quantify the changes in the term structure of market *E*, in response to one unit shock to the supply, to the demand intercept, and to the short rate in market *C*, or vice versa. In this section, I discuss how my model generates important policy implications for the government to assess (1) the impact of issuing or buying back sovereign bonds with a particular maturity in one market on the sovereign bond yield curve in the other market, and (2) examine how the respective market participants adjust their bond allocations to internalize the shocks.

Suppose market C experiences an exogenous positive supply shock of $\Delta S^C_{t,\tau^*}$ at maturity τ^*. As shown in Proposition 6, in the absence of the cross-market arbitrageur, the yield $y^C_{t,\tau}$ increases by an amount of $\Delta y^C_{t,\tau} \equiv \kappa^C_r \Delta \bar{r}^C \left(\tau^{-1} \int_0^\tau A^C_r(s)ds\right)\Delta S^C_{t,\tau^*}$, which is linear in $\tau^{-1} \int_0^\tau A^C_r(s)ds$. In the absence of the cross-market arbitrageur, changes in yields are fully determined by the parameters governing the term structure of market C and are not influenced by those of market E. Once the cross-market arbitrageur intervenes, however, changes in yields are also affected by the parameters governing the term structure of market E, as long as the cross-market arbitrageur is risk-averse. In Proposition 11, I show that a positive supply shock induces an increase in $y^{C,A}_{t,\tau}$ by $\Delta y^{C,A}_{t,\tau}$, which is proportional to $\Delta S^C_{t,\tau^*}$ and is affine in $\tau^{-1} \int_0^\tau A^{C,A}_r(s)ds$. The coefficient in the front of the expression of $\Delta y^{C,A}_{t,\tau}$ manifests the dependence on the parameters governing the term structure of market E. This intuitively induces the cross-market arbitrageur to purchase more sovereign bonds in market C, meanwhile reducing her allocation of the market–E bonds. A reduction in the aggregate demand at maturity clientele τ in market E will lead to a rise in $y^{E,A}_{t,\tau}$, causing an upward shift of $\Delta y^{E,A}_{t,\tau}$ in the term structure of market E. The increase $\Delta y^{E,A}_{t,\tau}$ is proportional to $\Delta S^C_{t,\tau^*}$ and is affine in $\tau^{-1} \int_0^\tau A^{E,A}_r(s)ds$ and $\tau^{-1} \int_0^\tau A^{E,A}_\beta(s)ds$, which correspond to the two risk factors governing the term structure of market E. Again, the coefficient in the front of the expression of $\Delta y^{E,A}_{t,\tau}$ manifests the dependence on the parameters governing the term structure of market C. Corollary 6 is given below. In Proposition 12, I analyze the transmission of an exogenous supply shock of $\Delta S^E_{t,\tau^*}$ at maturity τ^* from market E to market C.

***Proposition 11.* (*Supply shock transmission, market C → E*)**

In the weak-form of market segmentation, under the presence of the cross-market arbitrageur, an exogenous positive shock that increases supply from S^C_{t,τ^*} to $S^C_{t,\tau^*} + \Delta S^C_{t,\tau^*}$ at maturity τ^* in market C affects yields in both markets when the three arbitrageurs are all risk-averse. The yield $y^C_{t,\tau}$ increases at any maturity τ as the term structure in market C is pushed up. The cross-market arbitrageur increases her demand for market–C sovereign bonds and decreases her demand for market–E sovereign bonds. This subsequently pushes up the term structure in market E.

Proof. See Appendix.

***Proposition 12.* (*Supply shock transmission, market E → C*)**

In the weak-form of market segmentation, under the presence of the cross-market arbitrageur, an exogenous positive shock that increases supply from S^E_{t,τ^*} to $S^E_{t,\tau^*} + \Delta S^E_{t,\tau^*}$ at maturity τ^* in market E affects yields in both markets when the three arbitrageurs are all risk-averse. The yield $y^E_{t,\tau}$ increases at any maturity τ as the term structure in market E is pushed up. The cross-market arbitrageur increases her demand for *sovereign bonds in market E and decreases her demand for sovereign bonds in market C. This subsequently pushes up the term structure in market C.*

Proof. See Appendix.

I now analyze the impact of shocks to the demand intercept on the term structures of the two markets. Suppose market *C* experiences a shock to the demand intercept $\Delta d^C(\tau)$. A positive shock $\Delta d^C(\tau) > 0$ (< 0) will induce a change of the intercept tem $C^{C,A}(\tau)$ by $\Delta C^{C,A}(\Delta d^C(\tau))$, which is affine in $\int_0^T A_r^{C,A}(\tau)d\tau$. Similar to Corollary 2, I show that the sign of $\Delta y_{t,\tau}^{C,A}(\Delta d^C(\tau))$ cannot be precisely pinned down at any maturity $\tau \geq T$ and depends on the coefficient of $\int_0^\tau \mathrm{A}_\mathrm{r}^{\mathrm{C,A}}(\mathrm{s})\mathrm{ds}$ that involves an integral term of $\Delta d^C(\tau)$. Suppose $\Delta d^C(\tau_1) > 0$ and $\Delta d^C(\tau_2) < 0$ for some $0 < \tau_1, \tau_2 < T$, the coefficient can take either positive or negative values. In market *E*, the induced term structure movement due to $\Delta d^C(\tau)$ is $\Delta y_{t,\tau}^{E,A}(\Delta d^C(\tau))$, which is affine in $\int_0^T A_r^{E,A}(\tau)d\tau$ and $\int_0^T A_\beta^{E,A}(\tau)d\tau$, the two sensitivity terms that correspond to the two price factors in market *E*. The dependence of their respective coefficient on an integral term of $\Delta d^C(\tau)$ also indicates that the sign of $\Delta y_{t,\tau}^{E,A}(\Delta d^C(\tau))$ cannot be pinned down if $\Delta d^C(\tau)$ is neither uniformly positive nor uniformly negative. The analyses of the impact of a shock to the demand intercept from market *E* is similar. These discussions lead to Proposition 13 and Proposition 14.

***Proposition 13.* (*Demand shock transmission, market C → E*)**

In the weak-form of market segmentation, under the presence of the cross-market arbitrageur, consider a shock that shifts the demand intercept from $d^C(\tau)$ to $d^C(\tau) + \Delta d^C(\tau)$ of the preferred habitat investors' demand function in market C. For a positive (negative) demand shock, i.e. $\forall\tau$, $\Delta d^C(\tau) < 0$ ($\Delta d^C(\tau) > 0$), the yield $y_{t,\tau}^C$ increases (decreases) at maturity τ as the entire term structure is pushed up (down). Responding to such changes in $y_{t,\tau}^C$, the cross-market arbitrageur increases (decreases) her market–C sovereign bond holdings and decreases (increases) her market–E sovereign bond holdings, which subsequently pushes up (down) the term structure in market E. For a shock to the demand intercept with unequal signs of $\Delta d^C(\tau)$ at two distinct maturities, $\int_0^T \Delta d^C A_r^{C,A} d\tau$ can take either positive or negative values, and the yield $y_{t,\tau}^C$ increases or decreases at all maturities.

Proof. See Appendix.

***Proposition 14.* (*Demand shock transmission, market E → C*)**

In the weak-form of market segmentation, under the presence of the cross-market arbitrageur, consider a shock that shifts the demand intercept from $d^E(\tau)$ to $d^E(\tau) + \Delta d^E(\tau)$ of the preferred habitat investors' demand function in market E. For a positive (negative) demand shock, i.e. $\forall\tau$, $\Delta d^E(\tau) < 0$ ($\Delta d^E(\tau) > 0$), the yield $y_{t,\tau}^E$ increases (decreases) at maturity τ as the entire term structure is pushed up (down). Responding *to such changes in $y_{t,\tau}^{E}$, the cross-market arbitrageur increases (decreases) her market–E sovereign bond holdings and decreases (increases) her market–C sovereign bond holdings, which subsequently pushes up (down) the term structure in market E. For a shock to the demand intercept with unequal signs of $\Delta d^{E}(\tau)$ at two distinct maturities, $\int_0^T \Delta d^E A_r^{E,A} d\tau$ and $\int_0^T \Delta d^E A_\beta^{E,A} d\tau$ can be either positive or negative, and the yield $y_{t,\tau}^{E}$ increases or decreases at all maturities.*

Proof. See Appendix.

Under the semi-form of market segmentation where the two parallel markets remain segmented, I have analyzed the impact of an exogenous supply shock and the impact of a shock to the demand intercept on the term structures of each market. I have shown that an exogenous supply shock exerts a global impact on the term structures in both markets (Proposition 6 and Proposition 9), and a shock to the demand intercept function exerts a global impact on the term structure in market *C* (Proposition 10) and a localized impact on the term structure in market *E* (Proposition 10). The intervention of the cross-market arbitrageur disturb the equilibrium as the existing representative agents adjust their bond allocations. In particular, one may conjecture that the global effects in the semi-form case now become more localized as the cross-market arbitrageur connects the two markets. In Corollary 1, however, I show that the results in the semi-form case carries over into the weak-form of market segmentation, though this may appear to be counter-intuitive at the first glance.

Corollary 1.

1) *An exogenous supply shock, generated from either of the two markets, has a global effect on the term structures in both markets.*
2) *An exogenous shock to the demand intercept, generated from either of the two markets, has a global effect on the market-C term structure and a localized effect on the market-E term structure.*

Proof. See proofs of Proposition 6, Proposition 7, Proposition 9 and Proposition 10 in the Appendix.

Having analyzed the transmission of exogenous supply shocks and demand shocks to the term structure movements across the two markets, I move on to discuss some theoretical implications pertinent to the impact of different levels of the cross-market arbitrageur's risk aversion on the transmission of shocks across the two markets. In Theorem 1, I consider the *weak-form* of market segmentation under which the *CMA* has already intervened. Theorem 1 elaborates the situations of supply shock internalization and transmission under different levels of *CMA*'s risk aversion. In Theorem 2, I consider the introduction of the *CMA* to the *semi-form* of market segmentation.

Theorem 2 depicts the changes in yield curves due to the *CMA*'s arbitraging behavior under distinct levels of risk aversion. I later link these theoretical implications to the calibration results.

***Theorem 1.* (*Shock internalization and transmission*)**

1) *For an infinitely risk-averse CMA, under the weak-form equilibrium her induced allocations over the two markets due to an exogenous market-C supply shock are zeros. The shock is fully internalized by the market-C preferred-habitat investors and is not transmitted to market E. This would cause a maximal shift in the market-C yield curve and zero change in the market-E yield curve.*
2) *For a risk-neutral CMA, under the weak-form equilibrium she internalizes fully an exogenous supply shock and does not transmit it to the other market. Yield curves in both markets are left unaffected.*
3) *The CMA transmits an exogenous supply shock partially to the other market under finite risk-aversion.*

Proof. See Appendix.

***Theorem 2.* (*CMA's intervention*)**

1) *If the cross-market arbitrageur is infinitely risk-averse, her optimal allocations in the two markets are both zeros. This case leads to the semi-form of market segmentation.*
2) *If the cross-market arbitrageur is risk-neutral, i.e.,* $\gamma^A = 0$ *and infinite short-selling is allowed, she will short as much as she could from the market of the more expensive sovereign bonds and use all of the proceeds to long the cheaper sovereign bonds in the other market. In this case, the yield differential between the two term structures will be fully closed. The case leads to the full integration scenario.*
3) *If the cross-market arbitrageur is risk-averse, i.e.,* $\gamma^A > 0$ *and infinite short-selling is allowed, she will not be able to trade as much as she wishes. The gap between the two term structures is partially closed.*
4) *If the cross-market arbitrageur is risk-neutral, i.e.,* $\gamma^A = 0$ *and the short-selling constraint* $B_t \geq -\lambda X_t^*$ *is imposed, i.e., the CMA cannot borrow more than a fraction* $\lambda \in (0,1)$ *of her optimal allocation* X_t^* *at time* t *in part 2). In this case, the yield differential between the two term structures is either full or partially closed, representing the weak-form of market segmentation.*

Proof. See Appendix.

6 Calibration

In this section, I perform a calibration of the model on the parallel sovereign bond markets in China. I first describe the data I use for this calibration exercise. I then describe the detail steps in estimating the parameters governing the equilibrium term structures of market-*C* and market-*E*. Although the *weak-form* of market segmentation is the most interesting case due to its involvement of the cross-market arbitrageur, whose presence links the term structure of interest rates across the two markets, fundamental parameters governing the two term structures are estimated under the case of *semi-form* market segmentation.

6.1 *Data*

The data set I use covers a period of eleven years, from January 2007 to December 2017, of the Chinese sovereign bond markets and contains detailed issuance variables in the primary market and trading information variables in the secondary market. The secondary market dataset contains detailed information about all end-of-day sovereign bond transactions, including daily trading volume, highest and lowest traded prices (clean and dirty), opening and closing prices (clean and dirty), and yield to maturity based on the closing dirty price. Because of the PBOC's data regulations, the China Foreign Exchange Trade System (CFETS) provides only the end-of-day transaction data. In other words, regardless of whether a bond is traded once or more than once during a day, it appears only once in our data set as a bond-day observation. The key variable employed in my study is sovereign bonds' yields to maturity. Since sovereign bonds in China were not infrequently traded during the seminal stage of development of the Chinese bond markets, I use the subsample of sovereign bond transactions from January 2013 to December 2017 to estimate some of the parameters whose accuracy is potentially affected by trading infrequency. I make clear these changes in the calibration exercise.

It should be noted that sovereign bonds in China have maturities ranging from three months to 30 years. In this paper, I bootstrap the coupon-implied sovereign bond yields to maturity to obtain the corresponding zero rates. I then apply the three-factor Nelson-Siegel model to generate the entire zero yield curve for each trading day. In this way, I create a panel of sovereign bond zero rates, a $N \times 40$ grid of zero rates. Figure 2 plots the term structure of interest rates, yields to maturity averaged over the 11-year sample period from January 2007 to December 2017, in the exchange market and the interbank OTC market.

[Insert Figure 2 Here]

6.2 *Semi–form market–C term structure parameters*

The equilibrium term structure of market-C is specified by the parameters $(\bar{r}^C, \kappa_r^C, \sigma_r)$ of the short rate process under the risk-neutral measure, the *WMA.C*'s risk-aversion coefficient γ^C, and the functions $\left(\alpha^C(\tau), d^C(\tau)\right)$. Note that $(\bar{r}^C, \kappa_r^C)$ further depends on (κ_r, γ^C), the parameters governing the short rate process under the physical measure, and the function $\alpha^C(\tau)$ that describes the demand slope of the market-C preferred-habitat investors. Note that the long-run mean of the short rate process $\bar{r}$ and the demand intercept $d^C(\tau)$ affect only the long-run average of yields and now the response of shocks to yields. Thus, they do not come into the analysis that concerns the response to shocks.

[Insert Figure 3 Here]

For the functional form of $\alpha^C(\tau)$, I adopt a similar exponential specification to the V&V model and set $\alpha^C(\tau) = \alpha^C \exp(-\delta^C \tau)$. To estimate the parameters required for determining the term structure of market-C in the absence of the cross-market arbitrageur, which corresponds to the *semi-form* of market segmentation, I use the 3-month spot rate as proxies for the short rate. Figure 3 plots the weekly aggregated values of the short rates over the entire sample period from January 2013 to December 2017. I use the maximum likelihood estimation (MLE) approach and follow Liptser and Shiryaev (2001). The log-likelihood function under the *Ornstein-Uhlenbeck* process, given a set of observable short rates $\{r_{t_i}\}$ for $i = 0,1,2,\dots,N$ is

$$L(\bar{r},\kappa_r,\sigma_r) = -\frac{1}{2}\sum_{i=1}^{N}\left[\log\left(2\pi V_{t_{i-1}}(t_i)\right) + \frac{\left(r_{t_i} - M_{t_{i-1}}(t_i)\right)^2}{V_{t_{i-1}}(t_i)}\right]$$

where

$$B(s,t) = \kappa_r^{-1}\left[1 - \exp\left(-\kappa_r(t-s)\right)\right]$$
$$M_{t_{i-1}}(t_i) = \bar{r}\kappa_r B(t_{i-1},t_i) + r_{t_{i-1}}\left(1 - \kappa_r B(t_{i-1},t_i)\right)$$
$$V_{t_{i-1}}(t_i) = \sigma_r^2\left[B(t_{i-1},t_i) - \frac{1}{2}\kappa_r B(t_{i-1},t_i)^2\right]$$

Since the time points $t_0, t_1, \dots, t_N$ are equidistant with spacing of one day, the maximum likelihood estimators of the three parameters of interest are

$$\hat{\bar{r}} = \frac{S_1 S_{00} - S_0 S_{01}}{S_0 S_1 - S_0^2 - S_{01} + S_{00}}, \quad \widehat{\kappa_r} = \log\left(\frac{S_0 - \bar{r}}{S_1 - \bar{r}}\right), \quad \widehat{\sigma_r} = \frac{2\kappa_r}{N[1 + \exp(-\kappa_r \Delta t)]}\sum_{i=1}^{N}\left[r_{t_i} - M_{t_{i-1}}(t_i)\right]^2,$$

where

$$S_0 = \frac{1}{N}\sum\nolimits_{i=1}^{N} r_{t_{i-1}}, \quad S_1 = \frac{1}{N}\sum\nolimits_{i=1}^{N} r_{t_i}, \quad S_{00} = \frac{1}{N}\sum\nolimits_{i=1}^{N} r_{t_{i-1}}^2, \quad S_{01} = \frac{1}{N}\sum\nolimits_{i=1}^{N} r_{t_{i-1}} r_{t_i},$$

The parameter values estimated using the Liptser and Shiryaev (2001) approach are $(\hat{\bar{r}}, \widehat{\kappa_r}, \widehat{\sigma_r}) = (0.029, 0.038, 0.029)$ for the market-C short rate process under the physical measure. To further estimate the remaining parameters, note that γ^C and α^C affect the equilibrium term structure only through their product, this leaves us a pair of parameters $(\gamma^C\alpha^C, \delta^C)$ to estimate. To obtain estimates of $(\gamma^C\alpha^C, \delta^C)$, I use a set of moments of bond yields: the unconditional volatility of bond yields. I consider yields at maturities $\tau = 5k, 6k, \dots, 40k$, $k = 0.25$, equidistant with a spacing of three months. I obtain 36 unconditional volatilities to be matched to the model-implied unconditional volatility of $y_{t,\tau}$:

$$\sigma\left(y_{t,\tau}^{C}\right) = \frac{1}{\tau}\sqrt{\frac{1}{2\kappa_r^C}\sigma_r^2 A_r^C(\tau)^2}$$

I determine the pair $(\gamma^C\alpha^C, \delta^C)$ through minimizing the sum of squared differences, $\mathfrak{J}^C$, which is a function of κ_r^C only and contains 36 moments in total. To avoid large variation of short-term spot rates, I omit those with maturities less than one year. The subscript {*model*, *data*} denotes whether the moment is obtained using the model or computed from the actual data. The subscript τ denotes a regression coefficient corresponding to maturity τ.

$$\mathfrak{J}^C(\kappa_r^C) \equiv \sum_{\tau=5}^{40}\left[\sigma_{model}\left(y_{t,\tau}^C\right) - \sigma_{data}\left(y_{t,\tau}^C\right)\right]^2$$

Figure 4 plots the model calibration results of the model-implied volatility curve $\sigma_{model}\left(y_{t,\tau}^C\right)$ in the interbank OTC market parameters and the data-based unconditional volatility curve $\sigma_{data}\left(y_{t,\tau}^C\right)$ at maturities $k = 1.25, 1.5, 1.75, \dots\dots, 10$. The value of κ_r^C that minimizes $\mathfrak{J}^C(\kappa_r^C)$ is $\kappa_r^C = 0.269$, which corresponds to a minimized value of $\mathfrak{J}^C$ at 3.19. I explain in the next section how I determine the value of each of the two parameters.

[Insert Figure 4 Here]

6.3 *Semi–form market–E term structure parameters*

The equilibrium term structure of market-*E* is specified by the parameters $(\bar{r}^E, \kappa_r^E, \sigma_r)$ of the short rate process and $\left(\kappa_\beta, \sigma_\beta\right)$ of the net demand process under the risk-neutral measure, the *WMA.E*'s risk-aversion coefficient γ^E, and the functions $\left(\alpha^E(\tau), d^E(\tau), \theta(\tau)\right)$.[6] Note from the expression of $Y_{t,\tau}^E$ that $\theta(\tau)$ and β_t only affect the demand function of the preferred-habitat investors of market *E* through their product, so do $\theta(\tau)$ and σ_β. I can then normalize σ_β to any arbitrary value. However, unlike the approach in the V&V model, which sets it equal to σ_r, I calibrate the volatility of β_t, along with its mean-reversion parameter, directly from an explicit time series of the net demand factor, calculated as the difference between the two distinct short rate time series of the two markets, that is, $\beta_t \equiv r_t^C - r_t^E$. For the functional forms of $\alpha^E(\tau)$ and $\theta(\tau)$, I set $\alpha^E(\tau) = \alpha^E \exp(-\delta^E\tau)$ and $\theta(\tau) = \theta[\exp(-\delta^E\tau) - \exp(-\delta_\theta\tau)]$.

[6] Note that $\bar{\beta}^E = 0$ so that the net demand process has a zero long-run mean.

Note also that I adopt a two-step estimation approach for market C. I first estimate the parameters governing the short rate process under the physical measure, $(\kappa_r, \bar{r})$, after which I estimate the remaining parameters governing the term structure. I follow the maximum likelihood estimator approach in Liptser and Shiryaev (2001) to estimate $(\kappa_r, \bar{r})$. Estimating these two fundamental parameters separately from the remaining parameters allows me to obtain maximum accuracy in their estimation. In market E, the term structure is specified by two factors. The parameters governing the short rate process in market E under the physical measure are estimated to be $(\hat{\bar{r}}, \widehat{\kappa_r}, \widehat{\sigma_r}) = (0.024, 0.364, 0.087)$. As the net demand factor is empirically observable, I follow the same approach as in market C and calibrate the parameters governing the net demand process in market E. The estimated parameter values are $\left(\hat{\bar{\beta}}, \widehat{\kappa_\beta}, \widehat{\sigma_\beta}\right) = (0.005, 0.570, 0.091)$.

The remaining parameters $(\gamma^E \alpha^E, \delta^E, \gamma^E \theta, \delta_\theta)$ to be estimated all come into the expression of matrix $\boldsymbol{A}$ in Proposition 2. To estimate them, I first estimate the parameters $(\psi_r, \psi_\beta, \kappa_r^E, \kappa_\beta^E)$ through minimizing the sum of squared differences, specified in $\mathfrak{I}^E$, similar to the second estimation step in the one-factor case of market C. However, instead of using the moments of unconditional volatility, I consider the Campbell-Shiller regression estimates and the Fama-Bliss regression estimates, which are susceptible to changes in the parameters associated with the net demand factor. In Campbell and Shiller (1991), they find that the slope of the term structure predicts changes in long rates. The regression they perform relates the behavior of bond risk premia to changes in long rates.

$$y^E_{t+\Delta\tau, \tau-\Delta\tau} - y^E_{t,\tau} = a^{CS} + b^{CS} \frac{\Delta\tau}{\tau - \Delta\tau} \left(y^E_{t,\tau} - y^E_{t,\Delta\tau}\right) + e^{CS}_{t+\Delta\tau}$$

The dependent variable is the change in yield of a zero coupon bond with maturity τ at time t, from time t to time $t + \Delta\tau$. The independent variable is the normalized difference between the spot rates at time t for maturities τ and $\Delta\tau$. Under the efficient market hypothesis, the regression estimate b is equal to one. I consider the change in the yield of a zero coupon bond with maturity τ over one year, such that $\Delta\tau = 1$. In this way, I obtain 36 regression coefficients of b to be matched to the model-implied regression estimates:

$$b^{CS}_{model} = \frac{\left(\frac{\tau - \Delta\tau}{\Delta\tau}\right)\left(K_r^{CS} \frac{\sigma_r^2}{\kappa_r} + K_\beta^{CS} \frac{\sigma_\beta^2}{\kappa_\beta}\right)}{\left[\frac{A_r^E(\tau)}{\tau} - \frac{A_r^E(\Delta\tau)}{\Delta\tau}\right]^2 \frac{\sigma_r^2}{\kappa_r} + \left[\frac{A_\beta^E(\tau)}{\tau} - \frac{A_\beta^E(\Delta\tau)}{\Delta\tau}\right]^2 \frac{\sigma_\beta^2}{\kappa_\beta}},$$

where, for $i = r, \beta$,

$$K_i^{CS} \equiv \left[\frac{A_i^E(\tau - \Delta\tau)}{\tau - \Delta\tau} e^{-\kappa_i \Delta\tau} - \frac{A_i^E(\tau)}{\tau}\right]\left[\frac{A_i^E(\tau)}{\tau} - \frac{A_i^E(\Delta\tau)}{\Delta\tau}\right]$$

I then follow Fama and Bliss (1987) which documents the empirical fact of positive premia-slope relationship. They perform the regression

$$\frac{1}{\Delta\tau}\log\left(\frac{P^E_{t+\Delta\tau,\tau-\Delta\tau}}{P^E_{t,\tau}}\right)-y^E_{t,\Delta\tau}=a^{FB}+b^{FB}\left(f^E_{t,\tau-\Delta\tau\to\tau}-y^E_{t,\Delta\tau}\right)+e^{FB}_{t+\Delta\tau}$$

The dependent variable is the holding period return of a zero coupon bond with maturity τ over a period $\Delta\tau$, subtracted by the spot rate at maturity $\Delta\tau$. The independent variable, $f^E_{t,\tau-\Delta\tau\to\tau}-y^E_{t,\Delta\tau}$, measures the difference between the forward rate between maturities $\tau-\Delta\tau$ and τ and the spot rate at maturity $\Delta\tau$. Fama and Bliss (1987) find that the coefficient estimate of b^{FB} remains positive and exceeds one for some maturities. Taking $\Delta\tau=1$, this is confirm in Figure 5. I obtain 36 regression coefficients of b to be matched to the model-implied regression estimates:

$$b^{FB}_{model}=\frac{\left(K^{FB}_r\frac{\sigma_r^2}{\kappa_r}+K^{FB}_\beta\frac{\sigma_\beta^2}{\kappa_\beta}\right)}{[A^E_r(\tau)-A^E_r(\tau-\Delta\tau)-A^E_r(\Delta\tau)]^2\frac{\sigma_r^2}{\kappa_r}+\left[A^E_\beta(\tau)-A^E_\beta(\tau-\Delta\tau)-A^E_\beta(\Delta\tau)\right]^2\frac{\sigma_\beta^2}{\kappa_\beta}},$$

where, for $i=r,\beta$,

$$K^{FB}_i\equiv\left[A^E_j(\tau)-A^E_j(\tau-\Delta\tau)e^{-\kappa_i\Delta\tau}-A^E_j(\Delta\tau)\right]\left[A^E_j(\tau)-A^E_j(\tau-\Delta\tau)-A^E_j(\Delta\tau)\right]$$

I estimate the parameters $(\psi_r,\psi_\beta,\kappa^E_r,\kappa^E_\beta)$ by minimizing the following sum of squared differences,

$$\mathfrak{I}^E(\psi_r,\psi_\beta,\kappa^E_r,\kappa^E_\beta)\equiv\sum_{\tau=5}^{40}\left[b^{CS}_{model}(\tau)-b^{CS}_{data}(\tau)\right]^2+\sum_{\tau=5}^{40}\left[b^{FB}_{model}(\tau)-b^{FB}_{data}(\tau)\right]^2$$

[Insert Figure 5 Here]

Figure 5 plots the calibration results of the exchange market parameters, based on minimizing the sum of the squared differences between Campbell-Shiller regression estimates $b^{CS}_{data}(\tau)$ and Fama-Bliss regression estimates $b^{FB}_{data}(\tau)$ and their respective model-implied values, $b^{CS}_{model}(\tau)$ and $b^{FB}_{model}(\tau)$. The calibrated parameters are $\left(\widehat{\psi_r},\widehat{\psi_\beta},\widehat{\kappa^E_r},\widehat{\kappa^E_\beta}\right)=(10.8,-9.6,0.495,0.175)$. The minimized $\mathfrak{I}^E$ is 10.67. My model can thus can hence match simultaneously these two sets of moments with a small means squared error term. Figure 5 demonstrates the close match between model-implied moments, plotted in blue dotted curves, and their data-based regression counterparts, plotted in black dotted curves.

With the calibrated values of parameters determining the functional forms of $A^E_r(\tau)$ and $A^E_\beta(\tau)$, I further use the expressions of A_{11} and A_{12}, derivation of which given in the proof of Proposition 2 of the appendix, to pin down the values of $(\gamma^E\alpha^E,\delta^E)$.

$$A_{11}(\gamma^E\alpha^E,\delta^E)=\frac{y_2\kappa_\beta^E-y_1\kappa_r^E}{y_1-y_2}+\kappa_r \quad \& \quad A_{12}(\gamma^E\alpha^E,\delta^E)=\frac{y_1\left(\kappa_r^E-\kappa_\beta^E\right)}{y_1-y_2}\left(\frac{\psi_r}{\psi_\beta}\right)$$

The calibrated parameters are found to be $\left(\widehat{\gamma^E\alpha^E},\widehat{\delta^E}\right)=(4.7,0.04)$, which I substitute into the expressions of A_{21} and A_{22} to estimate the parameters $(\gamma^E\theta,\delta_\theta)$, which are found to be $\left(\widehat{\gamma^E\theta},\widehat{\delta_\theta}\right)=(80.6,0.05)$.

$$A_{21}(\gamma^E\theta,\delta_\theta)=\frac{y_2\left(\kappa_\beta^E-\kappa_r^E\right)}{y_1-y_2}\left(\frac{\psi_\beta}{\psi_r}\right) \quad \& \quad A_{22}(\gamma^E\theta,\delta_\theta)=\frac{y_2\kappa_r^E-y_1\kappa_\beta^E}{y_1-y_2}+\kappa_\beta$$

6.4 *Estimation of term structure parameters under the weak-form*

Estimating the parameters governing the equilibrium term structure of interest rates over the two markets can be pivotal because these parameters directly get involved in the expressions of examining the transmission of shocks to the short rate and to the demand factor through the cross-market arbitrageur from one market to the other, or vice versa. I add a superscript A to the parameters to be estimated to denote the involvement of the cross-market arbitrageur. First, note that the parameters that appear in the expressions of the preferred-habitat investors' demand functions, which are not susceptible to changes after involving the cross-market arbitrageur, remain the same as in the *semi-form* of market segmentation. Such parameters include $(\alpha^C,\alpha^E,\delta^C,\delta^E,\delta_\theta)$.

I estimate the group of parameters $\left(\kappa_r^{C,A},\kappa_r^{E,A},\kappa_\beta^{E,A},\psi_r^A,\psi_\beta^A,y_1^A,y_2^A\right)$ under the *weak-form* of market segmentation. Fortunately, the model setup facilitates the computational burden inasmuch as the induced demand shocks triggered by the cross-market arbitrageur's adjusted bond allocations affect only the long run averages of yields and the net demand factor, but not the system of equations that delineate the responses of yields to shocks.[7] Therefore, the seven parameters to be estimated remain the same as in the *semi-form* case. This is also verified from the equations that lead to their solutions. The expressions of equations (3.2.1) and (3.3.3) tell that $\kappa_r^{C,A}=\kappa_r^C$, and a comparison of the equations (3.2.3 ~ 3.2.4) and (3.3.5 ~ 3.3.6) tell that $\left(\kappa_r^{E,A},\kappa_\beta^{E,A}\right)=\left(\kappa_r^E,\kappa_\beta^E\right)$ and $\left(\psi_r^A,\psi_\beta^A,y_1^A,y_2^A\right)=\left(\psi_r,\psi_\beta,y_1,y_2\right)$. With these parameters in hand, in the next section I proceed to answer the question asked at the beginning, namely, how do shocks to the short rate, supply shocks and demand shocks from one market affect the term structure of interest rates in the other market, and to what extent.

[7] See Appendix for the expressions of long run means of the short rate and the net demand factor: $\bar{r}^{C,A}$, $\bar{r}^{E,A}$ and $\bar{\beta}^{E,A}$.

7 Shock Transmissions

In this section, I report the calibration results and discuss some theoretical implications generated from my model. Specifically, I focus on analyzing how shocks to the rate, exogenous supply shocks and demand shocks from one market can affect the term structure of interest rates in the other market. I discuss the global effects and localized effects from these shocks on the term structure. In particular, I quantify the movements on yield curves due to such shocks and examine their differential impact on distinct maturities of the term structure. In addition, I delineate the induced demand shocks from the cross-market arbitrageur's trading behavior and analyze how the net demand factor, proxied by the difference between the two short rates at time t, propagates to longer maturity clienteles.

7.1 *Responses of yields to maturity to shocks*

Figure 6 plots the responses of yields to maturity to shocks to the short rate and to the net demand factor under the equilibria of different forms of market segmentation. Figure 6.1 plots the function $A_r^C(\tau)/\tau$, which describes the response of yields to maturity to shocks to the short rate under the *semi-form* and the *weak-form* of market segmentations in market C. For small values of τ, a unit short to the short rate passes on almost fully to short-term yields. However, the response to yields declines steadily as maturity increases, though a unit shock to r_t remains positive and raises yields at all maturities. In the V&V model, it is found that this effect turns slightly negative for maturities longer than nineteen years due to hedging activities from arbitrageurs. This effect remains positive in my model due to the selection of 10 years as the longest maturity of the term structure. Figure 6.2 plots the function $A_\beta^E(\tau)/\tau$ that describes the response of yields to shocks to the net demand factor under the *extreme-form* of market segmentations in market E. Under segmentation equilibrium, a unit shock to β_t remains positive and raises yields at all maturities, despite the low percentage of shocks transmitted to yields. This effect declines to below 10% as maturity exceeds four years. Figure 6.3 plots the functions $A_r^E(\tau)/\tau$ and $A_\beta^E(\tau)/\tau$, which delineate the response of yields to shocks to the short rate and to the net demand factor under the *semi-form* and *weak-form* of market segmentations in market E. A shock to the net demand factor exhibits a larger effect on yields in the segmentation equilibrium than in the presence of arbitrageurs, which remains mostly negative for maturities longer than one year. Averaging across maturities, the effect of a unit of the net demand shock is more than ten times larger than in the absence of the cross-market arbitrageur.

[Insert Figure 6 Here]

7.2 *Responses of demand functions to jumps in the short rate*

The analysis of responses of demand functions to jumps in the short rate process does not involve the cross-market arbitrageur. Figure 7 plots the response of demand function of the market-*C* preferred-habitat investors *PHI.C*s to expected jumps in the short rate process, under different values of the risk-aversion parameter of the market-*C* within-market arbitrageur *WMA.C*, γ^C, and the jump intensity of the short rate, λ_r. While the effect remains close to zero at maturities less than one year, yields at long maturities are more sensitive to jumps in the short rate. Shown in the Appendix, imposing the market clearing conditions tells that the effect on the *WMA.C*'s demand function is opposite to the effect on the *PHI.C*s' demand functions. This indicates that the *WMA.C*, being risk-averse, reduces her bond allocations under more uncertainty in the short rate. The *PHI.C*s fully absorb such effect by investing in the private technology. Moreover, such effect is linear in λ_r, which leads to more pronounced effects of jumps with higher intensities, all else being equal. On the other hand, holding the jump intensity constant, the less risk-averse the within market arbitrageur is, the more herself and the *PHI.C*s respond to uncertainty in the short rate; such effect is amplified at longer maturities with steeper slope of the response curve. In other words, equilibrium allocations of more risk-averse within-market arbitrageurs are less sensitive to uncertainty in short rate jumps. This complies with the intuition of CARA utilities that more risk-averse investors adjust less to obtain the same change in utility than do less risk-averse investors.

[Insert Figure 7 Here]

7.3 *Supply effects and demand effects*

One of the key implications of my model is that the effects of exogenous supply shocks are always global, while the effects of demand shocks are global in the one-factor model case of market *C* and localized in the two-factor model case of market *E*.

7.3.1 *Market–C generated shocks*

Figure 8.1 plots the changes in the market-*C* yield curve due to supply shocks of one unit, 0.5 unit and 0.3 unit, respectively, from market *C*. It shows that yields at longer maturities are more affected by the supply shock as $\Delta y^C_{t,\tau}$ is increasing in τ. Moreover, the larger the supply shock is, the larger the effect on the yield curves. In the case of market *C* whose term structure is governed fully by the short rate factor, the supply effects are global in the sense that the relative effect across distinct maturities does not depend on the origin of the shock τ^*. Consider two maturities τ_1 and τ_2 with $\tau_1 < \tau_2$, the ratio of the supply effect is

$$\frac{\Delta y^C_{t,\tau_2}\left(\Delta S^C_{t,\tau^*}\right)}{\Delta y^C_{t,\tau_1}\left(\Delta S^C_{t,\tau^*}\right)} = \frac{\int_0^{\tau_2} A^C_r(s)ds}{\int_0^{\tau_1} A^C_r(s)ds} = \int_{\tau_1}^{\tau_2} A^C_r(s)ds > 1$$

The above expression depends neither on the origin of the shock τ^* nor on the magnitude of the supply shock $\Delta S^C_{t,\tau^*}$, but only on the two reference maturities and the integral of the coefficient of r_t in the term structure expression, negative to the short rate factor sensitivity, as mentioned in Lemma 1 of Section 3.2.

[Insert Figure 8 Here]

Figure 8.2 plots the changes in yield curve due to shocks to the preferred-habitat investor's demand intercept under three functional forms, specified in the legend box. Note that a positive $\Delta d^C(\tau)$ indicate a negative demand shock, as manifested by the expression of the preferred-habitat investor *PHI.C*'s demand function. The first demand shock, $\Delta d^C(\tau) = 1/\tau$, depicts a situation in which demand over short-term bonds declines drastically, followed by less and steady decline in demand over longer maturity bonds. The second demand shock, $\Delta d^C(\tau) = 1$ for $0 \leq \tau \leq 6$ and $\Delta d^C(\tau) = -1$ for $6 < \tau \leq 10$, delineates a quantitative easing situation in which the government buys back long-term bonds, inject money supply and liquidity in short-term borrowing and stimulate investment activities. The third demand shock, $\Delta d^C(\tau) = 0.01\tau$, depicts a decline in demand at all maturities that is proportional to the maturity length. The demand effects under the one-factor case of market C remain global with the relative effects across maturities also given by $\int_{\tau_1}^{\tau_2} A^C_r(s)ds$. The first and the third demand shocks remain positive in their functional forms, which indicates uniform decline in demand over all maturities. They correspond to an increase in the term structure as sovereign bond prices become cheaper, as shown in the figure. Buying back longer-term sovereign bonds and injecting money supply to the short-term investments may have either a positive or a negative impact on the term structure, depending on the relative magnitude. In the case of the second demand shock, it leads to a decline in the yield curve, indicating that the impact on the term structure of buying back longer-maturity bonds exceeds that of injecting money supply to the shorter-maturity end.

7.3.2 *Market–E generated shocks*

Figure 9.1 plots the changes in the market-E yield curve due to supply shocks of one unit, 0.5 unit and 0.3 unit, respectively, from market E. The analysis follows similarly in the market-C case, with the supply effects being also global as the relative effects across maturities are $\int_{\tau_1}^{\tau_2} A^E_r(s)ds$. One interesting feature about the global supply effects in market E is the convexity of the response curves at the short end of the maturity spectrum, contrasting to the concave feature of the market-C response curves. This is because the response curves in market C are fully characterized by

$\tau^{-1}\int_0^\tau A_r^C(s)ds$, a concave function in τ, while those in market E are specified by both $\tau^{-1}\int_0^\tau A_r^E(s)ds$ and $\tau^{-1}\int_0^\tau A_\beta^E(s)ds$. The presence of the sensitivity term of the net demand factor renders the response curves convex at the short maturity end under suitable combinations. In other words, the presence of the net demand factor amplifies the supply effects at longer maturities as the exogenous supply shock increases.

[Insert Figure 9 Here]

I consider two types of shocks to the preferred-habitat investor *PHI.E*'s demand intercept. First, Figure 9.2 plots the changes in the market-E yield curve due to a unit shock to *PHI.E*'s demand function at maturity $\tau^* = 3, 5, 8$, respectively. For the same magnitude of demand shock, however, at different origins it induces unequal responses at all maturities. Besides, the demand effects have now become *localized* due to the presence of the additional term $\tau^{-1}\int_0^\tau A_\beta^E(s)ds$ in the expression of the relative effects across maturities, which depends on the relative contribution from the two sensitivity terms. Second, Figure 9.3 utilizes the same set of demand shocks in the market-C case. In market C, the demand shock $\Delta d^C(\tau) = 0.01\tau$ leads to a much lower response in yields than does the demand shock $\Delta d^C(\tau) = 1/\tau$ (about 20%). However, under the presence of the net demand factor, the demand shock $\Delta d^C(\tau) = 0.01\tau$ now induces a larger change in yields than does the demand shock $\Delta d^C(\tau) = 1/\tau$. Once again, the relative contribution of the two price factors matters in affecting the responses of yields to demand shocks. For the demand shock that depicts quantitative easing activities, yields respond more wildly under the two-factor model, with the magnitude of decline at all yields being three times to that under the one-factor model.

7.4 *Shock transmissions*

The analyses so far have focused only on the *semi-form* of market segmentation. From now on, I bring in the cross-market arbitrageur so that shocks to one market transmit to the other market through the cross-market arbitrageur.

7.4.1 *Supply shock transmission*

I first discuss the calibration results of supply shock transmissions. Figure 10.1 plots the changes in yields at all maturities in both markets due to one unit of market-C supply shock. Due to the global feature of supply effects, this shock can generate from any maturity. The red dotted line plots the impact on the market-C term structure, denoted by $\Delta y_{t,\tau}^{C,A}(1)$. The blue dotted line plots the impact on the market-E term structure, denoted by $\Delta y_{t,\tau}^{E,A}(1)$. The calibration results comply with the intuitions. A positive supply shock at τ^* in market C pushes down the prices of market-C sovereign bonds at τ^*, incentivizing the cross-market arbitrageur to short the more expensive market-E sovereign bonds at τ^* and purchase the cheaper market-C sovereign bonds at τ^*. This subsequently pushes down the price of market-E sovereign bonds at τ^* and lowers their yields. However, the response of the market-E term structure to the market-C generated supply shock accounts for up to 14% of the response of the market-C term structure at long maturities, indicating that only a small portion of the market-C supply shock transmits to the other market. In Figure 10.2, I further decompose the market-E term structure movement into two components, a short rate component and a net demand component. At very short maturities, changes in yields mostly come from the short rate component. As maturity increases, the net demand component accounts for more changes in yields and increases up to 50% of the aggregate variation.

[Insert Figure 10 Here]

Figure 11.1 plots the changes in yields at all maturities in both markets due to one unit of market-E supply shock. However, the response of the market-C term structure to the market-E generated supply shock accounts for up to only 6% of the response of the market-E term structure at long maturities, indicating an even smaller portion of the market-E supply shock transmission to the other market. Figure 11.2 plots the ratio of the change in the yield curve of market i to the change in the yield curve of market j, due to one unit of supply shock from market j ($i, j \in \{C, E\}$). From the one-factor case of market C to the two-factor case of market E, a unit of supply shock induces a steady increase in the percentage of shock transmission to the other market through the cross-market arbitrageur, from 1% at the very short maturity end to about 14% at the 10-year maturity end. However, from the two-factor case of market E to the one-factor case of market C, a unit of supply shock induces a drastic decline in the percentage of shock transmission at maturities less than five years, from nearly 60% to 10%, followed by a steady decline to 6% at maturities longer than five years. Due to the presence of the net demand factor, exogenous supply shocks are more transferrable to the other market through the cross-market arbitrageur than otherwise.

[Insert Figure 11 Here]

7.4.2 *Demand shock transmission*

I now discuss the calibration results of demand shock transmissions. Figure 12.1 plots the changes in yields at all maturities in both markets due to the three demand shocks from market C. The three dotted lines and the three dotted curves plot the impact on the market-C term structure and on the market-E term structure, denoted by $\Delta y_{t,\tau}^{C,A}(\Delta d^C)$ and $\Delta y_{t,\tau}^{E,A}(\Delta d^C)$, respectively, due to the three demand shocks. Similar to the findings in the transmission of supply shocks, market-C demand shocks transmit only partially to market E. However, a much higher proportion of market-C demand shocks transmits to market E than in the supply shock case. While the former accounts for as much as 22% in transmitting the demand shock $\Delta d^C(\tau) = 1/\tau$ to long maturities, the latter accounts for about 14%. Figure 12.2 plots the changes in yields at all maturities in both markets due to the three demand shocks from market E. As expected, the market-E response curves exhibit convex patterns at the short maturity end due to the presence of the net demand factor. Similar to the low transmission percentage at long maturities in the supply shock scenario, for each of the three market-E demand shocks the transmission percentage stays at about 7% at long maturities.

[Insert Figure 12 Here]

7.5 *Propagation of induced demand shocks*

The functions of the propagation of induced demand shocks, $\Delta\beta_{t,t}^{C}$ and $\Delta\beta_{t,t}^{E}$, are given below.[8]

$$\Delta\beta_{t,\tau}^{C} = \frac{M_3(\tau)M_5(\tau)}{M_1(\tau)M_5(\tau) - M_2(\tau)M_4(\tau)}\Delta S_{t,\tau^*}^{C}$$

$$\Delta\beta_{t,\tau}^{E} = -\frac{M_3(\tau)M_4(\tau)}{M_1(\tau)M_5(\tau) - M_2(\tau)M_4(\tau)}\Delta S_{t,\tau^*}^{C}$$

Figures 13.1 and 13.2 plot the propagation of induced demand shocks, $\Delta\beta_{t,t}^{C}(1)$ and $\Delta\beta_{t,\tau}^{E}(1)$, from the *CMA*'s adjusted allocations $X_{t,\tau}^{C}(1)$ and $X_{t,\tau}^{E}(1)$ in the two markets due to one unit of exogenous positive supply shock $\Delta S_{t,\tau^*}^{C} = 1$ from market C. The induced demand shocks exhibit exponential decaying features, which show drastic decline in magnitude at shorter maturities and quick convergence to a fixed value at longer maturities, which I call the long-run mean of the induced demand shock. Intuitively, a positive supply shock lowers the market-C maturity-τ^* sovereign bond prices and increases their yields, rendering themselves cheaper as compared to the market-E maturity-τ* sovereign bonds. This causes the *CMA* buy the market-C maturity-τ^* sovereign bonds, leading to an positive induced demand shock of $\Delta\beta_{t,\tau}^{C}$. Indeed, in market C the induced demand shock remains positive thanks to the positivity of the supply shock in the expression of $\Delta\beta_{t,\tau}^{C}$. At maturities less than one year, a unit of the induced demand shock transmits as much as nearly 70% to the maturity clientele, while this percentage decreases steadily to 50% as maturity increases. On the other hand, a negative demand shock is induced in market E due to decline the in demand for the maturity-τ^* sovereign bonds. Understandably, the percentage of transmitted shock is also much smaller as compared to that in market C, where the shock is generated.

Figures 13.3 and 13.4 plot the induced demand shocks from the *CMA*'s adjusted allocations $X_{t,\tau}^{C}(\Delta S_{t,\tau}^{E})$ and $X_{t,\tau}^{E}(\Delta S_{t,\tau}^{E})$ in the two markets due to one unit of exogenous supply shock $\Delta S_{t,\tau}^{E} = 1$ from market E. Figures 13.5 and 13.6 (13.7 and 13.8) plot the induced demand shocks from the

[8] Explicit expressions of the *M*'s are given in the proof of Proposition 11 in the Appendix.

CMA's adjusted allocations $X_{t,\tau}^{C}(\Delta d^{C})$ and $X_{t,\tau}^{E}(\Delta d^{C})$ ($X_{t,\tau}^{C}(\Delta d^{E})$ and $X_{t,\tau}^{E}(\Delta d^{E})$) in each market due to a shock to the preferred-habitat investors' demand intercept $\Delta d^{C}(\tau) = \tau^{-1}$ ($\Delta d^{E}(\tau) = \tau^{-1}$) from market *C* (*E*). These analyses follow similarly to the case of the supply shock generated from market-*C*, so I suppress the discussion.

[Insert Figure 13 Here]

As pointed out in Theorem 1, if the cross-market arbitrageur is infinitely risk-averse, her optimal allocations in the two markets are both zeros. In this case, the induced demand shocks should also be zeros. This statement is confirmed in the expression $\Delta\beta_{t,t}^{C}$ because

$$\lim_{\gamma^{A}\to\infty} \Delta\beta_{t,t}^{C} = \frac{\lim_{\gamma^{A}\to\infty} M_3(\tau)M_5(\tau)}{\lim_{\gamma^{A}\to\infty}[M_1(\tau)M_5(\tau) - M_2(\tau)M_4(\tau)]}\Delta S_{t,\tau^*}^{C} \sim \lim_{\gamma^{A}\to\infty} \frac{O(\gamma^{A})}{O\left({\gamma^{A}}^{2}\right)} = 0$$

Moreover, as also pointed out in Theorem 1, if the cross-market arbitrageur is risk-neutral, and infinite short-selling is allowed, she will short as much as she could from the market of the more expensive market-*E* maturity-τ^* sovereign bonds and use all of the proceeds to purchase the relatively cheaper market-*E* maturity-τ^* sovereign bonds. In this case, the induced demand shock will be fully absorbed by the cross-market arbitrageur. This statement is also confirmed in the expression $\Delta\beta_{t,t}^{C}$ because, for $\gamma^{A} = 0$, $M_2 = 0 = M_4$ and $M_1 = M_3$:

$$\Delta\beta_{t,t}^{C}\Big|_{\gamma^{A}=0} = \frac{M_3(\tau)M_5(\tau)}{M_1(\tau)M_5(\tau) - M_2(\tau)M_4(\tau)}\Delta S_{t,\tau^*}^{C} = \frac{M_3(\tau)M_5(\tau)}{M_1(\tau)M_5(\tau)}\Delta S_{t,\tau^*}^{C} = \frac{M_3(\tau)}{M_1(\tau)}\Delta S_{t,\tau^*}^{C} = \Delta S_{t,\tau^*}^{C}$$

Therefore, the supply shock is fully absorbed by the cross-market arbitrageur.

7.6 *Shock internalization and transmission at distinct risk aversions*

In this subsection, I examine the response of the term structures to supply shocks under different values of the cross-market arbitrageur's risk aversion, γ^{A}. Figures 14.1 and 14.2 plot the changes in yields at all maturities over the two markets due to one unit of supply shock $\Delta S_{t,\tau}^{C} = 1$ from market *C* at four different *CMA*'s risk aversion levels: $\gamma^{A} = 0, 1, 10$ and 100. As expected, the response curves in market *C* display monotonically increasing and concave patterns, while the responses in market *E* are monotonically increasing and exhibit convexity at the short end of the maturity spectrum. In market *C*, where the term structure is governed purely by the short rate factor, it is interesting to observe that responses to either supply or demand shocks are more pronounced when the *CMA* is more risk averse. As mentioned in Section 7.5, for a risk-neutral *CMA*, the induced demand shock due to her arbitraging behavior is one if infinite short-selling is permitted. If I set $F^{C}(\tau) = 1$, the grey dotted line would collapse to the zero line, indicating that the market-*C* term structure does not respond to the supply shock at all inasmuch as the shock is fully absorbed by the *CMA*.[9] As the transmitted shock only accounts for a small portion of the original shock, the close-to-zero grey dotted line shown in Figure 14.2 is anticipated.

Since I adopt a functional form of $F^C(\tau) = \exp(-\tau)$, the *CMA*'s full absorption of the supply shock transmits to long maturities in an exponential decay manner, causing the grey dotted line in Figure 14.1 to lie slightly above zero. The slight increase in the market-*C* term structure indicates that the *PHI.C*s internalize a small percentage of the supply shock. However, an exponentially decaying functional form of $F^C(\tau)$ would facilitate the analysis of the scenario when short-selling constraints are imposed. This corresponds to the fourth statement of Theorem 1, which states that the shock is not fully absorbed by the *CMA* when she cannot freely short-sell at her discretion.

For extremely risk-averse *CMA*, as pointed out in Section 7.5, the induced demand shocks are zeros, as she would prefer to remain idle. In this case, the *PHI.C*s fully internalize the supply shock in market *C*, bringing the largest responses of yields to the shock. This can be seen from the red dotted line ($\gamma^A = 100$) in Figure 14.1. Due to the *CMA*'s inaction, yields in market *E* do not respond to supply shocks from market *C*. This can be seen from the close-to-zero red dotted line in Figure 14.2. The yield response to the transmitted supply shock is non-zero because γ^A, though large, is finite and never approaches infinity.

Having understood the behavior of yield responses to the supply shock at the extreme values of the *CMA*'s risk aversion, the analyses of the other two scenarios ($\gamma^A = 1$ and 10) are standard and are suppressed for brevity.

[Insert Figure 14 Here]

7.7 *CMA's intervention at distinct risk aversions*

In this subsection, I examine the theoretical statements in Theorem 2, regarding the *CMA*'s intervention to the *semi-form* of market segmentation. I focus on Theorem 2.2 and Theorem 2.3, the more interesting cases.

Theorem 2.2 states that full integration is achieved if the *CMA* is risk-neutral and infinite short-selling is allowed. A theoretical proof of this statement is given in the Appendix, which contains an explicit analytical expression of the *CMA*'s positions $\{X_{t,\tau}^{C\,*}, X_{t,\tau}^{E\,*}\}$. To give a clear picture of the *CMA*'s invested position at each maturity clientele at equilibrium, I plot $X_{t,\tau}^{C\,*}$ using the estimated parameters from previous calibration results. To estimate the value of $\Delta S_{t,\tau}^{C}$ over the five-year period from 2013 to 2017, I consider the ratio of the amount of sovereign bonds outstanding to

[9] See the expression of $\Delta y_{t,\tau}^{C,A}(\Delta S_{t,\tau^*}^{C})$ in the proof of Proposition 11 in the Appendix for details.

China's GDP during that particular year. I divide the 10-year maturity spectrum into five maturity buckets: [0, 1], [1, 3], [3, 5], [5, 7] and [7, 10]. I then take the average of the five ratios in each bucket to obtain a step function of $\Delta S_{t,\tau}^{C}$. The five averaged ratios of the amount of sovereign bond outstanding in the corresponding maturity bucket to China's GDP amount are 2.01%, 3.42%, 2.98%, 2.43% and 2.00%. I also use the mean of r_t^C and r_t^E over the five-year period as estimates for these two short rates, which are 2.87% and 2.41%.

[Insert Figure 15 Here]

Figure 15 plots the curve of ${X_{t,\tau}^{C}}^*$, the *CMA*'s sovereign bond allocation ratios to clear the gap between the two term structures. The uniformly positive allocations in market *C* is as expected and is explained by the higher short rate in market *C*. The ratio firstly increases from 0.75 at the 3-month maturity clientele to 1.27 at the 15-month maturity clientele, before steadily declining to 0.33 at the 10-year maturity clientele. With the averaged ratios being only around 2% to 3%, it means that the *CMA* will have to short-sell as much as 40 times that of the current supply, say, at the 1- to 3-year maturity clienteles to close the gap between the two term structures. This, in reality, can be never possible and explains the persistent gap between the two term structures of the parallel sovereign bond markets in China. One point to note particularly is the close linkage between the shape of the curve in Figure 15 and the discrepancies between the two term structures in Figure 2. The mean yields at maturities shorter than 1.5 years in market *C* lie above those in market *E*, while the market-*E* term structure completely dominates the market-*C* term structure at maturities longer than 1.5 years. Interestingly, the peak value of ${X_{t,\tau}^{C}}^*$ occurs at the 1.5-year maturity clientele. As the discrepancy between the two term structures diminishes as maturity increases, ${X_{t,\tau}^{C}}^*$ decreases accordingly. This is in line with the intuition, as it requires the *CMA* to take a smaller arbitraging position to close the gap when the gap is smaller.

[Insert Figure 16 Here]

Theorem 2.3 states that, even if short-selling is allowed, a finitely risk-averse *CMA* would not be able to fully close the gap between the two term structures. Set $\gamma^A = 1$. Figure 16 plots the curves of the ratios of the induced sovereign bond allocations from the *CMA*'s arbitraging behavior to China's averaged GDP. As expected, the two curves display similar exponentially decay patterns to the induced demand shocks from *CMA*'s adjusted allocations in Figure 13, despite largely dissimilar solution algorithms, and converge to 0.66 and -0.10, respectively. Specifically, Figure 13 studies the induced demand shocks due to additional supply and demand shocks *<u>after</u>* the *weak-form* of market segmentation equilibrium is achieved, in which case the *CMA* is already present. On the other hand, Figure 16 studies the *CMA*'s initial sovereign bond allocations *<u>before</u>* the *weak-form* of market segmentation equilibrium is achieved. Figure 16.1 displays *CMA*'s sovereign bond allocations in market C, which are uniformly positive due to the higher initial value of the market-C short rate. The magnitude of the induced demand in market E is much smaller than that of the market-C because $S_{t,\tau}^{E}$ is much smaller than $S_{t,\tau}^{C}$ at all maturity clienteles, with the former accounting for about 15% of the aggregate supply over years. This number is very close to 13.5%, the percentage of the market-E induced demand shock. The interpretation of these two demand curves is that, for a risk-averse CMA with $\gamma^A = 1$ who is allowed to infinitely short-sell, the CMA would allocate these amounts of sovereign bonds with the corresponding maturity clienteles to solve her optimization problem in section 3.3. Note instantly that the two demand curves, which represent the ratio of the sovereign bond allocations to China's GDP, though mostly smaller than the demand curve in the above-mentioned risk-neutral CMA case, remain substantially larger than the actual sovereign bond supplies.

[Insert Figure 17 Here]

Theorem 2.4 states that, for a risk-neutral CMA, the imposition of short-selling constraints can lead to either full integration or partially integrated sovereign bond markets, depending on the relative magnitude of the CMA's endowment and the amount needed to achieve full integration. In the case of the parallel sovereign bond markets in China, calibration results show clearly that full integration can never be achieved due to the substantially large difference between the supply of sovereign bonds and CMA's equilibrium allocations in obtaining full integration. Figure 17 plots the mean values of the percentage of deviations from full integration when the CMA cannot freely short-sell. The horizontal axis plots the percentage of the amount needed to obtain full integration that the CMA can borrow.[10] As expected, the more the CMA can borrow, the more the two term structures converge to each other, as can be seen from the decline in the percentages of deviations from full integration from 35% to 0 in market E and from about 50% to 0 in market C.

8 Policy Implications

My model is also a toolbox for analyzing the effect of policy intervention. In this section, I discuss some practical applications of my model. In particular, I study some policy implications generated from my model on how it can be applied to investigate whether two segmented bond markets, which operate in tandem or across different universes, should be fully or partially integrated to enhance market efficiency and the aggregate welfare of market participants. For instance, the central bank can use my model to analyze the impact of its monetary policy intervention on the two term structures. The Chinese Ministry of Finance can study the impact of

[10] The exact amount at each maturity clientele needed to obtain

debt issuance on the two term structures. In particular, my model can be used to forecast the term structure comovements once the *Southbound* of the *Bond Connect* platform is opened. The government can use my model to determine the extent to which should they allow domestic investors to participate in the overseas bond markets. The government may control the degree of integration by allowing particular groups of institutional investors to trade in the *Southbound*; in other words, controlling for cross-market arbitrageur's risk-aversion, γ^A. This is easily obtainable as different groups of institutional investors characterize distinct risk aversion levels. I consider three specific situations and provide the decisions rules in the end.

8.1 *Optimal execution time of monetary policies*

Consider first a situation in which the government (central planner) in one country need to determine whether to implement some policy, and if yes, what time in the future to implement it. For instance, the government may determine what time to buy back sovereign bonds or other securities to increase money supply in the economy during quantitative easing or what time to issue a certain amount of maturity-τ sovereign bonds in one specific market. The central planner's optimization problem is to maximize the aggregate welfare gain over expenditure of each representative market participant

$$\max_T \sum_{i=1}^{2T+3} \theta_t^i \left[U_t^i(T) - U_t^i(\infty) - \alpha^i\left(\Delta S^i\right)^2\right]$$

where $\{\theta_t^i\}$ is the measure of representative agent i at time t and

$$U_t^i(T) = \int_t^T e^{-R_t^i(s-t)} e^{-\gamma^i W_s^i} ds + e^{-R_0^i T} \int_T^\infty e^{-R_T^i(s-T)} e^{-\gamma^i W_s^{i\prime}} ds$$

and the welfare gain for agent i is

$$U_t^i(T) - U_t^i(\infty) = e^{-R_0^i T} \int_T^\infty \left[e^{-R_T^i(s-T)} e^{-\gamma^i W_s^{i\prime}} - e^{-R_t^i(s-T)} e^{-\gamma^i W_s^i}\right] ds$$

Let the optimal policy implementation time be T^*, the solution to the above optimization problem.

Decision rule: If the objective function is positive at T^*, it would be optimal for the government to implement the policy at T^*. If the objective function is less than zero at T^*, the government should forfeit the policy.

8.2 *Choice of markets in bond issuances*

Consider another situation in which the government needs to decide in which market to issue a given amount of maturity-τ sovereign bonds at a certain future time point T to increase money supply. The central planner's optimization problem is to compare the aggregate welfare gains over expenditure of each representative market participant over the two markets

$$\max\left\{\sum_{i=1}^{2T+3}\theta_t^i\left[U_t^{i,C}(T)-U_t^{i,C}(\infty)-\alpha^{i,C}\left(\Delta S^{i,C}\right)^2\right],\sum_{i=1}^{2T+3}\theta_t^i\left[U_t^{i,E}(T)-U_t^{i,E}(\infty)-\alpha^{i,E}\left(\Delta S^{i,E}\right)^2\right]\right\}$$

where $\{\theta_t^i\}$ is the measure of agent i at time t and for $K = C, E$

$$U_t^{i,K}(T)=\int_t^T e^{-R_t^{i,K}(s-t)}e^{-\gamma^i W_s^{i,K}}ds+e^{-R_0^{i,K}T}\int_T^\infty e^{-R_T^{i,K}(s-T)}e^{-\gamma^i W_s^{i,K'}}ds$$

and the welfare gain for agent i is

$$U_t^{i,K}(T)-U_t^{i,K}(\infty)=e^{-R_0^{i,K}T}\int_T^\infty\left[e^{-R_t^{i,K}(s-T)}e^{-\gamma^i W_s^{i,K'}}-e^{-R_t^{i,K}(s-T)}e^{-\gamma^i W_s^{i,K}}\right]ds$$

Decision rule: If the objective function at the optimal solution is positive and is larger in market *C* (market *E*), it would be optimal for the government to issue the maturity-τ bonds in market *C* (market *E*). If the objective function is negative, the government should forfeit the issuance plan.

8.3 *Cross-country integration of segmented bond markets*

Consider another situation in which country *A*'s government and country *B*'s government need to determine whether they should integrate their originally segmented sovereign bond markets. The central planner from country *A* faces the optimization problem to maximize the aggregate welfare gain over expenditure of representative market participants in country *A* only, taking as given the actions $\left\{T^B,\lambda_t^B,\{S_{t,\tau}^B\}_{0<\tau\le T}\right\}$ from country *B*:

$$\max_{T^A,\lambda_t^A,\{S_{t,\tau}^A\}_{0<\tau\le T}}\sum_{i=1}^{T+2}\theta_t^{A,i}\left[U_t^{A,i}(T)-U_t^{A,i}(\infty)-\alpha^{A,i}\left(\Delta S^{A,i}\right)^2\right]$$

Similarly, the central planner from country *B* faces the optimization problem to maximize the aggregate welfare gain over expenditure of representative market participants in country *B* only, taking as given the actions $\left\{T^A,\lambda_t^A,\{S_{t,\tau}^A\}_{0<\tau\le T}\right\}$ from country *A*:

$$\max_{T^B,\lambda_t^B,\{S_{t,\tau}^B\}_{0<\tau\le T}}\sum_{i=1}^{T+2}\theta_t^{B,i}\left[U_t^{B,i}(T)-U_t^{B,i}(\infty)-\alpha^{B,i}\left(\Delta S^{B,i}\right)^2\right]$$

Again, $\{\theta_t^{A,i}\}$ and $\{\theta_t^{B,i}\}$ denote measures of agent i in country *A* and *B* at t, respectively.

Decision rule: If both objective functions are non-negative at their optimal solutions, cross-country sovereign bond market linkage should be set up. The central planners would abandon the connection if at least one of the two objective functions were less than zero.

9 Conclusion

In this paper, I model two term structures of interest rates that result from the interaction between three broad types of investors of two parallel bond markets. The three types of investors include (1) $2T$ representative preferred-habitat investors, T in each market, (2) two representative within-market arbitrageurs, one in each market, and (3) one representative cross-market arbitrageur. My model extends the one-market preferred-habitat model in Vayanos and Vila (2019) to a two-market setup with a pair of parallel bond markets. My model adapts to a modern no-arbitrage framework and generates a much richer set of theoretical implications than all existing term structure models that aim at modeling investors' bond allocations and trading behavior at equilibrium across multiple representative agents.

I later introduce three types of market segmentation, including (1) the *extreme-form*, (2) the *semi-form* and (3) the *weak-form* of market segmentations. These three forms of market segmentation closely delineate what happen in reality. Under the *extreme-form* of market segmentation, there only exists preferred-habitat investors who clear the markets at each maturity clientele through resorting to a private technology. I focus on the discussion of the *semi-form* and the *weak-form* of market segmentations, which generate much more implications than the *extreme-form* one, thanks to the participation of the risk-averse cross-market arbitrageur. Under the *semi-form* of market segmentation, I discuss the global supply effects and the global demand effects due to an exogenous supply shock and a shock to the demand intercept term, respectively, under the one-factor framework of market *C*. Under the two-factor framework of market *E* where both the short rate and a net demand factor function as price factors, I show that the demand effects have now become more localized, *i.e.* the relative effects across maturities depend on the origin of the shock. I extend similar analyses to the *weak-form* of market segmentation and show that the global and the localized features of supply and demand shock transmissions preserve.

I further perform a calibration exercise of my model on the parallel sovereign bond markets in China, the interbank OTC market and the exchange market. I focus on the discussion of the trading behavior of the cross-market arbitrageur, the key player in my model. I first examine the transmission of exogenous supply shocks and demand shocks through the cross-market arbitrageur from one market to another. I then examine the response of the term structures to supply shocks and discuss the internalization and transmission of these shocks under different values of the cross-market arbitrageur's risk aversion. Last, I study the yield differential between the two term structures of interest rates, with the cross-market arbitrageur being either risk-neutral or risk-averse and the short-selling constraint being imposed or not.

Last, I consider the real consequences of policy implications of my model. I discuss a number of practical applications that my model can be used to and how researchers can utilize it to study whether two segmented bond markets should be fully or partially integrated to enhance market efficiency and the aggregate welfare of market participants. In particular, I discuss three real-world problems that my model can answer, including the optimal execution time of a government policy, the choice of which market to issue a certain amount of sovereign bonds, and the cross-country integration of two countries' bond markets. One typical example I consider is the *Bond Connect* platform, the *Southbound* connection of which remains closed. Regulators can use my model to forecast term structure comovements once the *Southbound* is opened. The government may also control the degree of integration between the two bond markets by allowing specific investor groups to participate in the *Southbound*. Two other interesting extensions of my model can be extending it from a parallel-market structure to a *k*-market structure and generalizing it by adding a yield spread so that corporate bond investors may also use it as a benchmark model.

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Table 1. Parallel market structure of the Chinese sovereign bond markets

This table summarizes the major differences between my model and the two models based on which my models adopts the framework. My model admits a similar framework to the V&V model in Vayanos and Vila (2019) to model the term structure of interest rates within each market. At the same maturity clientele, my model adopts a modified framework to Goldstein, Li and Yang (2014).

	Cross-market Model Setup	*Within-market Model Setup*	*Parallel-market Model Setup*
	Goldstein, Li and Yang (2014)	Vayanos and Vila (2019)	My Paper
Markets	one asset market	one bond market	two parallel bond markets
Assets	2 risky assets 1 risk-free asset	1 long-term risky bond 1 short-term riskless bond	2 long-term risky bonds 1 short-term riskless bond
Market Participants	2 types of traders, with large and small investment opportunity sets	T preferred-habitat investors 1 risk-averse arbitrageur	$2T$ preferred-habitat investors 3 types of risk-averse arbitrageurs each has distinct investment opportunity set
Shock Transmission	short rates to long rates	N.A.	short rates to long rates demand/supply shocks across the markets

Figure 1. Parallel market structure of the Chinese sovereign bond markets

This figure gives a graphical representation of the model under the parallel market structure of the Chinese sovereign bond markets. Within each market, my model admits a similar framework to the V&V model in Vayanos and Vila (2019) to model the term structure of interest rates. At the same maturity clientele, my model adopts a modified framework to Goldstein, Li and Yang (2014). Household investors can only invest in market *E*, which represents the exchange market in China. Banks can only invest in market *C*, which represents the interbank OTC market in China. Under this framework, institutional investors function as the cross-market arbitrageurs who can invest in both the exchange market and the interbank OTC market.

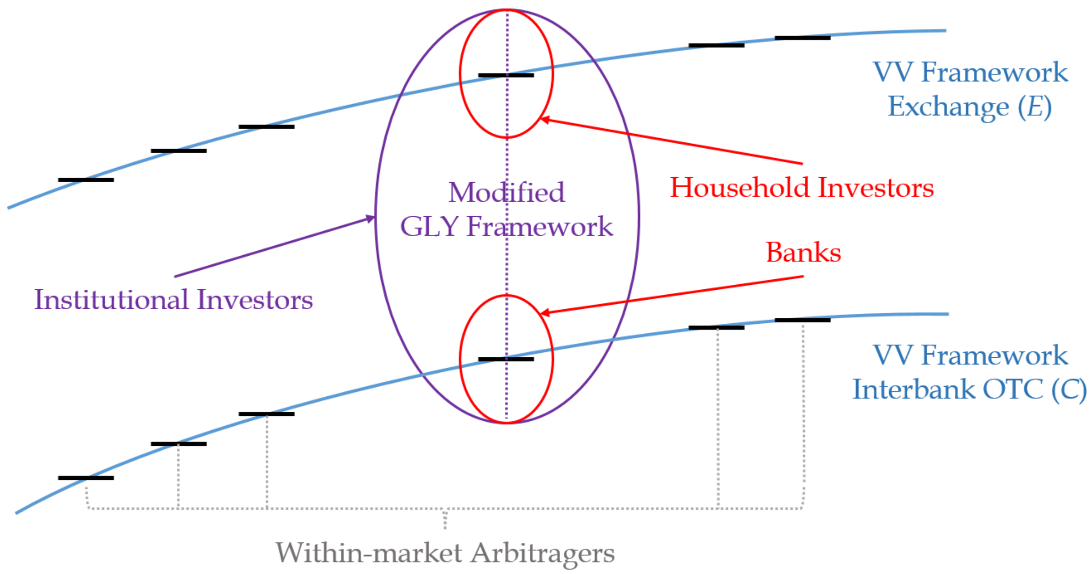


Figure 2. Mean yields across maturities from 2007 to 2017

This figure plots the mean values of yields to maturity over the entire sample period from January 2007 to December 2017. The black dotted line represents the term structure of interest rates in the exchange market. The blue starred line represents the term structure of interest rates in the interbank OTC market. Time to maturities range from three months to ten years, with an equidistant spacing at 3 months. I obtain the market transaction data from the China Foreign Exchange Trade System, ranging from January 2007 to December 2017.

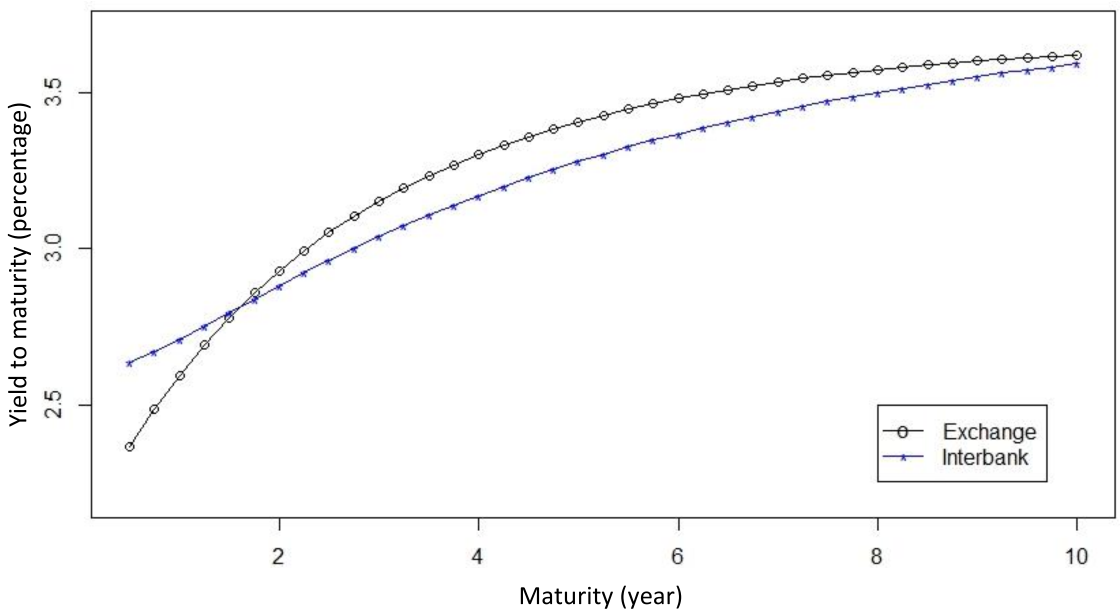


Figure 3. Weekly-aggregated short rate over time from 2013 ~ 2017

This figure plots the weekly aggregated values of the short rates, proxied by the 3-month spot rates, over the entire sample period from January 2013 to December 2017. The sample period covers 258 weeks, as indicated on the horizontal axis. The red solid line corresponds to the interbank OTC market (denoted by market *C*). The blue solid line corresponds to the exchange market (denoted by market *E*). I obtain the market transaction data from the China Foreign Exchange Trade System, ranging from January 2007 to December 2017.

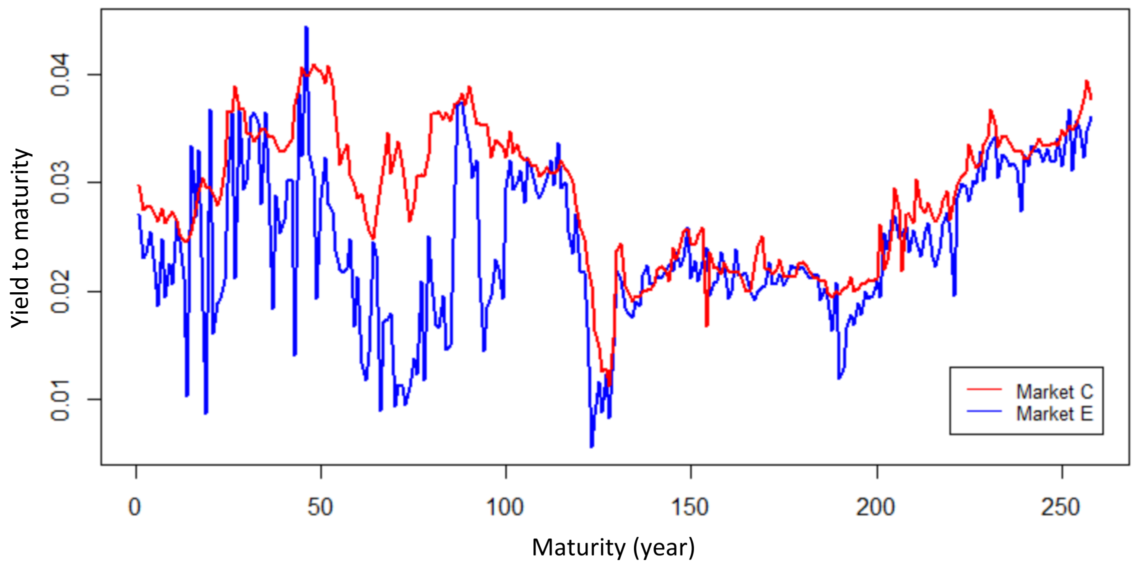


Figure 4. Unconditional volatility of yields (market-C parameters)

This figure plots the model calibration results of the interbank OTC market parameters, based on minimizing the sum of the squared differences between data-based unconditional volatilities to the model-implied volatilities at maturities $k = 1.25, 1.5, 1.75, \ldots \ldots, 10$. The black dotted line plots the actual unconditional volatilities. The blue dotted line plots the model-implied volatilities. Maturities on the horizontal axis are in years. I obtain the market transaction data from the China Foreign Exchange Trade System, ranging from January 2007 to December 2017.

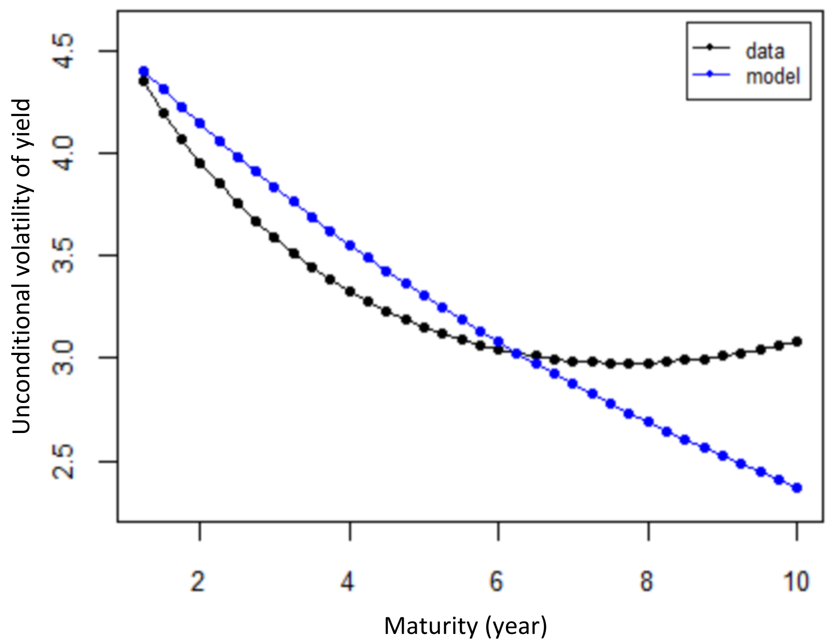
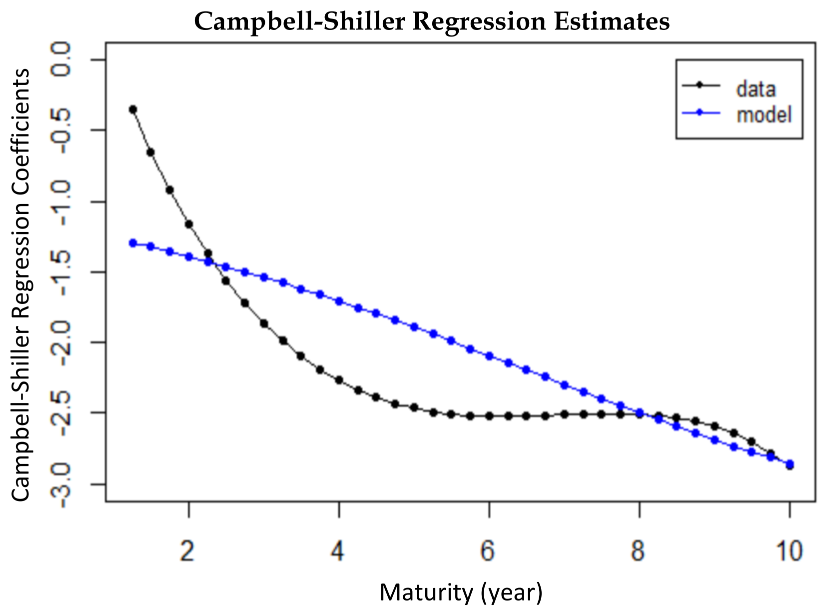


Figure 5. Campbell-Shiller and Fama-Bliss regressions (market-*E* parameters)

This figure plots the model calibration results of the exchange market parameters, based on minimizing the sum of the squared differences between Campbell-Shiller regression estimates and Fama-Bliss regression estimates and their corresponding model-implied values. In each sub-figure, the black dotted line plots the actual coefficient estimates from performing the regression. The blue dotted line plots the model-implied values. Maturities on the horizontal axis are in years. I obtain the market transaction data from the China Foreign Exchange Trade System, ranging from January 2007 to December 2017.



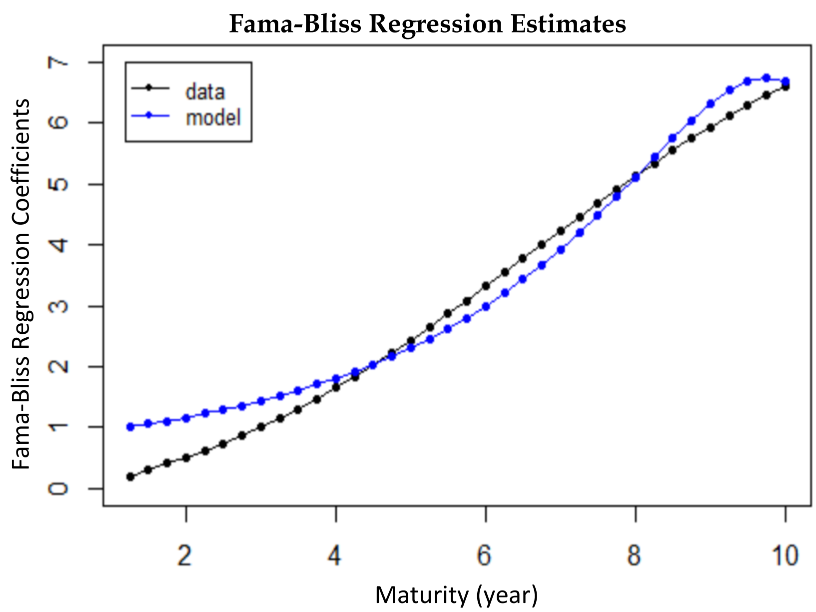
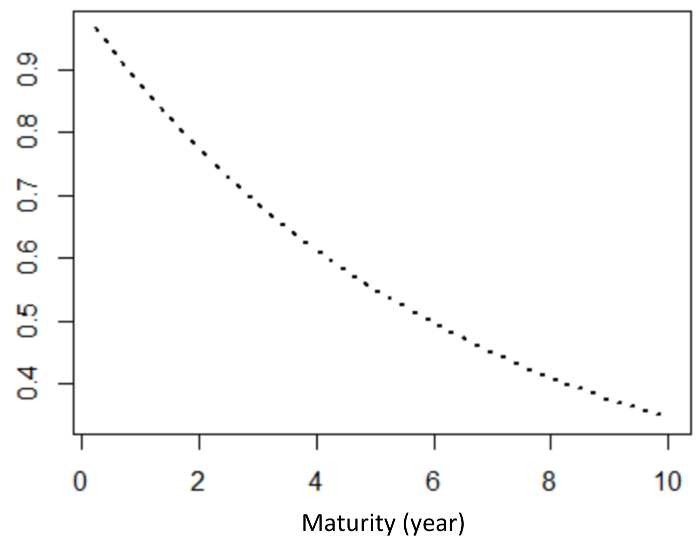


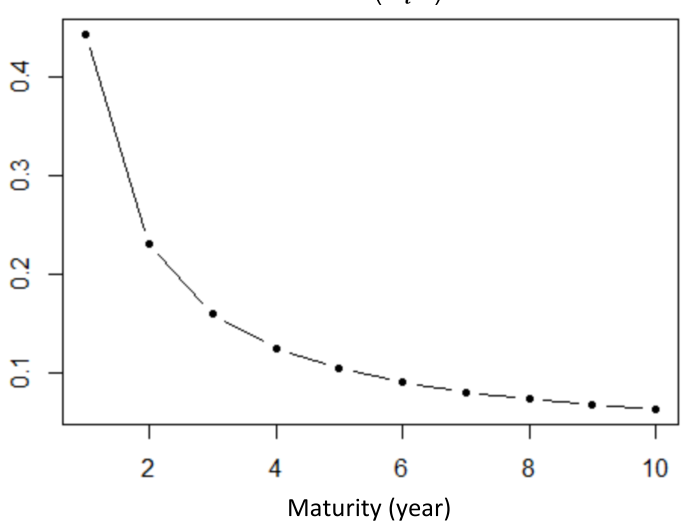
Figure 6. Response of yields to shocks to the short rate and the net demand factor

The first sub-figure (6.1) plots the function $A_r^C(\tau)/\tau$, representing the response of yields to the short rate shocks under the *semi-form* and the *weak-form* of market segmentations in market *C*. The second sub-figure (6.2) plots the function $A_\beta^E(\tau)/\tau$, which models the response of yields to the net demand factor under the *extreme-form* of market segmentation (segmentation equilibrium) in market *E*. The third sub-figure (6.3) plots the functions $A_r^E(\tau)/\tau$ and $A_\beta^E(\tau)/\tau$, which delineate the response of yields to shocks to the short rate (dotted curve) and to the net demand factor (dash-dotted line) under the *semi-form* and the *weak-form* of market segmentations in market *E*. Maturities on the horizontal axis are in years. I obtain the market transaction data from the China Foreign Exchange Trade System, ranging from January 2007 to December 2017.

(1) Market *C* response to r_t $\left(\frac{A_r^C(\tau)}{\tau}\right)$ (*semi- & weak-form*)



(2) Market *E* response to β_t $\left(\frac{A_\beta^E(\tau)}{\tau}\right)$ (*extreme-form*)



(3) Market *E* responses to r_t $\left(\frac{A_r^E(\tau)}{\tau}\right)$ and β_t $\left(\frac{A_\beta^E(\tau)}{\tau}\right)$ (*semi- & weak-form*)

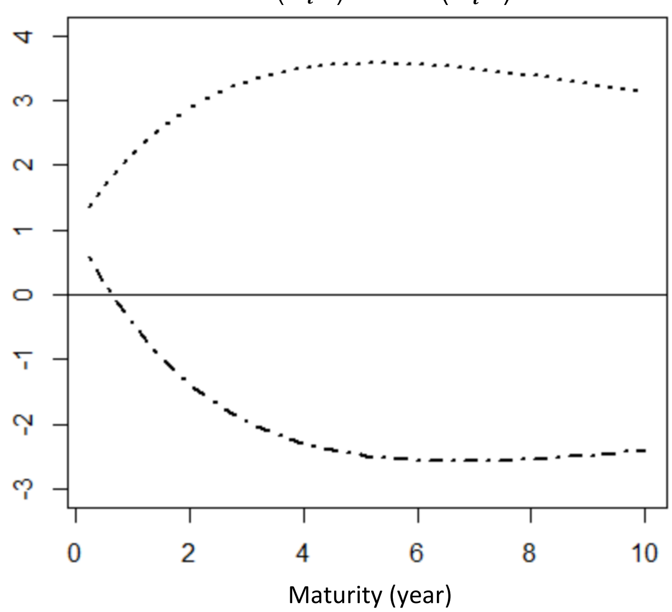


Figure 7. Response of demand functions to short rate jumps in market C

This figure plots the responses of the market-C preferred-habitat investors' demand functions to expected jumps in the short rate process. The three curves correspond to different parameters of the within-market arbitrageur's risk aversion parameter γ^C and the Poisson intensity of the short rate jump process. Maturities on the horizontal axis are in years. I obtain the market transaction data from the China Foreign Exchange Trade System, ranging from January 2007 to December 2017.

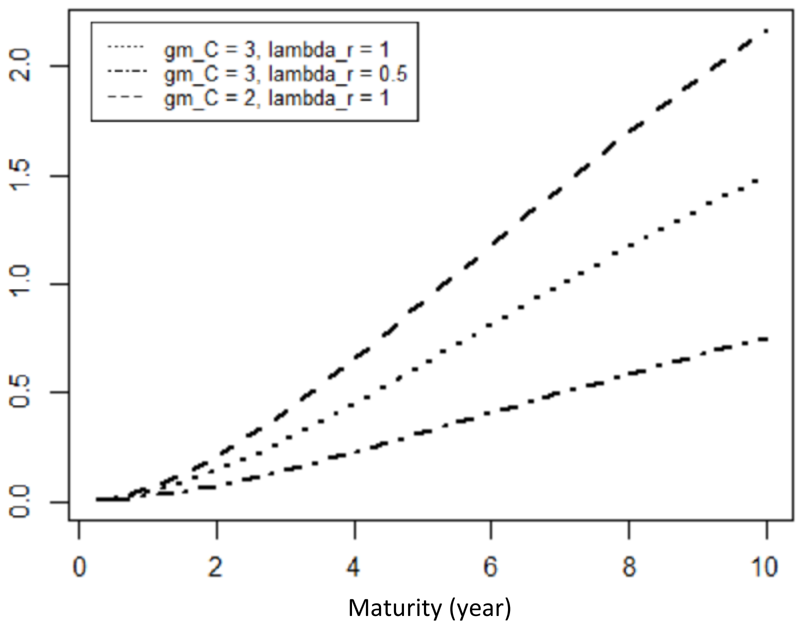
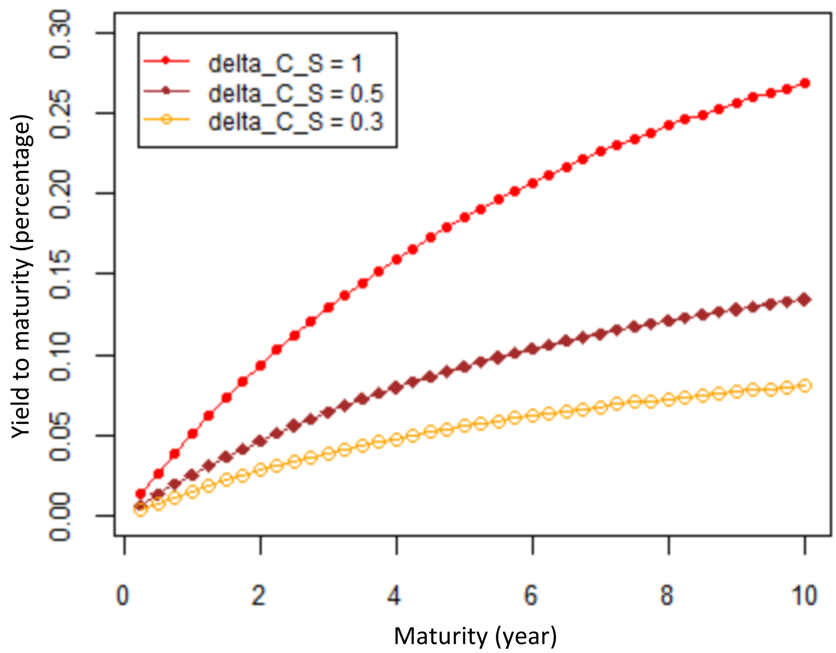


Figure 8. Global supply effects and global demand effects from market *C*

The first sub-figure (8.1) plots the changes in yield curve due to supply shocks of one unit, 0.5 unit and 0.3 unit, respectively, in market *C*. The second sub-figure (8.2) plots the changes in yield curve due to demand shocks to preferred-habitat investors' demand intercept under three functional forms, specified in the legend box. Maturities on the horizontal axis are in years. I obtain the market transaction data from the China Foreign Exchange Trade System, ranging from January 2007 to December 2017.



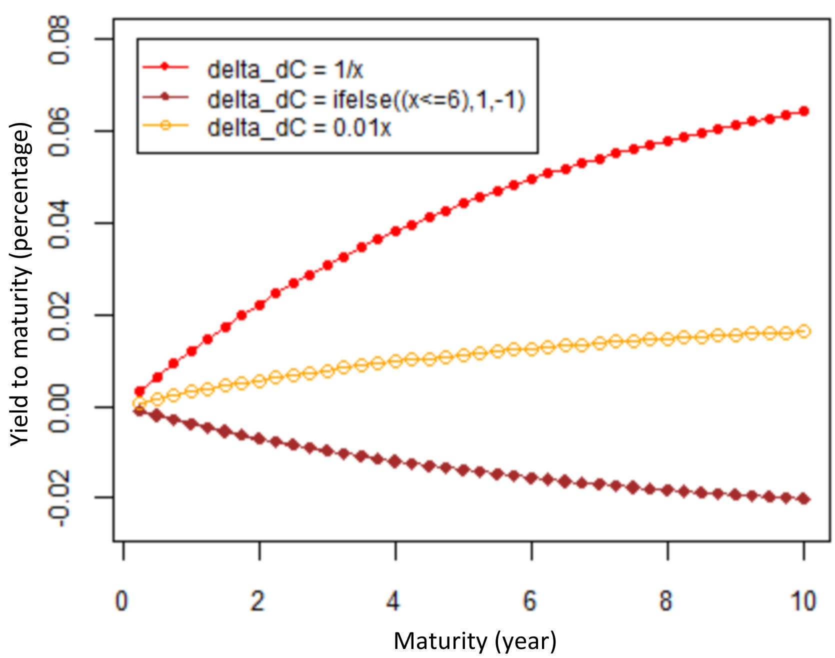


Figure 9. Global supply effects and localized demand effects from market *E*

The first sub-figure (9.1) plots the changes in yield curve due to supply shocks of one unit, 0.5 unit and 0.3 unit, respectively, in market *E*. The second sub-figure (9.2) plots the changes in yield curve due to one unit of demand shock to preferred-habitat investors' demand intercept at three different maturity clienteles. The third sub-figure plots the changes in yield curve due to demand shocks to preferred-habitat investors' demand intercept under three functional forms, specified in the legend box. Maturities on the horizontal axis are in years. I obtain the market transaction data from the China Foreign Exchange Trade System, ranging from January 2007 to December 2017.

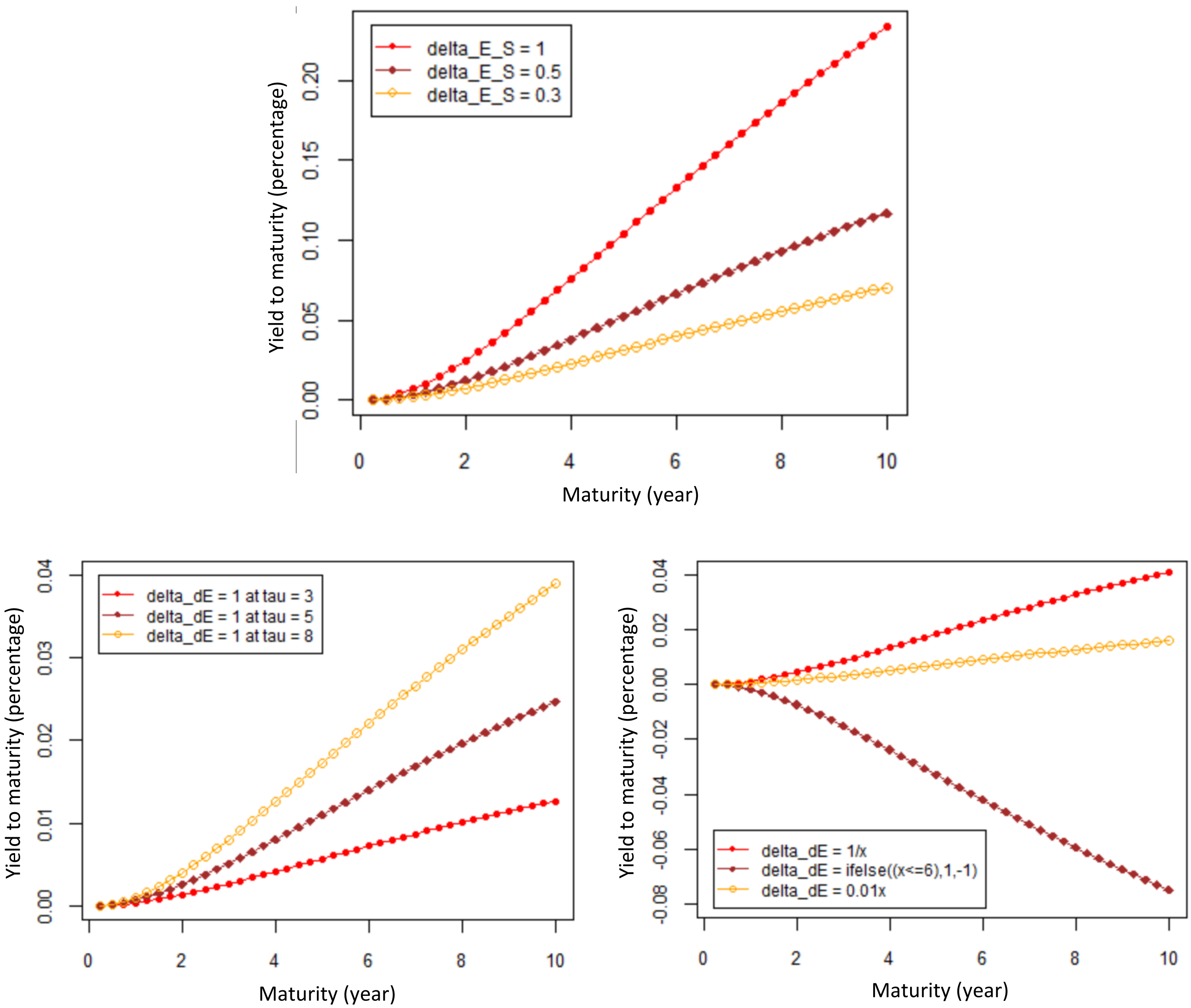


Figure 10. Supply shock transmission from market *C* to market *E*

This figure plots the changes in yields at distinct maturities in both markets due to one unit of supply shock from market *C*. In the first sub-figure (10.1), the red dotted line plots the impact on the market-*C* term structure, denoted by $\Delta y_{t,\tau}^{C,A}(\Delta S_{t,\tau^*}^{C})$; the blue dotted line plots the impact on the market-*E* term structure, denoted by $\Delta y_{t,\tau}^{E,A}(\Delta S_{t,\tau^*}^{C})$. In the second sub-figure (10.2), I further decompose the change in the market-*E* term structure into two components: (1) the short rate component and (2) the net demand component. The risk aversion of the cross-market arbitrageur is set to be one. Maturities on the horizontal axis are in years. I obtain the market transaction data from the China Foreign Exchange Trade System, ranging from January 2007 to December 2017.

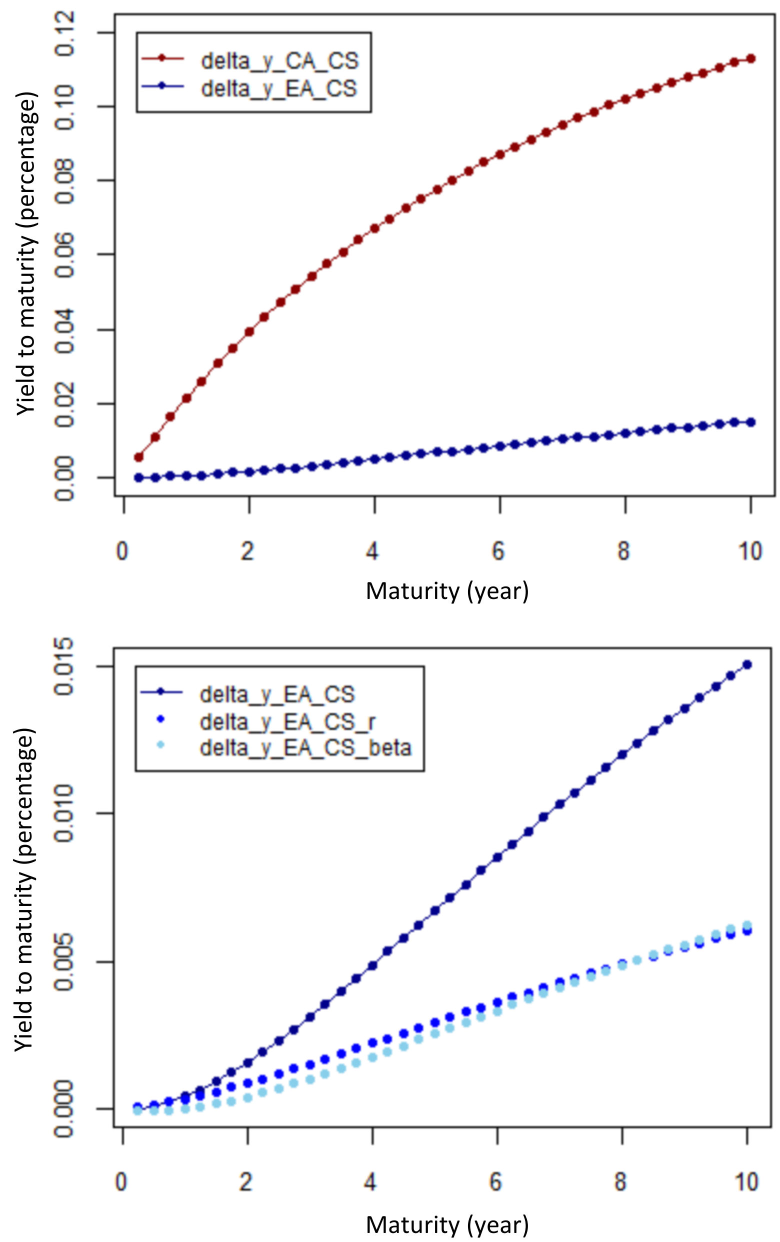
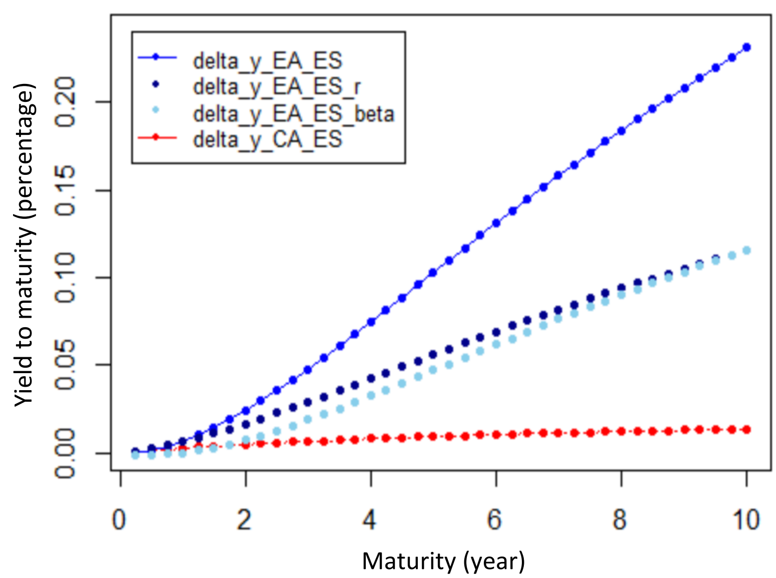


Figure 11. Supply shock transmission from market E to market C

This figure plots the changes in yields at distinct maturities in both markets due to one unit of supply shock from market E. In the first sub-figure (11.1), the red dotted line plots the impact on the market-C term structure, denoted by $\Delta y_{t,\tau}^{C,A}(\Delta S_{t,\tau^*}^{E})$; the blue dotted line plots the impact on the market-E term structure, denoted by $\Delta y_{t,\tau}^{E,A}(\Delta S_{t,\tau^*}^{E})$. The two dotted curves plot the compositions of the impact on the market-E term structure. The second sub-figure (11.2) plots the ratio of the change in the yield curve of market i to the change in the yield curve of market j, due to one unit of supply shock from market j (i, $j \in \{C, E\}$). The risk aversion of the cross-market arbitrageur is set to be one. Maturities on the horizontal axis are in years. I obtain the market transaction data from the China Foreign Exchange Trade System, ranging from January 2007 to December 2017.



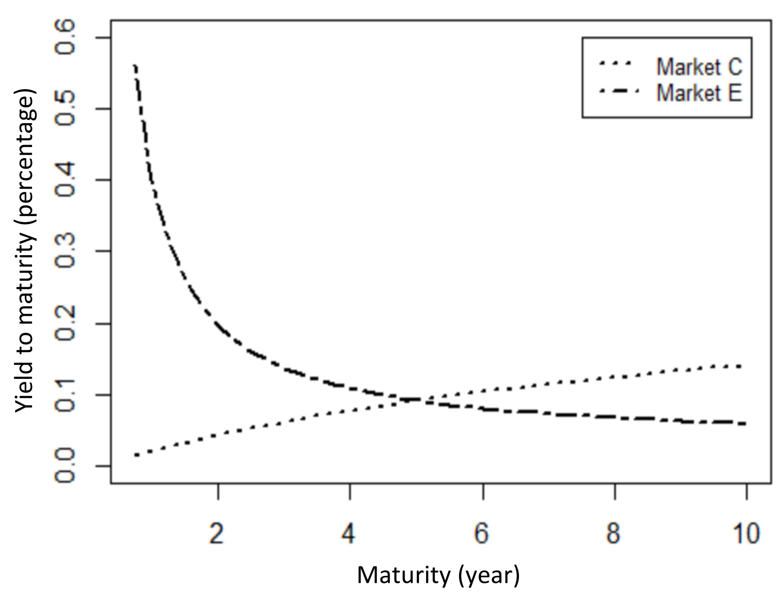
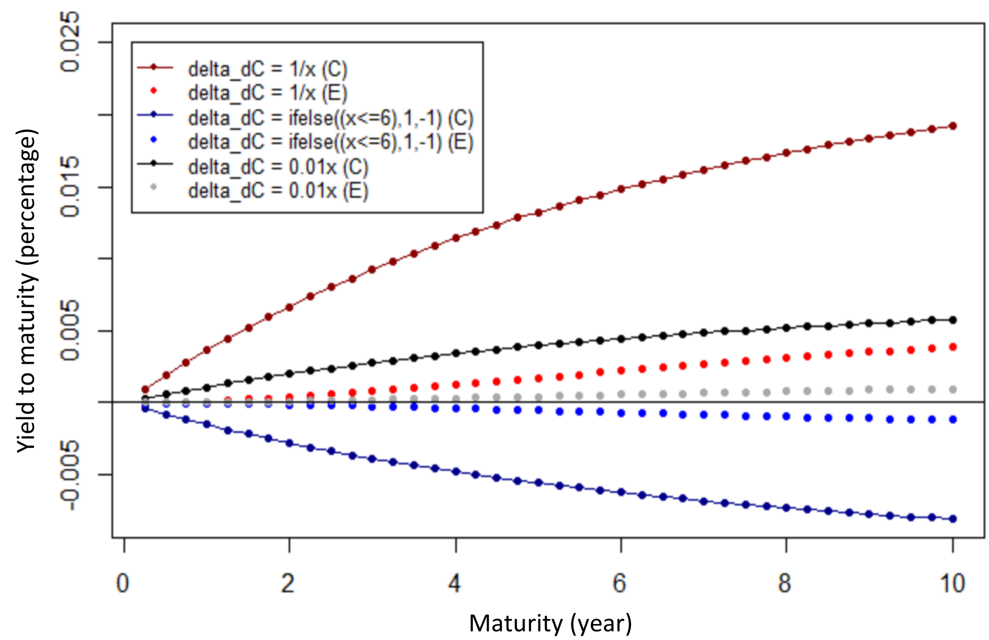


Figure 12. Demand shock transmission

The first sub-figure (12.1) plots the changes in yields at distinct maturities in both markets due to three different types of demand shock from market C. The second sub-figure (12.2) plots the changes in yields at distinct maturities in both markets due to three different types of demand shock from market E. The three dotted lines plot the impact on the market-I term structure, denoted by $\Delta y_{t,\tau}^{I,A}(\Delta d^I)$, due to the three demand shocks from market I, respectively. The three dotted curves plot the impact on the market-J term structure, denoted by $\Delta y_{t,\tau}^{J,A}(\Delta d^I)$, due to the three demand shocks from market I, respectively. Indices $I, J \in \{C, E\}$. The risk aversion of the cross-market arbitrageur is set to be one. Maturities on the horizontal axis are in years. I obtain the market transaction data from the China Foreign Exchange Trade System, ranging from January 2007 to December 2017.



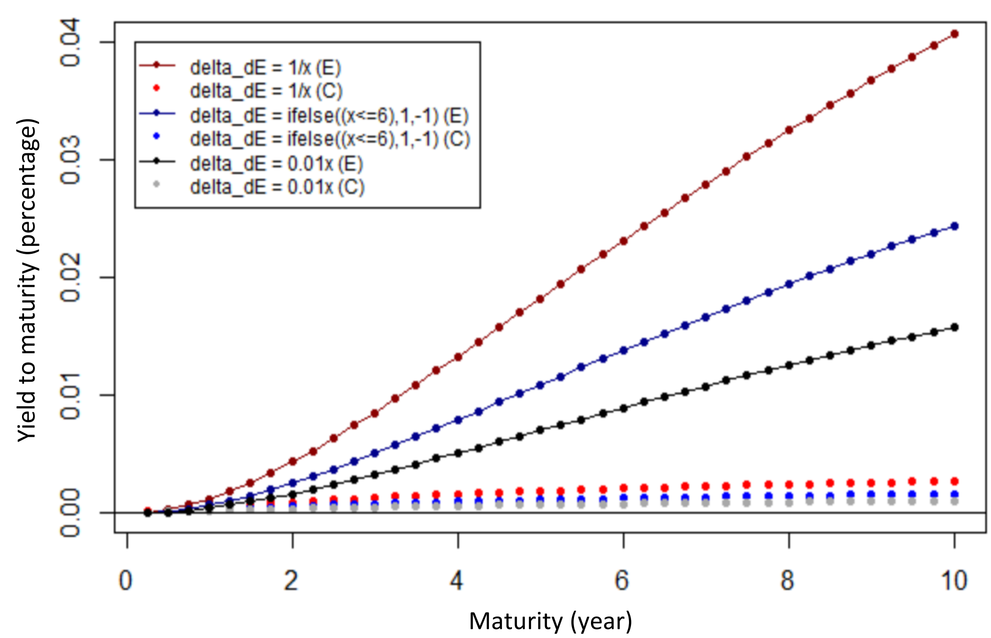


Figure 13. Induced demand shocks from *CMA*'s adjusted allocations

Figures 13.1 and 13.2 (13.3 and 13.4) plot the induced demand shocks from the *CMA*'s adjusted allocations $X^C_{t,\tau}(\Delta S^C_{t,\tau})$ and $X^E_{t,\tau}(\Delta S^C_{t,\tau})$ [$X^C_{t,\tau}(\Delta S^E_{t,\tau})$ and $X^E_{t,\tau}(\Delta S^E_{t,\tau})$] in the two markets due to one unit of exogenous supply shock $\Delta S^C_{t,\tau} = 1$ ($\Delta S^E_{t,\tau} = 1$) from market *C* (market *E*). Figures 13.5 and 13.6 (13.7 and 13.8) plot the induced demand shocks from the *CMA*'s adjusted allocations $X^C_{t,\tau}(\Delta d^C)$ and $X^E_{t,\tau}(\Delta d^C)$ [$X^C_{t,\tau}(\Delta d^E)$ and $X^E_{t,\tau}(\Delta d^E)$] in the two markets due to a shock to the preferred-habitat investors' demand intercept $\Delta d^C(\tau) = \tau^{-1}$ ($\Delta d^E(\tau) = \tau^{-1}$) from market *C* (*E*). The risk aversion of the cross-market arbitrageur is set to be one. Maturities on the horizontal axis are in years. I obtain the market transaction data from the China Foreign Exchange Trade System, ranging from January 2007 to December 2017.

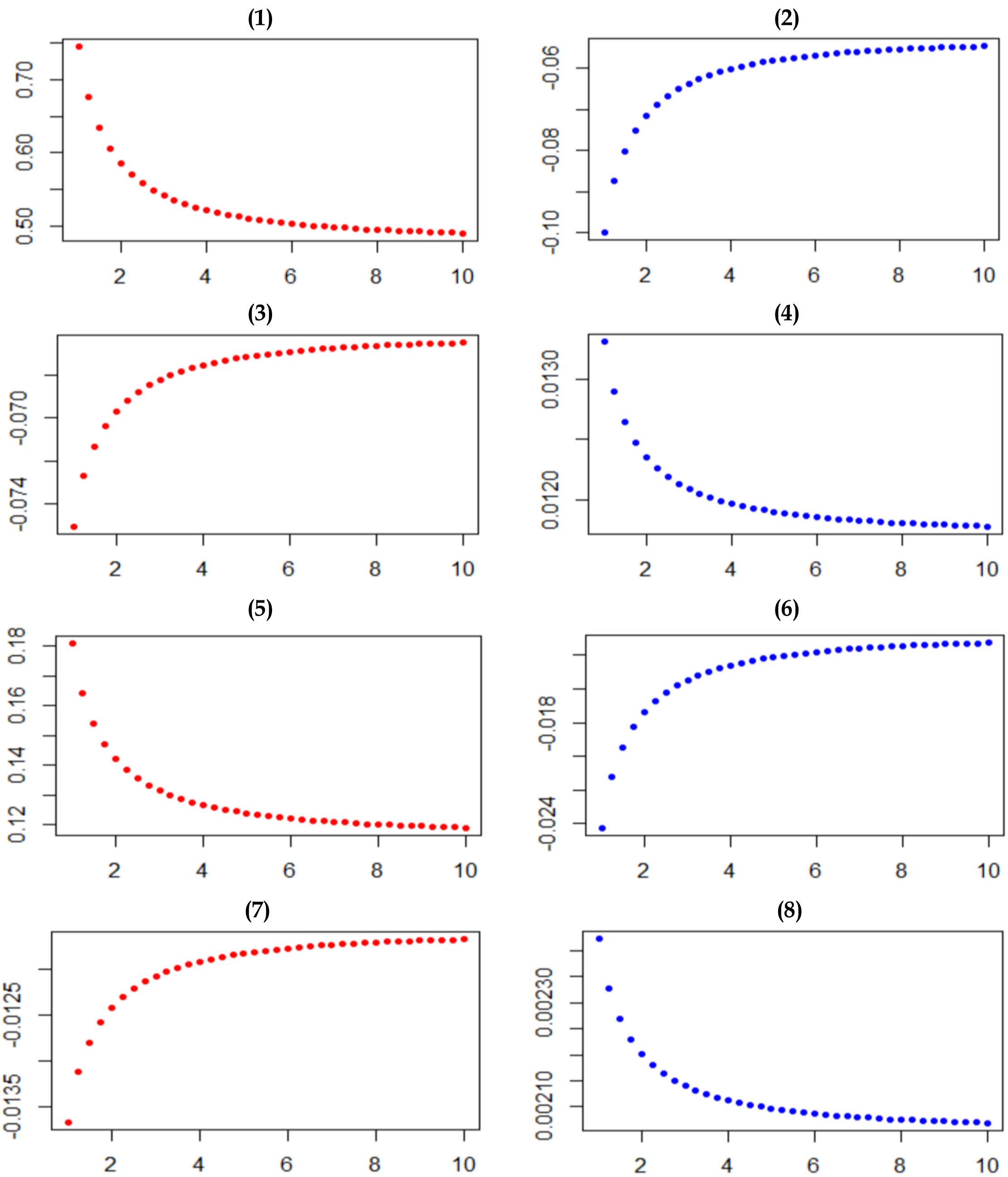
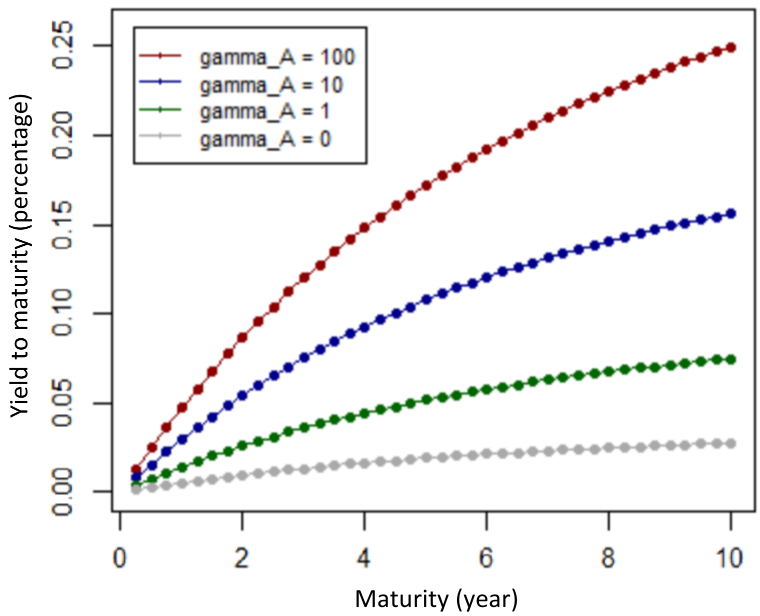


Figure 14. Shock internalization and transmission under different γ^A's

This figure plots the changes in yields at distinct maturities over the two markets due to one unit of supply shock $\Delta S^C_{t,\tau} = 1$ from market C, with four different CMA's risk aversion levels at $\gamma^A = 0, 1, 10, 100$. The first sub-figure (14.1) plots the impact of $\Delta S^C_{t,\tau}$ on the market-C term structure. The second sub-figure (14.2) plots the impact of $\Delta S^C_{t,\tau}$ on the market-E term structure. Maturities on the horizontal axis are in years. I obtain the market transaction data from the China Foreign Exchange Trade System, ranging from January 2007 to December 2017.



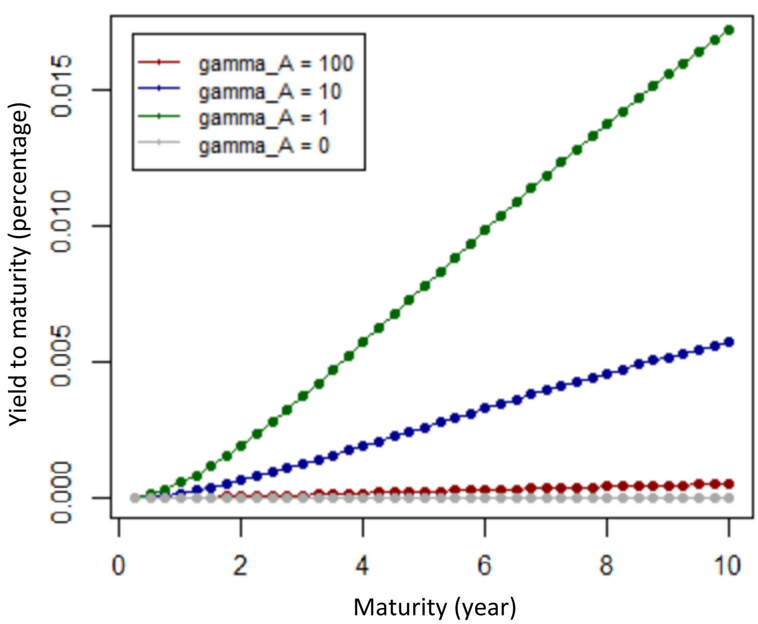


Figure 15. Risk-neutral *CMA*'s allocation under infinite short-selling

This figure plots a risk-neutral *CMA*'s sovereign bond allocations at all maturity clienteles in market *C* that are required to obtain the full integration scenario, in which case the gap between two term structures in the *semi-form* case of market segmentation is completely closed. Maturities on the horizontal axis are in years. I obtain the market transaction data from the China Foreign Exchange Trade System, ranging from January 2007 to December 2017.

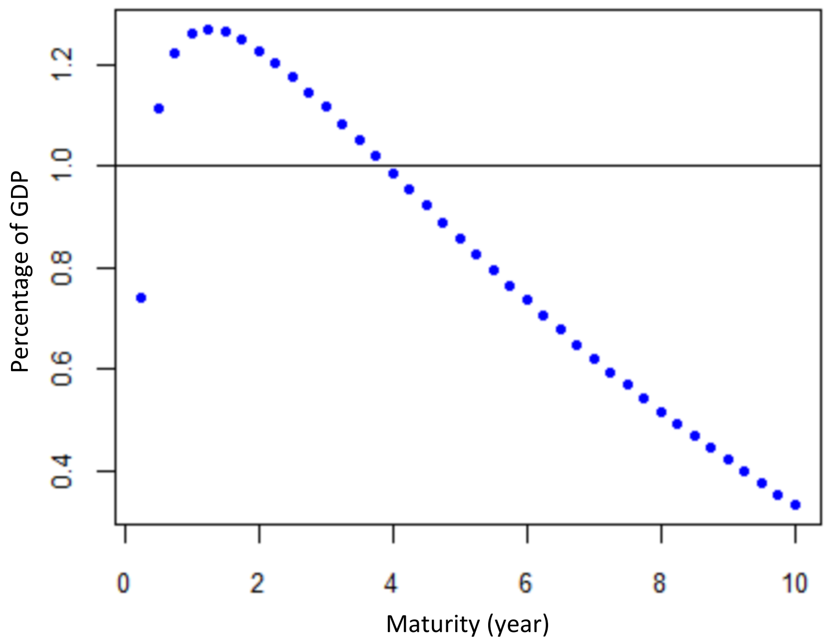


Figure 16. Risk-averse *CMA*'s intervention

This figure plots a risk-averse *CMA*'s optimal sovereign bond allocations at all maturity clienteles in both markets after she intervenes and arbitrages from the two sovereign bond markets that are originally segmented (i.e. *semi-form*). Figure 16.1 displays *CMA*'s sovereign bond allocations in market *C*. Figure 16.2 displays *CMA*'s sovereign bond allocations in market *E*. Maturities on the horizontal axis are in years. I obtain the market transaction data from the China Foreign Exchange Trade System, ranging from January 2007 to December 2017.

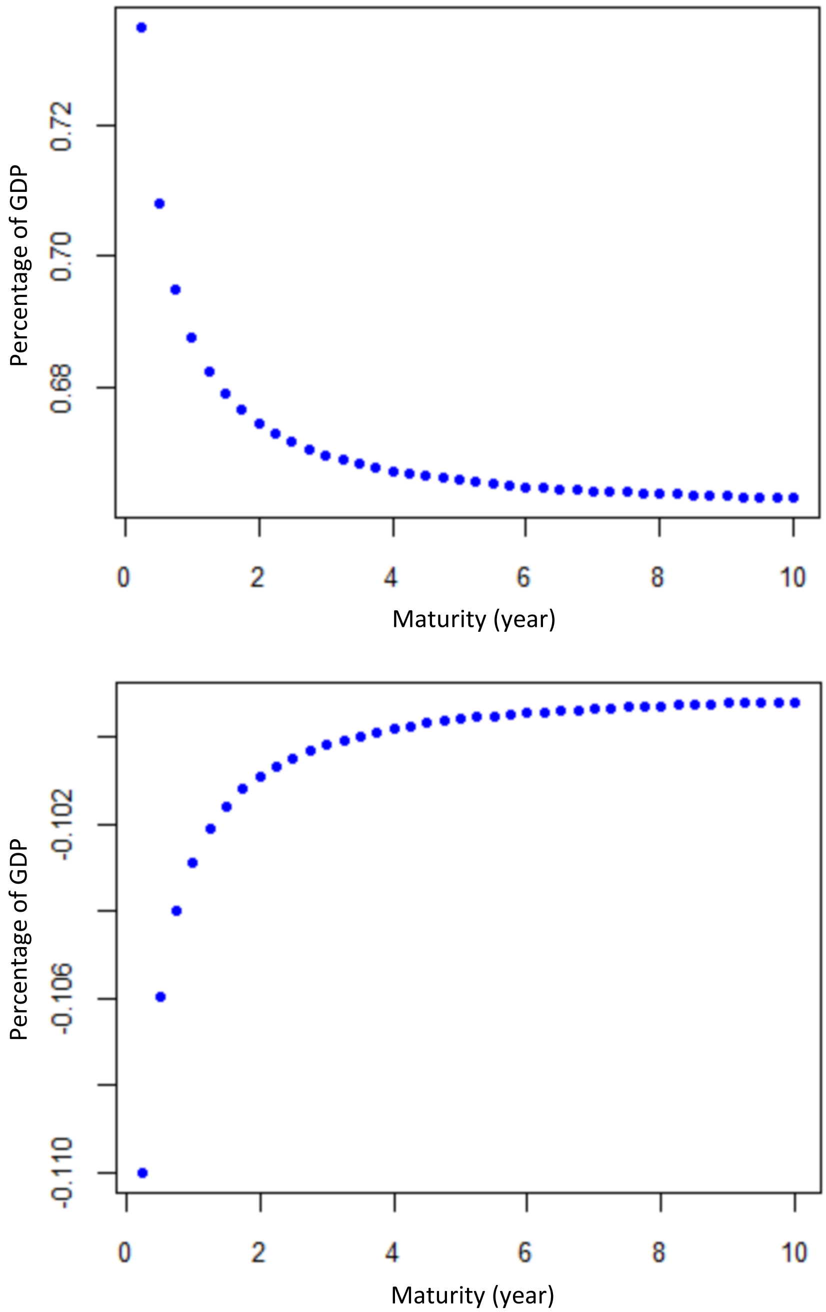
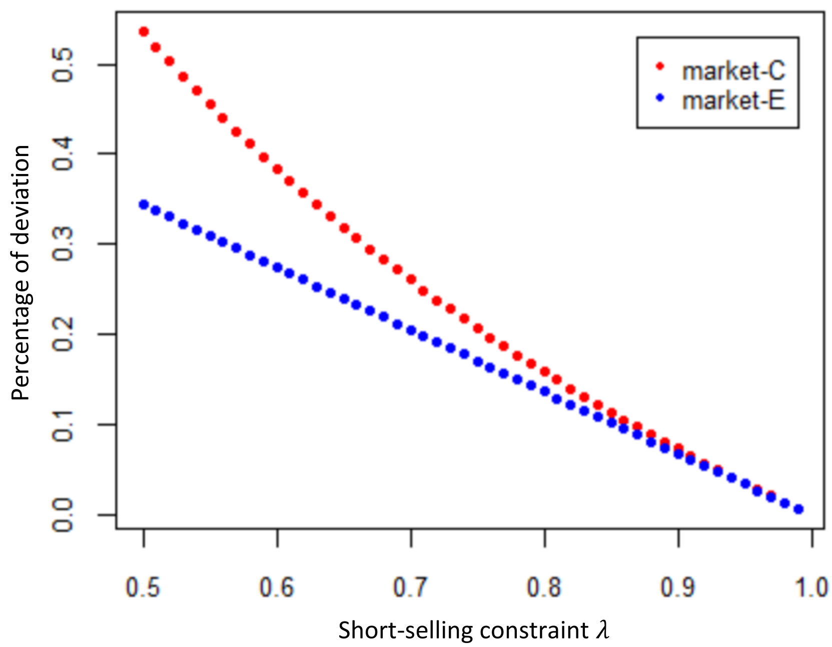


Figure 17. Short-selling constraints and deviation from full integration

This figure plots the mean values of the percentage of deviations from full integration when the *CMA* cannot freely short-sell. The horizontal axis denotes the percentages of the curve of the full integration case in Figure 15 that the *CMA* can short-sell. Percentages that can be short-sold up to are on the horizontal axis. I obtain the market transaction data from the China Foreign Exchange Trade System, ranging from January 2007 to December 2017.



Appendix

Proof of Lemma 1. (Semi-form, WMA.C)

Substitute into the budget constraint of the following expression

$$\frac{dP^C(r_t,\tau)}{P^C(r_{t-},\tau)} = \left[A_r^{C\prime} r_t + C^{C\prime} - \mu_r(r_{t-})A_r^C + \frac{1}{2}\big(\sigma_r^2 + \mathrm{Var}(\Delta r)\big){A_r^C}^2\right]dt - A_r^C\big(\sigma_r dW_{r,t} + \Delta r dJ_t\big)$$

The *WMA.C*'s optimization problem is tantamount to maximize

$$\aleph^C \equiv W_t r_t - r_t\int_0^T Z_{t,\tau}^C d\tau + \int_0^T Z_{t,\tau}^C\left(A_r^{C\prime} r_t + C^{C\prime} - \mu_r(r_{t-})A_r^C + \frac{1}{2}\big(\sigma_r^2 + \mathrm{Var}(\Delta r)\big){A_r^C}^2\right)d\tau$$
$$-\frac{1}{2}\gamma^C\big(\sigma_r^2 + \mathrm{Var}(\Delta r)\big)\int_0^T {Z_{t,\tau}^C}^2 {A_r^C}^2 d\tau$$

Taking the first order derivative w.r.t. *WMA.C*'s allocation $Z_{t,\tau}^C$ to obtain the first order conditions

$$0 = \frac{\partial\aleph^C}{\partial Z_{t,\tau}^C} = -r_t + A_r^{C\prime} r_t + C^{C\prime} - [\kappa_r(\bar{r} - r_t) + \lambda_r E(\Delta r_t)]A_r^C + \frac{1}{2}[\sigma_r^2 + \mathrm{Var}(\Delta r)]{A_r^C}^2$$
$$-\gamma^C[\sigma_r^2 + \mathrm{Var}(\Delta r)]A_r^C\int_0^T \big[S_{t,\tau}^C - \alpha^C(A_r^C r_t + C^C) + d^C\big]A_r^C d\tau$$

Collect terms of the short rate factor r_t and the constant term to obtain the system of two first order ordinary differential equations in (3.2.1) and (3.2.2).

Proof of Proposition 1. (Semi-form, WMA.C)

Rewrite (3.2.1) into

$$A_r^{C\prime} + \left\{\kappa_r + \gamma^C[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^T \alpha^C {A_r^C}^2 d\tau\right\}A_r^C - 1 = 0$$

Set $\kappa_r^C \equiv \kappa_r + \gamma^C[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^T \alpha^C {A_r^C}^2 d\tau$ to obtain the solution

$$A_r^C(\tau) = \frac{1 - e^{-\kappa_r^C\tau}}{\kappa_r^C}$$

Rewrite (3.2.2) into

$$C^{C\prime} - z_r^C A_r^C + \frac{1}{2}[\sigma_r^2 + \mathrm{Var}(\Delta r)]{A_r^C}^2 = 0$$

where

$$z_r^C \equiv [\kappa_r\bar{r} + \lambda_r E(\Delta r_t)] + \gamma^C[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^T A_r^C\big[S_{t,\tau}^C - \alpha^C C^C + d^C\big]d\tau$$

Integrate both sides and rearrange to obtain

$$C^C(\tau) = z_r^C\int_0^\tau A_r^C(s)ds - \frac{1}{2}[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^\tau A_r^C(s)^2 ds$$

Substitute the above expression of $C^C(\tau)$ into (3.2.2) to obtain an equation of z_r^C only. Collect terms to obtain an expression of z_r^C:

$$z_r^C = \frac{1}{1+\gamma^C[\sigma_r^2+\mathrm{Var}(\Delta r)]\int_0^T \alpha^C\left[\int_0^\tau A_r^C(s)ds\right]A_r^C d\tau}$$
$$\times\left\{\left(\kappa_r+\gamma^C[\sigma_r^2+\mathrm{Var}(\Delta r)]\int_0^T \tau\alpha^C A_r^C d\tau\right)\bar{r}+\lambda_r E(\Delta r_t)\right.$$
$$+\gamma^C[\sigma_r^2+\mathrm{Var}(\Delta r)]\int_0^T\left(S_{t,\tau}^C+d^C-\tau\alpha^C\bar{r}\right)A_r^C d\tau$$
$$\left.+\frac{1}{2}\gamma^C[\sigma_r^2+\mathrm{Var}(\Delta r)]^2\int_0^T\alpha^C\left[\int_0^\tau A_r^C(s)^2 ds\right]A_r^C d\tau\right\}$$

Define $z_r^C=\kappa_r^C\bar{r}^C$. Substitute into the above equation to obtain and expression for the risk-neutral long-run mean $\bar{r}^C$:

$$\bar{r}^C=\bar{r}+\frac{1}{\kappa_r^C\left\{1+\gamma^C[\sigma_r^2+\mathrm{Var}(\Delta r)]\int_0^T\alpha^C\left[\int_0^\tau A_r^C(s)ds\right]A_r^C d\tau\right\}}$$
$$\times\left\{\lambda_r E(\Delta r_t)+\gamma^C[\sigma_r^2+\mathrm{Var}(\Delta r)]\int_0^T\left(S_{t,\tau}^C+d^C-\tau\alpha^C\bar{r}\right)A_r^C d\tau\right.$$
$$\left.+\frac{1}{2}\gamma^C[\sigma_r^2+\mathrm{Var}(\Delta r)]^2\int_0^T\alpha^C\left[\int_0^\tau A_r^C(s)^2 ds\right]A_r^C d\tau\right\}$$

The parameters $\{\kappa_r^C,\bar{r}^C\}$ characterizes the dynamics of the short rate under the risk-neutral measure: $dr_t=\kappa_r^C(\bar{r}^C-r_t)dt+\sigma_r dW_{r,t}^C$.

Proof of Lemma 2. (Semi-form, WMA.E)

Substitute into the budget constraint of the following expression

$$\frac{dP^E(r_t,\tau)}{P^E(r_{t-},\tau)}=\left[A_r^{E\prime}r_t+A_\beta^{E\prime}\beta_t+C^{E\prime}-\mu_r(r_{t-})A_r^E+\frac{1}{2}\left(\sigma_r^2+\mathrm{Var}(\Delta r)\right)A_r^{E^2}-\kappa_\beta\beta_t A_\beta^E+\frac{1}{2}\sigma_\beta^2 A_\beta^{E^2}\right]dt$$
$$-A_r^E\left(\sigma_r dW_{r,t}+\Delta r dJ_t\right)-A_\beta^E\sigma_\beta W_{\beta,t}$$

The *WMA.E*'s optimization problem is tantamount to maximize

$$\aleph^E\equiv W_t r_t-(r_t+\beta_t)\int_0^T Z_{t,\tau}^E d\tau$$
$$+\int_0^T Z_{t,\tau}^E\left(A_r^{E\prime}r_t+A_\beta^{E\prime}\beta_t+C^{E\prime}-\mu_r(r_{t-})A_r^E+\frac{1}{2}\left(\sigma_r^2+\mathrm{Var}(\Delta r)\right)A_r^{E^2}-\kappa_\beta\beta_t A_\beta^E+\frac{1}{2}\sigma_\beta^2A_\beta^{E^2}\right)d\tau$$
$$-\frac{1}{2}\gamma^E\left(\sigma_r^2+\mathrm{Var}(\Delta r)\right)\int_0^T Z_{t,\tau}^{E^2}A_r^{E^2}d\tau-\frac{1}{2}\gamma^E\sigma_\beta^2\int_0^T Z_{t,\tau}^{E^2}A_\beta^{E^2}d\tau$$

Taking the first order derivative w.r.t. *WMA.E*'s allocation $Z_{t,\tau}^E$ to obtain the FOC:

$$0=\frac{\partial\aleph^E}{\partial Z_{t,\tau}^E}=-(r_t+\beta_t)+A_r^{E\prime}r_t+A_\beta^{E\prime}\beta_t+C^{E\prime}-[\kappa_r(\bar{r}-r_t)+\lambda_r E(\Delta r_t)]A_r^E+\frac{1}{2}[\sigma_r^2+\mathrm{Var}(\Delta r)]A_r^{E^2}-\kappa_\beta\beta_t A_\beta^E$$
$$+\frac{1}{2}\sigma_\beta^2A_\beta^{E^2}-\gamma^E\sigma_\beta^2A_\beta^E\int_0^T\left[S_{t,\tau}^E-\alpha^E\left(A_r^E r_t+A_\beta^E\beta_t+C^E\right)+d^E+\theta\beta_t\right]A_\beta^E d\tau$$
$$-\gamma^E[\sigma_r^2+\mathrm{Var}(\Delta r)]A_r^E\int_0^T\left[S_{t,\tau}^E-\alpha^E\left(A_r^E r_t+A_\beta^E\beta_t+C^E\right)+d^E+\theta\beta_t\right]A_r^E d\tau$$

Collect terms of the short rate factor r_t, of the net demand factor β_t, and of the constant term to obtain the system of three first order equations in (3.2.3), (3.2.4) and (3.2.5).

Proof of Proposition 2. (Semi-form, WMA.E)

Rewrite (3.2.3) and (3.2.4) into a nonhomogeneous system of differential equations

$$\boldsymbol{X}' = \boldsymbol{A}\boldsymbol{X} + \boldsymbol{g}$$

where

$$\boldsymbol{X} \equiv \begin{bmatrix} A_r^E \\ A_\beta^E \end{bmatrix}, \boldsymbol{A} = -\begin{bmatrix} k_r + \gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^T \alpha^E {A_r^E}^2 d\tau & \gamma^E \sigma_\beta^2 \int_0^T \alpha^E A_\beta^E A_r^E d\tau \\ \gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^T A_r^E\left(\alpha^E A_\beta^E - \theta\right) d\tau & \kappa_\beta + \gamma^E \sigma_\beta^2 \int_0^T A_\beta^E\left(\alpha^E A_\beta^E - \theta\right) d\tau \end{bmatrix} \text{ and } \boldsymbol{g} \equiv \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Rewrite (3.2.5) into

$$C^{E'} - z_r^E A_r^E - z_\beta^E A_\beta^E + \frac{1}{2}[\sigma_r^2 + \mathrm{Var}(\Delta r)]{A_r^E}^2 + \frac{1}{2}\sigma_\beta^2 {A_\beta^E}^2 = 0$$

where

$$z_r^E \equiv [\kappa_r \bar{r} + \lambda_r E(\Delta r_t)] + \gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^T \left[S_{t,\tau}^E - \alpha^E C^E + d^E\right] A_r^E d\tau$$

$$z_\beta^E \equiv \gamma^E \sigma_\beta^2 \int_0^T \left[S_{t,\tau}^E - \alpha^E C^E + d^E\right] A_\beta^E d\tau$$

Integrate both sides and rearrange to obtain

$$C^E(\tau) = z_r^E \int_0^\tau A_r^E(s) ds + z_\beta^E \int_0^\tau A_\beta^E(s) ds - \frac{1}{2}[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^\tau A_r^E(s)^2 ds - \frac{1}{2}\sigma_\beta^2 \int_0^\tau A_\beta^E(s)^2 ds$$

Substitute the above expression of $C^E(\tau)$ into (3.2.5) to obtain an equation of z_r^E and z_β^E. Collect terms to obtain expressions of z_r^E and z_β^E:

$$\begin{aligned} z_r^E = &\frac{1}{1 + \gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^T \alpha^E\left[\int_0^\tau A_r^E ds\right] A_r^E d\tau} \\ &\times\left\{\left(\kappa_r + \gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^T \tau\alpha^E A_r^E d\tau\right)\bar{r} + \lambda_r E(\Delta r_t)\right. \\ &+ \gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^T \left(S_{t,\tau}^E + d^E - \tau\alpha^E \bar{r}\right) A_r^E d\tau \\ &\left.+ \frac{1}{2}\gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)]^2 \int_0^T \alpha^E\left[\int_0^\tau A_r^E(s)^2 ds\right] A_r^E d\tau\right\} \end{aligned}$$

$$z_\beta^E = \frac{1}{1 + \gamma^E \sigma_\beta^2 \int_0^T \alpha^E\left[\int_0^\tau A_\beta^E ds\right] A_\beta^E d\tau} \times \left\{\gamma^E \sigma_\beta^2 \int_0^T \left(S_{t,\tau}^E + d^E\right) A_\beta^E d\tau + \frac{1}{2}\gamma^E \sigma_\beta^2 \int_0^T \alpha^E\left[\int_0^\tau A_\beta^E(s)^2 ds\right] A_\beta^E d\tau\right\}$$

Define $z_r^E = \kappa_r^E \bar{r}^E$ and $z_\beta^E = \kappa_\beta^E \bar{\beta}^E$. Substitute into the above equations to obtain expressions for the risk-neutral long-run means $\bar{r}^E$ and $\bar{\beta}^E$:

$$\bar{r}^E = \bar{r} + \frac{1}{\kappa_r^E \left\{1 + \gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T \alpha^E \left[\int_0^\tau A_r^E ds\right] A_r^E d\tau\right\}}$$
$$\times \left\{\lambda_r E(\Delta r_t) + \gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T \left(S_{t,\tau}^E + d^E - \tau\alpha^E \bar{r}\right) A_r^E d\tau\right.$$
$$\left. + \frac{1}{2}\gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T \alpha^E \left[\int_0^\tau A_r^E(s)^2 ds\right] A_r^E d\tau\right\}$$
$$\bar{\beta}^E = \frac{1}{\kappa_\beta^E \left\{1 + \gamma^E \sigma_\beta^2 \int_0^T \alpha^E \left[\int_0^\tau A_\beta^E ds\right] A_\beta^E d\tau\right\}} \times \left\{\gamma^E \sigma_\beta^2 \int_0^T \left(S_{t,\tau}^E + d^E\right) A_\beta^E d\tau + \frac{1}{2}\gamma^E \sigma_\beta^2 \int_0^T \alpha^E \left[\int_0^\tau A_\beta^E(s)^2 ds\right] A_\beta^E d\tau\right\}$$

The parameters $\{\kappa_r^E, \bar{r}^E\}$ and $\{\kappa_\beta^E, \bar{\beta}^E\}$ characterize the dynamics of the short rate factor and the net demand factor under the risk-neutral measure in market-*E*:

$$dr_t = \kappa_r^E(\bar{r}^E - r_t)dt + \sigma_r dW_{r,t}^E$$
$$d\beta_t = \kappa_\beta^E\left(\bar{\beta}^E - \beta_t\right)dt + \sigma_\beta dW_{\beta,t}^E$$

To derive the functions $A_r^E(\tau)$ *and* $A_\beta^E(\tau)$, substitute the conjectured functional forms into (3.2.3) and (3.2.4). Since these two equations hold for all τ, the coefficients of r_t and β_t and the constant term have to be zeros.

$$(3.2.3)\begin{cases} \psi_r - \dfrac{\kappa_r \psi_r}{\kappa_\beta^E} + \dfrac{\psi_r}{\kappa_\beta^E} A_{11} + \dfrac{\psi_\beta}{\kappa_\beta^E} A_{12} = 0 \\ y_1\left(\psi_r - \dfrac{\kappa_r \psi_r}{\kappa_r^E} + \dfrac{\psi_r}{\kappa_r^E} A_{11}\right) + \dfrac{\psi_\beta}{\kappa_r^E} A_{12} = 0 \\ 1 - \kappa_r \psi_r \left(\dfrac{1}{\kappa_\beta^E} - \dfrac{y_1}{\kappa_r^E}\right) + \psi_r \left(\dfrac{1}{\kappa_\beta^E} - \dfrac{y_1}{\kappa_r^E}\right) A_{11} + \psi_r \left(\dfrac{1}{\kappa_\beta^E} - \dfrac{y_2}{\kappa_r^E}\right) A_{12} = 0 \end{cases}$$

$$(3.2.4)\begin{cases} \psi_\beta - \dfrac{\kappa_\beta \psi_\beta}{\kappa_\beta^E} + \dfrac{\psi_r}{\kappa_\beta^E} A_{21} + \dfrac{\psi_\beta}{\kappa_\beta^E} A_{22} = 0 \\ y_2\left(\psi_\beta - \dfrac{\kappa_\beta \psi_\beta}{\kappa_r^E} + \dfrac{\psi_\beta}{\kappa_r^E} A_{22}\right) + y_1 \dfrac{\psi_r}{\kappa_r^E} A_{21} = 0 \\ 1 - \kappa_\beta \psi_\beta \left(\dfrac{1}{\kappa_\beta^E} - \dfrac{y_1}{\kappa_r^E}\right) + \psi_r \left(\dfrac{1}{\kappa_\beta^E} - \dfrac{y_1}{\kappa_r^E}\right) A_{21} + \psi_\beta \left(\dfrac{1}{\kappa_\beta^E} - \dfrac{y_2}{\kappa_r^E}\right) A_{22} = 0 \end{cases}$$

Solving the first two equations corresponding to (3.2.3) to get

$$A_{11} = \frac{y_2 \kappa_\beta^E - y_1 \kappa_r^E}{y_1 - y_2} + \kappa_r \quad \& \quad A_{12} = \frac{y_1\left(\kappa_r^E - \kappa_\beta^E\right)}{y_1 - y_2}\left(\frac{\psi_r}{\psi_\beta}\right)$$

Substitute these expressions into the last equation corresponding to (3.2.3) to get $\psi_r(y_1 - 1) = 1$.

Solving the first two equations corresponding to (3.2.4) to get

$$A_{21} = \frac{y_2\left(\kappa_\beta^E - \kappa_r^E\right)}{y_1 - y_2}\left(\frac{\psi_\beta}{\psi_r}\right) \quad \& \quad A_{22} = \frac{y_2 \kappa_r^E - y_1 \kappa_\beta^E}{y_1 - y_2} + \kappa_\beta$$

Substitute these expressions into the last equation corresponding to (3.2.4) to get $\psi_\beta(y_2 - 1) = 1$.

Proof of Lemma 3. (Weak-form, CMA)

Substitute into the budget constraint of the following expression

$$\frac{dP^{C,A}(r_t,\tau)}{P^{C,A}(r_{t-},\tau)} = \left[A_r^{C,A'} r_t + C^{C,A'} - \mu_r(r_{t-})A_r^{C,A} + \frac{1}{2}\left(\sigma_r^2 + \mathrm{Var}(\Delta r)\right){A_r^{C,A}}^2\right]dt - A_r^{C,A}\left(\sigma_r dW_{r,t} + \Delta r dJ_t\right)$$

$$\frac{dP^{E,A}(r_t,\tau)}{P^{E,A}(r_{t-},\tau)} = \left[A_r^{E,A'} r_t + A_\beta^{E,A'}\beta_t + C^{E,A'} - \mu_r(r_{t-})A_r^{E,A} + \frac{1}{2}\left(\sigma_r^2 + \mathrm{Var}(\Delta r)\right){A_r^{E,A}}^2 - \kappa_\beta \beta_t A_\beta^{E,A} + \frac{1}{2}\sigma_\beta^2 {A_\beta^{E,A}}^2\right]dt$$
$$- A_r^{E,A}\left(\sigma_r dW_{r,t} + \Delta r dJ_t\right) - A_\beta^{E,A}\sigma_\beta W_{\beta,t}$$

The market-E CMA's optimization problem is tantamount to maximize

$$\begin{aligned}
\aleph^A \equiv\ & W_t r_t - r_t\int_0^T X_{t,\tau}^C d\tau - (r_t+\beta_t)\int_0^T X_{t,\tau}^E d\tau \\
& + \int_0^T X_{t,\tau}^C\left[A_r^{C,A'} r_t + C^{C,A'} - \mu_r(r_{t-})A_r^{C,A} + \frac{1}{2}\left(\sigma_r^2 + \mathrm{Var}(\Delta r)\right){A_r^{C,A}}^2\right]d\tau \\
& + \int_0^T X_{t,\tau}^E\left[A_r^{E,A'} r_t + A_\beta^{E,A'}\beta_t + C^{E,A'} - \mu_r(r_{t-})A_r^{E,A} + \frac{1}{2}\left(\sigma_r^2 + \mathrm{Var}(\Delta r)\right){A_r^{E,A}}^2 - \kappa_\beta\beta_t A_\beta^{E,A}\right. \\
& \left. + \frac{1}{2}\sigma_\beta^2{A_\beta^{E,A}}^2\right]d\tau - \frac{1}{2}\gamma^A[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^T {X_{t,\tau}^C}^2{A_r^{C,A}}^2 d\tau - \frac{1}{2}\gamma^A[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^T {X_{t,\tau}^E}^2{A_r^{E,A}}^2 d\tau \\
& - \frac{1}{2}\gamma^A\sigma_\beta^2\int_0^T {X_{t,\tau}^E}^2{A_\beta^{E,A}}^2 d\tau - \gamma^A[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^T X_{t,\tau}^E A_r^{E,A}d\tau\int_0^T X_{t,\tau}^C A_r^{C,A}d\tau
\end{aligned}$$

Taking the first order derivative w.r.t. the CMA's allocations $X_{t,\tau}^C$ and $X_{t,\tau}^E$ to obtain the FOCs:

$$\begin{aligned}
0 = \frac{\partial\aleph^A}{\partial X_{t,\tau}^C} =\ & -r_t + A_r^{C,A'}r_t + C^{C,A'} - [\kappa_r(\bar{r}-r_t) + \lambda_r E(\Delta r_t)]A_r^{C,A} + \frac{1}{2}\left(\sigma_r^2 + \mathrm{Var}(\Delta r)\right){A_r^{C,A}}^2 \\
& - \gamma^A[\sigma_r^2 + \mathrm{Var}(\Delta r)]A_r^{C,A}\left(\beta_t^C\int_0^T A_r^{C,A}F^C d\tau + \beta_t^E\int_0^T A_r^{E,A}F^E d\tau\right) \\
0 = \frac{\partial\aleph^A}{\partial X_{t,\tau}^E} =\ & -(r_t+\beta_t) + A_r^{E,A'}r_t + A_\beta^{E,A'}\beta_t + C^{E,A'} - [\kappa_r(\bar{r}-r_t) + \lambda_r E(\Delta r_t)]A_r^{E,A} + \frac{1}{2}\left(\sigma_r^2 + \mathrm{Var}(\Delta r)\right){A_r^{E,A}}^2 \\
& - \kappa_\beta\beta_t A_\beta^{E,A} + \frac{1}{2}\sigma_\beta^2{A_\beta^{E,A}}^2 - \gamma^A[\sigma_r^2 + \mathrm{Var}(\Delta r)]A_r^{E,A}\left(\int_0^T X_{t,\tau}^E A_r^{E,A}d\tau + \int_0^T X_{t,\tau}^C A_r^{C,A}d\tau\right) \\
& + \gamma^A\sigma_\beta^2 A_\beta^{E,A}\left(\int_0^T X_{t,\tau}^E A_\beta^{E,A}d\tau\right)
\end{aligned}$$

These two equations correspond to (3.3.1) and (3.3.2).

Proof of Lemma 4. (Weak-form, WMA.C)

Substitute into the budget constraint of the following expression

$$\frac{dP^{C,A}(r_t,\tau)}{P^{C,A}(r_{t-},\tau)} = \left[A_r^{C,A'} r_t + C^{C,A'} - \mu_r(r_{t-})A_r^{C,A} + \frac{1}{2}\left(\sigma_r^2 + \mathrm{Var}(\Delta r)\right){A_r^{C,A}}^2\right]dt - A_r^{C,A}\left(\sigma_r dW_{r,t} + \Delta r dJ_t\right)$$

The *CMA*'s optimization problem is tantamount to maximize

$$\begin{aligned}
\aleph^{C,A} \equiv\ & W_t r_t - r_t\int_0^T Z_{t,\tau}^{C,A}d\tau + \int_0^T Z_{t,\tau}^{C,A}\left[A_r^{C,A'}r_t + C^{C,A'} - \mu_r(r_{t-})A_r^{C,A} + \frac{1}{2}\left(\sigma_r^2 + \mathrm{Var}(\Delta r)\right){A_r^{C,A}}^2\right]d\tau \\
& - \frac{1}{2}\gamma^C[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^T {Z_{t,\tau}^{C,A}}^2{A_r^{C,A}}^2 d\tau
\end{aligned}$$

Taking the first order derivative w.r.t. *WMA.C*'s allocation $Z_{t,\tau}^{C,A}$ to obtain the first order conditions

$$0=\frac{\partial ℵ^{C,A}}{\partial X_{t,\tau}^{C}}=-r_t+A_r^{C,A\prime}r_t+C^{C,A\prime}-\mu_r(r_{t-})A_r^{C,A}+\frac{1}{2}\left(\sigma_r^2+\mathrm{Var}(\Delta r)\right)A_r^{C,A^2}$$
$$-\gamma^C[\sigma_r^2+\mathrm{Var}(\Delta r)]A_r^{C,A}\int_0^T A_r^{C,A}\left[S_{t,\tau}^C-F^C\beta_t^C-\alpha^C\left(A_r^{C,A}r_t+C^{C,A}\right)+d^C\right]d\tau$$

Collect terms of the short rate factor r_t and of the constant term $C^{C,A}$ to obtain the system of two first order equations in (3.3.3) and (3.3.4).

Proof of Proposition 4. (Weak-form, WMA.C)

Rewrite (3.3.3) into

$$A_r^{C,A\prime}+\left\{\kappa_r+\gamma^C[\sigma_r^2+\mathrm{Var}(\Delta r)]\int_0^T\alpha^C A_r^{C,A^2}d\tau\right\}A_r^{C,A}-1=0$$

Set $\kappa_r^{C,A}\equiv\kappa_r+\gamma^C[\sigma_r^2+\mathrm{Var}(\Delta r)]\int_0^T\alpha^C A_r^{C,A^2}d\tau$ to obtain the solution

$$A_r^{C,A}(\tau)=\frac{1-e^{-\kappa_r^{C,A}\tau}}{\kappa_r^{C,A}}$$

Rewrite (3.3.4) into

$$C^{C,A\prime}-z_r^{C,A}A_r^{C,A}+\frac{1}{2}[\sigma_r^2+\mathrm{Var}(\Delta r)]A_r^{C,A^2}=0$$

where

$$z_r^{C,A}\equiv[\kappa_r\bar{r}+\lambda_r E(\Delta r_t)]+\gamma^C[\sigma_r^2+\mathrm{Var}(\Delta r)]\int_0^T A_r^{C,A}\left[S_{t,\tau}^C-F^C\beta_t^C-\alpha^C C^{C,A}+d^C\right]d\tau$$

Integrate both sides and rearrange to obtain

$$C^{C,A}(\tau)=z_r^{C,A}\int_0^\tau A_r^{C,A}(s)ds-\frac{1}{2}[\sigma_r^2+\mathrm{Var}(\Delta r)]\int_0^\tau A_r^{C,A}(s)^2ds$$

Substitute the above expression of $C^{C,A}(\tau)$ into (3.2.2) to obtain an equation of $z_r^{C,A}$ only. Collect terms to obtain an expression of $z_r^{C,A}$:

$$z_r^{C,A}=\frac{1}{1+\gamma^C[\sigma_r^2+\mathrm{Var}(\Delta r)]\int_0^T\alpha^C\left[\int_0^\tau A_r^{C,A}(s)ds\right]A_r^{C,A}d\tau}$$
$$\times\left\{\left(\kappa_r+\gamma^C[\sigma_r^2+\mathrm{Var}(\Delta r)]\int_0^T\tau\alpha^C A_r^{C,A}d\tau\right)\bar{r}+\lambda_r E(\Delta r_t)\right.$$
$$+\gamma^C[\sigma_r^2+\mathrm{Var}(\Delta r)]\int_0^T\left(S_{t,\tau}^C-F^C\beta_t^C+d^C-\tau\alpha^C\bar{r}\right)A_r^{C,A}d\tau$$
$$\left.+\frac{1}{2}\gamma^C[\sigma_r^2+\mathrm{Var}(\Delta r)]^2\int_0^T\alpha^C\left[\int_0^\tau A_r^{C,A}(s)^2ds\right]A_r^{C,A}d\tau\right\}$$

Define $z_r^{C,A}=\kappa_r^{C,A}\bar{r}^{C,A}$. Substitute into the above equation to obtain and expression for the risk-neutral long-run mean $\bar{r}^C$:

$$\bar{r}^{C,A} = \bar{r} + \frac{1}{\kappa_r^C \left\{1 + \gamma^C[\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T \alpha^C \left[\int_0^\tau A_r^{C,A}(s)ds\right] A_r^{C,A} d\tau\right\}}$$
$$\times \left\{\lambda_r E(\Delta r_t) + \gamma^C[\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T \left(S_{t,\tau}^C - F^C \beta_t^C + d^C - \tau\alpha^C \bar{r}\right) A_r^{C,A} d\tau \right.$$
$$\left. + \frac{1}{2}\gamma^C[\sigma_r^2 + \mathrm{Var}(\Delta r)]^2 \int_0^T \alpha^C \left[\int_0^\tau A_r^{C,A}(s)^2 ds\right] A_r^{C,A} d\tau \right\}$$

The parameters $\{\kappa_r^{C,A}, \bar{r}^{C,A}\}$ characterizes the dynamics of the short rate under the risk-neutral measure: $dr_t = \kappa_r^{C,A}(\bar{r}^{C,A} - r_t)dt + \sigma_r dW_{r,t}^C$.

Proof of Lemma 5. (Weak-form, WMA.E)

Substitute into the budget constraint of the following expression

$$\frac{dP^{E,A}(r_t,\tau)}{P^{E,A}(r_{t-},\tau)} = \left[A_r^{E,A\prime} r_t + A_\beta^{E,A\prime} \beta_t + C^{E,A\prime} - \mu_r(r_{t-}) A_r^{E,A} + \frac{1}{2}\left(\sigma_r^2 + \mathrm{Var}(\Delta r)\right) A_r^{E,A^2} - \kappa_\beta \beta_t A_\beta^{E,A} + \frac{1}{2}\sigma_\beta^2 A_\beta^{E,A^2}\right] dt$$
$$- A_r^{E,A}\left(\sigma_r dW_{r,t} + \Delta r dJ_t\right) - A_\beta^{E,A} \sigma_\beta W_{\beta,t}$$

The *WMA.E*'s optimization problem is tantamount to maximize

$$\aleph^{E,A} \equiv W_t r_t - (r_t + \beta_t) \int_0^T Z_{t,\tau}^{E,A} d\tau$$
$$+ \int_0^T Z_{t,\tau}^{E,A} \left[A_r^{E,A\prime} r_t + A_\beta^{E,A\prime} \beta_t + C^{E,A\prime} - \mu_r(r_{t-}) A_r^{E,A} + \frac{1}{2}\left(\sigma_r^2 + \mathrm{Var}(\Delta r)\right) A_r^{E,A^2} - \kappa_\beta \beta_t A_\beta^{E,A}\right.$$
$$\left. + \frac{1}{2}\sigma_\beta^2 A_\beta^{E,A^2}\right] d\tau - \frac{1}{2}\gamma^E\left(\sigma_r^2 + \mathrm{Var}(\Delta r)\right) \int_0^T Z_{t,\tau}^{E,A^2} A_r^{E,A^2} d\tau - \frac{1}{2}\gamma^E \sigma_\beta^2 \int_0^T Z_{t,\tau}^{E,A^2} A_\beta^{E,A^2} d\tau$$

Taking the first order derivative w.r.t. *WMA.E*'s allocation $Z_{t,\tau}^{E,A}$ to obtain the FOC:

$$0 = \frac{\partial \aleph^{E,A}}{\partial Z_{t,\tau}^{E,A}} = -(r_t + \beta_t) + A_r^{E,A\prime} r_t + A_\beta^{E,A\prime} \beta_t + C^{E,A\prime} - [\kappa_r(\bar{r} - r_t) + \lambda_r E(\Delta r_t)] A_r^{E,A} - \kappa_\beta \beta_t A_\beta^{E,A}$$
$$+ \frac{1}{2}[\sigma_r^2 + \mathrm{Var}(\Delta r)] A_r^{E,A^2} + \frac{1}{2}\sigma_\beta^2 A_\beta^{E,A^2}$$
$$- \gamma^E[\sigma_r^2$$
$$+ \mathrm{Var}(\Delta r)] A_r^{E,A} \int_0^T \left[S_{t,\tau}^E - F^E \beta_t^E - \alpha^E\left(A_r^{E,A} r_t + A_\beta^{E,A} \beta_t + C^{E,A}\right) + d^E + \theta\beta_t\right] A_r^{E,A} d\tau$$
$$- \gamma^E \sigma_\beta^2 A_\beta^{E,A} \int_0^T \left[S_{t,\tau}^E - F^E \beta_t^E - \alpha^E\left(A_r^{E,A} r_t + A_\beta^{E,A} \beta_t + C^{E,A}\right) + d^E + \theta\beta_t\right] A_\beta^{E,A} d\tau$$

Collect terms of the short rate factor r_t, of the net demand factor β_t, and of the constant term $C^{E,A}$ to obtain the system of three first order equations in (3.3.5), (3.3.6) and (3.3.7).

Proof of Proposition 5. (Weak-form, WMA.E)

Rewrite (3.3.5) and (3.3.6) into a nonhomogeneous system of differential equations

$$\boldsymbol{X}' = \boldsymbol{A}\boldsymbol{X} + \boldsymbol{g}$$

where

$$\boldsymbol{X} \equiv \begin{bmatrix} A_r^{E,A} \\ A_\beta^{E,A} \end{bmatrix}, \boldsymbol{A} \equiv -\gamma^E \begin{bmatrix} [\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T \alpha^E A_r^{E,A^2} d\tau & \sigma_\beta^2 \int_0^T \alpha^E A_\beta^{E,A} A_r^{E,A} d\tau \\ [\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T A_r^{E,A} \left(\alpha^E A_\beta^{E,A} - \theta\right) d\tau & \sigma_\beta^2 \int_0^T A_\beta^{E,A} \left(\alpha^E A_\beta^{E,A} - \theta\right) d\tau \end{bmatrix} \text{ and } \boldsymbol{g} \equiv \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Rewrite (3.3.7) into

$$C^{E,A'} - z_r^{E,A} A_r^{E,A} - z_\beta^{E,A} A_\beta^{E,A} + \frac{1}{2}[\sigma_r^2 + \mathrm{Var}(\Delta r)] A_r^{E,A^2} + \frac{1}{2}\sigma_\beta^2 A_\beta^{E,A^2} = 0$$

where

$$z_r^{E,A} \equiv [\kappa_r \bar{r} + \lambda_r E(\Delta r_t)] + \gamma^E [\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T \left[S_{t,\tau}^E - F^E \beta_t^E - \alpha^E C^{E,A} + d^E\right] A_r^{E,A} d\tau$$

$$z_\beta^{E,A} \equiv \gamma^E \sigma_\beta^2 \int_0^T \left[S_{t,\tau}^E - F^E \beta_t^E - \alpha^E C^{E,A} + d^E\right] A_\beta^{E,A} d\tau$$

Integrate both sides and rearrange to obtain

$$C^{E,A}(\tau) = z_r^{E,A} \int_0^\tau A_r^{E,A}(s) ds + z_\beta^{E,A} \int_0^\tau A_\beta^{E,A}(s) ds - \frac{1}{2}[\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^\tau A_r^{E,A}(s)^2 ds - \frac{1}{2}\sigma_\beta^2 \int_0^\tau A_\beta^{E,A}(s)^2 ds$$

Substitute the above expression of $C^{E,A}(\tau)$ into (3.3.7) to obtain an equation of $z_r^{E,A}$ and $z_\beta^{E,A}$. Collect terms to obtain expressions of $z_r^{E,A}$ and $z_\beta^{E,A}$:

$$\begin{aligned} z_r^{E,A} = & \frac{1}{1 + \gamma^E [\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T \alpha^E \left[\int_0^\tau A_r^{E,A} ds\right] A_r^{E,A} d\tau} \\ & \times \Bigg\{ \left(\kappa_r + \gamma^E [\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T \tau \alpha^E A_r^{E,A} d\tau\right) \bar{r} + \lambda_r E(\Delta r_t) \\ & + \gamma^E [\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T \left(S_{t,\tau}^E - F^E \beta_t^E + d^E - \tau \alpha^E \bar{r}\right) A_r^{E,A} d\tau \\ & + \frac{1}{2}\gamma^E [\sigma_r^2 + \mathrm{Var}(\Delta r)]^2 \int_0^T \alpha^E \left[\int_0^\tau A_r^{E,A}(s)^2 ds\right] A_r^{E,A} d\tau \Bigg\} \\ z_\beta^{E,A} = & \frac{1}{1 + \gamma^E \sigma_\beta^2 \int_0^T \alpha^E \left[\int_0^\tau A_\beta^{E,A} ds\right] A_\beta^{E,A} d\tau} \\ & \times \left\{ \gamma^E \sigma_\beta^2 \int_0^T \left(S_{t,\tau}^E - F^E \beta_t^E + d^E\right) A_\beta^{E,A} d\tau + \frac{1}{2}\gamma^E \sigma_\beta^2 \int_0^T \alpha^E \left[\int_0^\tau A_\beta^{E,A}(s)^2 ds\right] A_\beta^{E,A} d\tau \right\} \end{aligned}$$

Define $z_r^{E,A} = \kappa_r^{E,A} \bar{r}^{E,A}$ and $z_\beta^{E,A} = \kappa_\beta^{E,A} \bar{\beta}^{E,A}$. Substitute into the above equations to obtain the risk-neutral long-run means $\bar{r}^{E,A}$ and $\bar{\beta}^{E,A}$:

$$\begin{aligned} \bar{r}^{E,A} = \bar{r} + & \frac{1}{\kappa_r^{E,A} \left\{1 + \gamma^E [\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T \alpha^E \left[\int_0^\tau A_r^{E,A} ds\right] A_r^{E,A} d\tau\right\}} \\ & \times \Bigg\{ \lambda_r E(\Delta r_t) + \gamma^E [\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T \left(S_{t,\tau}^E - F^E \beta_t^E + d^E - \tau \alpha^E \bar{r}\right) A_r^{E,A} d\tau \\ & + \frac{1}{2}\gamma^E [\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T \alpha^E \left[\int_0^\tau A_r^{E,A}(s)^2 ds\right] A_r^{E,A} d\tau \Bigg\} \end{aligned}$$

$$\bar{\beta}^{E,A} = \frac{1}{\kappa_\beta^{E,A}\left\{1 + \gamma^E \sigma_\beta^2 \int_0^T \alpha^E \left[\int_0^\tau A_\beta^{E,A} ds\right] A_\beta^{E,A} d\tau\right\}} \times \left\{\gamma^E \sigma_\beta^2 \int_0^T \left(S_{t,\tau}^E - F^E \beta_t^E + d^E\right) A_\beta^{E,A} d\tau + \frac{1}{2}\gamma^E \sigma_\beta^2 \int_0^T \alpha^E \left[\int_0^\tau A_\beta^{E,A}(s)^2 ds\right] A_\beta^{E,A} d\tau\right\}$$

The parameters $\{\kappa_r^{E,A}, \bar{r}^{E,A}\}$ and $\{\kappa_\beta^{E,A}, \bar{\beta}^{E,A}\}$ characterize the dynamics of the short rate factor and the net demand factor under the risk-neutral measure in market-E:

$$dr_t = \kappa_r^{E,A}(\bar{r}^{E,A} - r_t)dt + \sigma_r dW_{r,t}^E$$

$$d\beta_t = \kappa_\beta^{E,A}\left(\bar{\beta}^{E,A} - \beta_t\right)dt + \sigma_\beta dW_{\beta,t}^E$$

Proof of Proposition 6. (Semi-form, market C, global supply effects)

An exogenous supply shock $\Delta S_{t,\tau^*}^C$ of bonds with maturity τ^* in market C induces a change in z_r^C of

$$\Delta z_r^C = \frac{\gamma^C[\sigma_r^2 + \mathrm{Var}(\Delta r)]\Delta S_{t,\tau^*}^C \int_0^T A_r^C(\tau)d\tau}{1 + \gamma^C[\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T \alpha^C(\tau) A_r^C(\tau)\left[\int_0^\tau A_r^C(s)ds\right]d\tau}$$

$$\Delta C^C(\tau) = \Delta z_r^C \int_0^\tau A_r^C(s)ds = \frac{\gamma^C[\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T A_r^C(\tau)d\tau}{1 + \gamma^C[\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T \alpha^C(\tau) A_r^C(\tau)\left[\int_0^\tau A_r^C(s)ds\right]d\tau} \int_0^\tau A_r^C(s)ds\, \Delta S_{t,\tau^*}^C$$

Since $\tau y_{t,\tau}^C = A_r^C(\tau) r_t + C^C(\tau)$, the change in yield is

$$\Delta y_{t,\tau}^C = \frac{1}{\tau}\Delta C^C(\tau) = \frac{\gamma^C[\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T A_r^C(\tau)d\tau}{1 + \gamma^C[\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T \alpha^C(\tau) A_r^C(\tau)\left[\int_0^\tau A_r^C(s)ds\right]d\tau}\left(\frac{1}{\tau}\int_0^\tau A_r^C(s)ds\right)\Delta S_{t,\tau^*}^C$$
$$\equiv \phi_C(\kappa_r^C, \sigma_r, \gamma^C)\left(\frac{1}{\tau}\int_0^\tau A_r^C(s)ds\right)\Delta S_{t,\tau^*}^C$$

Since $\Delta y_{t,\tau}^C$ is increasing in τ, yields at longer maturities are more affected by the supply shock. Consider two maturities τ_1 and τ_2 with $\tau_1 < \tau_2$, the supply effect is ***global*** as the relative effect across different maturities does not depend on the shock origin τ^*:

$$\frac{\Delta y_{t,\tau_2}^C}{\Delta y_{t,\tau_1}^C} = \frac{\int_0^{\tau_2} A_r^C(s)ds}{\int_0^{\tau_1} A_r^C(s)ds} = \int_{\tau_1}^{\tau_2} A_r^C(s)ds > 1$$

Since $Y_{t,\tau}^C = \alpha^C(\tau)\tau y_{t,\tau}^C - d^C(\tau)$, the clientele investor at maturity τ changes her demand by

$$\Delta Y_{t,\tau}^C = \alpha^C(\tau)\Delta C^C(\tau) = \phi_C(\kappa_r^C, \sigma_r, \gamma^C)\alpha^C\left(\int_0^\tau A_r^C(s)ds\right)\Delta S_{t,\tau^*}^C$$

and the within-market arbitrageurs' change in allocation is

$$\Delta Z_{t,\tau}^C = \begin{cases}\left[1 - \phi_C \alpha^C \int_0^\tau A_r^C(s)ds\right]\Delta S_{t,\tau^*}^C & \tau = \tau^* \\ -\phi_C \alpha^C \int_0^\tau A_r^C(s)ds\, \Delta S_{t,\tau^*}^C & \tau \neq \tau^*\end{cases}$$

Since $\alpha^C(\tau)$, $A_r^C(\tau)$ and $\int_0^\tau A_r^C(s)ds$ are increasing in τ, $\forall\, \tau \in [0, T]$

$$
\begin{aligned}
\int_0^T \alpha^C(\tau)A_r^C(\tau)\left[\int_0^\tau A_r^C(s)ds\right]d\tau &\geq \int_0^T \alpha^C(\tau)A_r^C(\tau)d\tau \times \int_0^T \int_0^\tau A_r^C(s)ds\, d\tau \\
&\geq \int_0^T \alpha^C(\tau)d\tau \times \int_0^T A_r^C(\tau)d\tau \times \int_0^T \int_0^\tau A_r^C(s)ds\, d\tau \\
&\geq \alpha^C(T) \times \int_0^T A_r^C(\tau)d\tau \times \int_0^T A_r^C(s)ds \\
&\geq \alpha^C(\tau) \int_0^T A_r^C(\tau)d\tau \int_0^\tau A_r^C(s)ds
\end{aligned}
$$

This implies that $\phi_C\alpha^C \int_0^\tau A_r^C(s)ds \in [0,1)$ for all $0 \leq \tau \leq T$. Thus, a positive exogenous supply shock at $\tau = \tau^*$ leads to a higher demand of the clientele investor at any maturity τ. It also induces a lower demand of the within-market arbitrageur at any maturity τ, except at $\tau = \tau^*$.

Proof of Proposition 7. (Semi-form, market C, global demand effects)

An exogenous demand shock $\Delta d^C(\tau)$ at maturity τ^* in market C induces a change in z_r^C of

$$
\Delta z_r^C = \frac{\gamma^C[\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T \Delta d^C(\tau)A_r^C(\tau)d\tau}{1 + \gamma^C[\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T \alpha^C(\tau)A_r^C(\tau)\left[\int_0^\tau A_r^C(s)ds\right]d\tau}
$$

$$
\Delta C^C(\tau) = \Delta z_r^C \int_0^\tau A_r^C(s)ds = \frac{\gamma^C[\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T \Delta d^C(\tau)A_r^C(\tau)d\tau}{1 + \gamma^C[\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T \alpha^C(\tau)A_r^C(\tau)\left[\int_0^\tau A_r^C(s)ds\right]d\tau} \int_0^\tau A_r^C(s)ds
$$

Since $\tau y_{t,\tau}^C = A_r^C(\tau)r_t + C^C(\tau)$, the change in yield is

$$
\begin{aligned}
\Delta y_{t,\tau}^C &= \frac{\gamma^C[\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T \Delta d^C(\tau)A_r^C(\tau)d\tau}{1 + \gamma^C[\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T \alpha^C(\tau)A_r^C(\tau)\left[\int_0^\tau A_r^C(s)ds\right]d\tau} \left(\frac{1}{\tau}\int_0^\tau A_r^C(s)ds\right) \\
&\equiv \phi_r^C(\Delta d^C; \kappa_r^C, \sigma_r, \gamma^C) \left(\frac{1}{\tau}\int_0^\tau A_r^C(s)ds\right)
\end{aligned}
$$

Now I examine how the sign of $\Delta d^C(\tau)$ affects changes in yields.

<u>Case 1</u>: If $\Delta d^C(\tau) > 0$ (< 0) for all τ, it leads to a decrease (an increase) in $Y_{t,\tau}^C$ at all maturities and indicates a negative (positive) demand shock. Since $\int_0^T \Delta d^C(\tau)A_r^C(\tau)d\tau > 0$ (< 0) for $\Delta d^C(\tau) > 0$ (< 0), a negative (positive) demand shock *always* leads to an increase (a decrease) in yield.

<u>Case 2</u>: If $\Delta d^C(\tau_1) > 0$ and $\Delta d^C(\tau_2) < 0$ for some $0 < \tau_1, \tau_2 < T$, $\int_0^T \Delta d^C(\tau)A_r^C(\tau)d\tau$ could be either positive or negative. Suppose $\int_0^T \Delta d^C(\tau)A_r^C(\tau)d\tau > 0$ (< 0), the yield $y_{t,\tau}^C$ increases (decreases) at all maturities, even for those maturity clienteles with $\Delta d^C(\tau) < 0$ (< 0).

Since $\Delta y_{t,\tau}^C$ is increasing in τ, yields with longer maturities are more affected by the demand shock. Moreover, yields of longer maturities are more affected as $\Delta y_{t,\tau}^C$ is increasing in τ. Consider two maturities τ_1 and τ_2 with $\tau_1 < \tau_2$, the demand effects are ***global*** as the relative effect across different maturities does not depend on the origin τ^*:

$$\frac{\Delta y_{t,\tau_2}^C\left(\Delta d^C(\tau_2)\right)}{\Delta y_{t,\tau_1}^C\left(\Delta d^C(\tau_1)\right)} = \frac{\int_0^{\tau_2} A_r^C(s)ds}{\int_0^{\tau_1} A_r^C(s)ds} = \int_{\tau_1}^{\tau_2} A_r^C(s)ds > 1$$

Since $Y_{t,\tau}^C = \alpha^C(\tau)\tau y_{t,\tau}^C - d^C(\tau)$, the clientele investor at of maturity τ changes her demand by

$$\Delta Y_{t,\tau}^C = \alpha^C(\tau)\Delta C^C(\tau) - \Delta d^C(\tau) = \phi(\Delta d^C; \kappa_r^C, \sigma_r, \gamma^C)\alpha^C \int_0^\tau A_r^C(s)ds - \Delta d^C(\tau)$$

and the within-market arbitrageur *WMA.C*'s change in allocation is

$$\Delta Z_{t,\tau}^C = -\Delta Y_{t,\tau}^C$$

Proof of Proposition 8. (Semi-form, market C, jumps in the short rate)

Taking the first order derivatives of $Y_{t,\tau}^C$ and $Z_{t,\tau}^C$ w.r.t. $E(\Delta r)$ to obtain

$$\frac{\partial Y_{t,\tau}^C}{\partial E(\Delta r)} = \alpha^C(\tau)\frac{\partial C^C(\tau)}{\partial E(\Delta r)} = \frac{\lambda_r \alpha^C(\tau)\int_0^\tau A_r^C(s)ds}{1+\gamma^C[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^T \alpha^C(\tau)A_r^C(\tau)\left[\int_0^\tau A_r^C(s)ds\right]d\tau} > 0$$

$$\frac{\partial Z_{t,\tau}^C}{\partial E(\Delta r)} = -\frac{\partial Y_{t,\tau}^C}{\partial E(\Delta r)} = -\frac{\lambda_r \alpha^C(\tau)\int_0^\tau A_r^C(s)ds}{1+\gamma^C[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^T \alpha^C(\tau)A_r^C(\tau)\left[\int_0^\tau A_r^C(s)ds\right]d\tau} < 0$$

Because $\int_0^\tau A_r^C(s)ds$ is increasing in τ, $\partial Y_{t,\tau}^C/\partial E(\Delta r)$ is, so yields of longer-maturity bonds are more sensitive to expected jumps in the short rate. Note also that $\partial Z_{t,\tau}^C/\partial E(\Delta r)$ is decreasing in magnitude in γ^C, indicating that equilibrium allocations of more risk-averse within-market arbitrageurs are less sensitive to the uncertainty due to jumps in the short rate. This complies with the intuition of CARA utilities that more risk-averse investors adjust less to obtain the same change in utility than do less risk-averse investors.

Proof of Proposition 9. (Semi-form, market E, global supply effects)

An exogenous supply shock $\Delta S_{t,\tau^*}^E$ of bonds with maturity τ^* in market E induces a change in z_r^E of

$$\Delta z_r^E = \frac{\gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)]\Delta S_{t,\tau^*}^E \int_0^T A_r^E d\tau}{1+\gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^T \alpha^E\left[\int_0^\tau A_r^E ds\right]A_r^E d\tau}$$

$$\Delta z_\beta^E = \frac{\gamma^E \sigma_\beta^2 \Delta S_{t,\tau^*}^E \int_0^T A_\beta^E d\tau}{1+\gamma^E\sigma_\beta^2 \int_0^T \alpha^E\left[\int_0^\tau A_\beta^E ds\right]A_\beta^E d\tau}$$

$$\begin{aligned}\Delta C^E(\tau) &= \Delta z_r^E \int_0^\tau A_r^E(s)ds + \Delta z_\beta^E \int_0^\tau A_\beta^E(s)ds \\ &= \frac{\gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)]\Delta S_{t,\tau^*}^E \int_0^T A_r^E d\tau \int_0^\tau A_r^E(s)ds}{1+\gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^T \alpha^E\left[\int_0^\tau A_r^E ds\right]A_r^E d\tau} + \frac{\gamma^E\sigma_\beta^2 \Delta S_{t,\tau^*}^E \int_0^T A_\beta^E d\tau \int_0^\tau A_\beta^E(s)ds}{1+\gamma^E\sigma_\beta^2\int_0^T \alpha^E\left[\int_0^\tau A_\beta^E ds\right]A_\beta^E d\tau}\end{aligned}$$

Since $\tau y_{t,\tau}^E = A_r^E(\tau)r_t + A_\beta^E(\tau)\beta_t + C^E(\tau)$, the change in yield is

$$\Delta y_{t,\tau}^E\left(\Delta S_{t,\tau^*}^E\right) = \frac{1}{\tau}\Delta C^E(\tau) = \frac{\gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^T A_r^E d\tau}{1+\gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^T \alpha^E\left[\int_0^\tau A_r^E ds\right]A_r^E d\tau}\Delta S_{t,\tau^*}^E\left[\frac{1}{\tau}\int_0^\tau A_r^E(s)ds\right]$$

$$+\frac{\gamma^E\sigma_\beta^2\int_0^T A_\beta^E d\tau}{1+\gamma^E\sigma_\beta^2\int_0^T\alpha^E\left[\int_0^\tau A_\beta^E ds\right]A_\beta^E d\tau}\Delta S_{t,\tau^*}^E\left[\frac{1}{\tau}\int_0^\tau A_\beta^E(s)ds\right]$$
$$\equiv\phi_r^E(\kappa_r^E,\sigma_r,\gamma^E)\Delta S_{t,\tau^*}^E\left[\frac{1}{\tau}\int_0^\tau A_r^E(s)ds\right]+\phi_\beta^E(\kappa_\beta^E,\sigma_\beta,\gamma^E)\Delta S_{t,\tau^*}^E\left[\frac{1}{\tau}\int_0^\tau A_\beta^E(s)ds\right]$$

Since $\Delta y_{t,\tau}^E(\Delta S_{t,\tau^*}^E)$ is increasing in τ, yields with longer maturities are more affected by the supply shock. In addition, the supply effect is ***global*** as the relative effect across different maturities does not depend on origin of the supply shock. Consider two maturities τ_1 and τ_2 with $\tau_1\neq\tau_2$, the relative change in yield due to $\Delta S_{t,\tau^*}^E$ is independent of the shock origin τ^*:

$$\frac{\Delta y_{t,\tau_2}^E(\Delta S_{t,\tau^*}^E)}{\Delta y_{t,\tau_1}^E(\Delta S_{t,\tau^*}^E)}=\left(\frac{\tau_2}{\tau_1}\right)\frac{\phi_r^E\int_0^{\tau_1}A_r^E(s)ds+\phi_\beta^E\int_0^{\tau_1}A_\beta^E(s)ds}{\phi_r^E\int_0^{\tau_2}A_r^E(s)ds+\phi_\beta^E\int_0^{\tau_2}A_\beta^E(s)ds}>1$$

Since $Y_{t,\tau}^E=\alpha^E(\tau)\tau y_{t,\tau}^E-d^E(\tau)-\theta(\tau)\beta_t$, the clientele investor at maturity τ changes her demand by

$$\Delta Y_{t,\tau}^E(\Delta S_{t,\tau^*}^E)=\alpha^E(\tau)\Delta C^E(\tau)=\left(\phi_r^E\int_0^\tau A_r^E(s)ds+\phi_\beta^E\int_0^\tau A_\beta^E(s)ds\right)\Delta S_{t,\tau^*}^E\equiv\phi^E(\tau)\Delta S_{t,\tau^*}^E$$

Moreover, $\phi^E(\tau)\geq 0$ and is increase in τ, so a positive $\Delta S_{t,\tau^*}^E$ induces a higher demand of the clientele investors, while the demand of the within-market arbitrageurs changes by

$$\Delta Z_{t,\tau}^E=\begin{cases}[1-\phi^E(\tau)]\Delta S_{t,\tau^*}^E>0 & \tau=\tau^*,\phi^E(\tau)<1\\ [1-\phi^E(\tau)]\Delta S_{t,\tau^*}^E<0 & \tau=\tau^*,\phi^E(\tau)\geq 1\\ -\phi^E(\tau)\Delta S_{t,\tau^*}^E<0 & \tau\neq\tau^*\end{cases}$$

Proof of Proposition 10. (Semi-form, market E, local demand effects)

An exogenous shock to the demand intercept $\Delta d^E(\tau)$ in market E induces a change in z_r^E of

$$\Delta z_r^E=\frac{\gamma^E[\sigma_r^2+\mathrm{Var}(\Delta r)]\int_0^T\Delta d^E A_r^E d\tau}{1+\gamma^E[\sigma_r^2+\mathrm{Var}(\Delta r)]\int_0^T\alpha^E\left[\int_0^\tau A_r^E ds\right]A_r^E d\tau}$$
$$\Delta z_\beta^E=\frac{\gamma^E\sigma_\beta^2\int_0^T\Delta d^E A_\beta^E d\tau}{1+\gamma^E\sigma_\beta^2\int_0^T\alpha^E\left[\int_0^\tau A_\beta^E ds\right]A_\beta^E d\tau}$$

$$\Delta C^E(\tau)=\Delta z_r^E\int_0^\tau A_r^E(s)ds+\Delta z_\beta^E\int_0^\tau A_\beta^E(s)ds$$
$$=\frac{\gamma^E[\sigma_r^2+\mathrm{Var}(\Delta r)]\int_0^T\Delta d^E A_r^E d\tau\int_0^\tau A_r^E(s)ds}{1+\gamma^E[\sigma_r^2+\mathrm{Var}(\Delta r)]\int_0^T\alpha^E\left[\int_0^\tau A_r^E ds\right]A_r^E d\tau}+\frac{\gamma^E\sigma_\beta^2\int_0^T\Delta d^E A_\beta^E d\tau\int_0^\tau A_\beta^E(s)ds}{1+\gamma^E\sigma_\beta^2\int_0^T\alpha^E\left[\int_0^\tau A_\beta^E ds\right]A_\beta^E d\tau}$$

The change in yields due to $\Delta d^E(\tau)$ is given by

$$\Delta y_{t,\tau}^E(\Delta d^E)=\frac{1}{\tau}\Delta C^E(\tau)$$
$$=\frac{\gamma^E[\sigma_r^2+\mathrm{Var}(\Delta r)]\int_0^T\Delta d^E A_r^E d\tau}{1+\gamma^E[\sigma_r^2+\mathrm{Var}(\Delta r)]\int_0^T\alpha^E\left[\int_0^\tau A_r^E ds\right]A_r^E d\tau}\left[\frac{1}{\tau}\int_0^\tau A_r^E(s)ds\right]$$
$$+\frac{\gamma^E\sigma_\beta^2\int_0^T\Delta d^E A_\beta^E d\tau}{1+\gamma^E\sigma_\beta^2\int_0^T\alpha^E\left[\int_0^\tau A_\beta^E ds\right]A_\beta^E d\tau}\left[\frac{1}{\tau}\int_0^\tau A_\beta^E(s)ds\right]>0$$

I then examine the impact of $\Delta d^E(\tau)$ on yields through the two terms $\int_0^T \Delta d^E A_r^E d\tau$ and $\int_0^T \Delta d^E A_\beta^E d\tau$, which characterize the sensitivity of arbitrageurs' portfolio to the short rate and the demand factor.

Case 1: If $\Delta d^E(\tau) > 0$ (< 0) for all τ, $Y_{t,\tau}^E$ decreases (increases) at all maturities, corresponding to a negative (positive) demand shock. Since $\int_0^T \Delta d^E(\tau) A_r^E(\tau) d\tau$, $\int_0^T \Delta d^E(\tau) A_\beta^E(\tau) d\tau > 0$ (< 0) if $\Delta d^E(\tau) > 0$ (< 0) for all τ, a negative (positive) demand shock *always* leads to an increase (a decrease) in yields.

Case 2: Suppose $\Delta d^E(\tau_1) > 0$ and $\Delta d^E(\tau_2) < 0$ for some $0 < \tau_1, \tau_2 < T$, both terms $\int_0^T \Delta d^E(\tau) A_r^E(\tau) d\tau$ and $\int_0^T \Delta d^E(\tau) A_\beta^E(\tau) d\tau$ can take either positive or negative values. If both terms are positive (negative), the yield $y_{t,\tau}^E$ increases (decreases) at all maturities, even for maturity clienteles with $\Delta d^E(\tau) < 0$ (< 0). In addition, the demand effect is now ***localized*** as the relative effect across maturities do depend on the origin of the demand shock. To be specific, it depends on the relative magnitude of $\tau^{-1} \int_0^\tau A_r^E(s) ds$ and $\tau^{-1} \int_0^\tau A_\beta^E(s) ds$.

Consider τ_1 and τ_2 with $\tau_1 \neq \tau_2$, the relative change in yield is now dependent on the origin of Δd^E:

$$\frac{\Delta y_{t,\tau_2}^E}{\Delta y_{t,\tau_1}^E} = \frac{\Delta y_{t,\tau_1}^E\left(\Delta d^E(\tau)\right)}{\Delta y_{t,\tau_2}^E\left(\Delta d^E(\tau)\right)}$$

Since $Y_{t,\tau}^E = \alpha^E(\tau)\tau y_{t,\tau}^E - d^E(\tau) - \theta(\tau)\beta_t$, the clientele investor at maturity τ changes her demand by

$$\Delta Y_{t,\tau}^E\left(\Delta d^E(\tau)\right) = \alpha^E(\tau)\left[A_\beta^E(\tau)\Delta\beta_t + \Delta C^E(\tau)\right] - \Delta d^E(\tau) - \theta(\tau)\Delta\beta_t$$

and the within-market arbitrageur changes her demand for maturity-τ bonds by

$$\Delta Z_{t,\tau}^E\left(\Delta d^E(\tau)\right) = -\Delta Y_{t,\tau}^E\left(\Delta d^E(\tau)\right)$$

The signs of $\Delta Y_{t,\tau}^E\left(\Delta d^E(\tau)\right)$ and $\Delta Y_{t,\tau}^E\left(\Delta d^E(\tau)\right)$ remain unclear in this case.

Proof of Proposition 11. (Supply shock transmission, market C → E)

Consider the impact of an exogenous supply shock $\Delta S_{t,\tau^*}^C$ of bonds with maturity τ^* in market C. Suppose $\Delta S_{t,\tau^*}^C$ induces a change of $\Delta X_{t,\tau}^C = \Delta\beta_t^C F^C$ and $\Delta X_{t,\tau}^E = \Delta\beta_t^E F^E$ in the cross-market arbitrageur's demand.

$$C^{C,A'} - z_r^{C,A} A_r^{C,A} + \frac{1}{2}\left[\sigma_r^2 + \mathrm{Var}(\Delta r)\right] {A_r^{C,A}}^2 = 0$$

From the above equation,

$$\Delta\left(C^{C,A'}\right) = \Delta\left(z_r^{C,A}\right) A_r^{C,A} = \frac{\gamma^C\left[\sigma_r^2 + \mathrm{Var}(\Delta r)\right] A_r^{C,A} \int_0^T A_r^{C,A}\left(\Delta S_{t,\tau^*}^C - F^C \Delta\beta_t^C\right) d\tau}{1 + \gamma^C\left[\sigma_r^2 + \mathrm{Var}(\Delta r)\right] \int_0^T \alpha^C\left[\int_0^\tau A_r^{C,A}(s) ds\right] A_r^{C,A} d\tau}$$

Substitute into (3.3.1) and equate the induced changes on the L.H.S. and the R.H.S. to get

$$\frac{\gamma^C \Delta S_{t,\tau^*}^C \int_0^T A_r^{C,A} d\tau - \gamma^C \Delta\beta_t^C \int_0^T A_r^{C,A} F^C d\tau}{1 + \gamma^C\left[\sigma_r^2 + \mathrm{Var}(\Delta r)\right] \int_0^T \alpha^C\left[\int_0^\tau A_r^{C,A}(s) ds\right] A_r^{C,A} d\tau} = \gamma^A\left(\Delta\beta_t^C \int_0^T A_r^{C,A} F^C d\tau + \Delta\beta_t^E \int_0^T A_r^{E,A} F^E d\tau\right)$$

Equate the induced changes on the LHS and the RHS in (3.3.2) to get

$$
-\frac{\gamma^E[\sigma_r^2+\mathrm{Var}(\Delta r)]A_r^{E,A}\Delta\beta_t^E\int_0^T A_r^{E,A}F^E d\tau}{1+\gamma^E[\sigma_r^2+\mathrm{Var}(\Delta r)]\int_0^T\alpha^E\left[\int_0^\tau A_r^{E,A}ds\right]A_r^{E,A}d\tau}-\frac{\gamma^E\sigma_\beta^2A_\beta^{E,A}\Delta\beta_t^E\int_0^T A_\beta^{E,A}F^E d\tau}{1+\gamma^E\sigma_\beta^2\int_0^T\alpha^E\left[\int_0^\tau A_\beta^{E,A}ds\right]A_\beta^{E,A}d\tau}
$$

$$
=\gamma^A[\sigma_r^2+\mathrm{Var}(\Delta r)]A_r^{E,A}\left(\Delta\beta_t^E\int_0^T F^E A_r^{E,A}d\tau+\Delta\beta_t^C\int_0^T F^C A_r^{C,A}d\tau\right)+\gamma^A\sigma_\beta^2A_\beta^{E,A}\left(\Delta\beta_t^E\int_0^T F^E A_\beta^{E,A}d\tau\right)
$$

The above two equations can be written as

$$
\begin{aligned}
M_1\Delta\beta_t^C+M_2\Delta\beta_t^E&=M_3\Delta S_{t,\tau^*}^C\\
M_4\Delta\beta_t^C+M_5\Delta\beta_t^E&=0
\end{aligned}
$$

where

$$
\begin{aligned}
M_1&\equiv\left\{\gamma^A+\frac{\gamma^C}{1+\gamma^C[\sigma_r^2+\mathrm{Var}(\Delta r)]\int_0^T\alpha^C\left[\int_0^\tau A_r^{C,A}(s)ds\right]A_r^{C,A}d\tau}\right\}[\sigma_r^2+\mathrm{Var}(\Delta r)]\left(\int_0^T F^C A_r^{C,A}d\tau\right)A_r^{E,A}\\
M_2&\equiv\gamma^A[\sigma_r^2+\mathrm{Var}(\Delta r)]\left(\int_0^T F^E A_r^{E,A}d\tau\right)A_r^{E,A}\\
M_3&\equiv\frac{\gamma^C}{1+\gamma^C[\sigma_r^2+\mathrm{Var}(\Delta r)]\int_0^T\alpha^C\left[\int_0^\tau A_r^{C,A}(s)ds\right]A_r^{C,A}d\tau}[\sigma_r^2+\mathrm{Var}(\Delta r)]\left(\int_0^T A_r^{C,A}d\tau\right)A_r^{E,A}\\
M_4&\equiv\gamma^A[\sigma_r^2+\mathrm{Var}(\Delta r)]\left(\int_0^T F^C A_r^{C,A}d\tau\right)A_r^{E,A}\\
M_5&\equiv\left\{\gamma^A+\frac{\gamma^E}{1+\gamma^E[\sigma_r^2+\mathrm{Var}(\Delta r)]\int_0^T\alpha^E\left[\int_0^\tau A_r^{E,A}ds\right]A_r^{E,A}d\tau}\right\}[\sigma_r^2+\mathrm{Var}(\Delta r)]\left(\int_0^T F^E A_r^{E,A}d\tau\right)A_r^{E,A}\\
&\quad+\left\{\gamma^A+\frac{\gamma^E}{1+\gamma^E\sigma_\beta^2\int_0^T\alpha^E\left[\int_0^\tau A_\beta^{E,A}ds\right]A_\beta^{E,A}d\tau}\right\}\sigma_\beta^2\left(\int_0^T F^E A_\beta^{E,A}d\tau\right)A_\beta^{E,A}
\end{aligned}
$$

Note that $[\sigma_r^2+\mathrm{Var}(\Delta r)]A_r^{E,A}$ is multiplied to both sides of the first equation for the ease of later computation. Since $M_5>M_2$ and $M_1>M_4$, $M_1M_5-M_2M_4>0$, so

$$
\begin{aligned}
\Delta\beta_t^C&=\frac{M_3M_5}{M_1M_5-M_2M_4}\Delta S_{t,\tau^*}^C>0\\
\Delta\beta_t^E&=-\frac{M_3M_4}{M_1M_5-M_2M_4}\Delta S_{t,\tau^*}^C<0
\end{aligned}
$$

Secondly, I investigate how the supply shock $\Delta S_{t,\tau^*}^C$ in market C affects the term structures of market C and of market E. Because the mean-reversion parameters $\kappa_r^{C,A}$, $\kappa_r^{E,A}$ and $\kappa_\beta^{E,A}$ are independently pinned down and are not susceptible to supply and demand shocks, but only to shocks to the short-rate, supply and demand shocks affect the two term structures through the intercepts $C^{C,A}$ and $C^{E,A}$.

For market C, the intercept term *increases* by

$$
\begin{aligned}
\Delta C^{C,A}\left(\Delta S_{t,\tau^*}^C\right)&=\Delta z_r^{C,A}\int_0^\tau A_r^{C,A}(s)ds\\
&=\frac{\gamma^C[\sigma_r^2+\mathrm{Var}(\Delta r)]\int_0^T\left(\Delta S_{t,\tau^*}^C-F^C\Delta\beta_t^C\right)A_r^{C,A}d\tau}{1+\gamma^C[\sigma_r^2+\mathrm{Var}(\Delta r)]\int_0^T\alpha^C\left[\int_0^\tau A_r^{C,A}(s)ds\right]A_r^{C,A}d\tau}\int_0^\tau A_r^{C,A}(s)ds\\
&=\frac{\gamma^C[\sigma_r^2+\mathrm{Var}(\Delta r)]\int_0^T\left[1-\left(\frac{M_3M_5}{M_1M_5-M_2M_4}\right)F^C\right]A_r^{C,A}d\tau}{1+\gamma^C[\sigma_r^2+\mathrm{Var}(\Delta r)]\int_0^T\alpha^C\left[\int_0^\tau A_r^{C,A}(s)ds\right]A_r^{C,A}d\tau}\left[\int_0^\tau A_r^{C,A}(s)ds\right]\Delta S_{t,\tau^*}^C
\end{aligned}
$$

Now I show that $\Delta C^{C,A}\left(\Delta S^{C}_{t,\tau^*}\right) > 0$, which is equivalent to showing that

$$\int_0^T \left[1 - \left(\frac{M_3 M_5}{M_1 M_5 - M_2 M_4}\right) F^C\right] A_r^{C,A} d\tau > 0$$

Since

$$\int_0^T \left[1 - \left(\frac{M_3 M_5}{M_1 M_5 - M_2 M_4}\right) F^C\right] A_r^{C,A} d\tau \geq \int_0^T \left[1 - \left(\frac{M_3 M_5}{M_1 M_5 - M_2 M_4}\right) F^C\right] d\tau \int_0^T A_r^{C,A} d\tau$$

It remains to show that

$$\begin{aligned}
\int_0^T \left[1 - \left(\frac{M_3 M_5}{M_1 M_5 - M_2 M_4}\right) F^C\right] d\tau > 0 \quad &\Leftrightarrow \quad M_1 M_5 - M_2 M_4 > M_5 M_3 \int_0^T F^C d\tau \\
&\Leftrightarrow \quad \left(M_1 - M_3 \int_0^T F^C d\tau\right) M_5 > M_2 M_4 \\
&\Leftrightarrow \quad \left(M_1 - \mathrm{coeff}(M_3) \int_0^T A_r^{C,A} d\tau \int_0^T F^C d\tau\right) M_5 > M_2 M_4 \\
&\Leftrightarrow \quad \left(M_1 - \mathrm{coeff}(M_3) \int_0^T A_r^{C,A} F^C d\tau\right) M_5 > M_2 M_4 \\
&\Leftrightarrow \quad M_4 M_5 > M_2 M_4
\end{aligned}$$

Since $M_5 > M_2$, this finishes the proof of $\Delta C^{C,A}\left(\Delta S^{C}_{t,\tau^*}\right) > 0$.

Thus, yields *increase* by

$$\Delta y_{t,\tau}^{C,A} = \frac{1}{\tau} \Delta C^{C,A}\left(\Delta S^{C}_{t,\tau^*}\right) = \frac{\gamma^C [\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T \left[1 - \left(\frac{M_3 M_5}{M_1 M_5 - M_2 M_4}\right) F^C\right] A_r^{C,A} d\tau}{1 + \gamma^C [\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T \alpha^C \left[\int_0^\tau A_r^{C,A}(s) ds\right] A_r^{C,A} d\tau} \left[\frac{1}{\tau} \int_0^\tau A_r^{C,A}(s) ds\right] \Delta S^{C}_{t,\tau^*}$$

For market *E*, the intercept term *increases* by

$$\begin{aligned}
\Delta C^{E,A}\left(\Delta S^{C}_{t,\tau^*}\right) &= \Delta z_r^{E,A} \int_0^\tau A_r^{E,A}(s) ds + \Delta z_\beta^{E,A} \int_0^\tau A_\beta^{E,A}(s) ds \\
&= -\frac{\gamma^E [\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T F^E \Delta\beta_t^E A_r^{E,A} d\tau}{1 + \gamma^E [\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T \alpha^E \left[\int_0^\tau A_r^{E,A} ds\right] A_r^{E,A} d\tau} \int_0^\tau A_r^{E,,A}(s) ds \\
&\quad - \frac{\gamma^E \sigma_\beta^2 \int_0^T F^E \Delta\beta_t^E A_\beta^{E,A} d\tau}{1 + \gamma^E \sigma_\beta^2 \int_0^T \alpha^E \left[\int_0^\tau A_\beta^{E,A} ds\right] A_\beta^{E,A} d\tau} \int_0^\tau A_\beta^{E,A}(s) ds \\
&= \frac{\gamma^E [\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T \left(\frac{M_3 M_4}{M_1 M_5 - M_2 M_4}\right) F^E A_r^{E,A} d\tau}{1 + \gamma^E [\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T \alpha^E \left[\int_0^\tau A_r^{E,A} ds\right] A_r^{E,A} d\tau} \left[\int_0^\tau A_r^{E,A}(s) ds\right] \Delta S^{C}_{t,\tau^*} \\
&\quad + \frac{\gamma^E \sigma_\beta^2 \int_0^T \left(\frac{M_3 M_4}{M_1 M_5 - M_2 M_4}\right) F^E A_\beta^{E,A} d\tau}{1 + \gamma^E \sigma_\beta^2 \int_0^T \alpha^E \left[\int_0^\tau A_\beta^{E,A} ds\right] A_\beta^{E,A} d\tau} \left[\int_0^\tau A_\beta^{E,A}(s) ds\right] \Delta S^{C}_{t,\tau^*}
\end{aligned}$$

so that yields *increase* by

$$\Delta y_{t,\tau}^{E,A} = \frac{1}{\tau} \Delta C^{E,A}\left(\Delta S^{C}_{t,\tau^*}\right)$$

$$= \frac{\gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T \left(\frac{M_3 M_4}{M_1 M_5 - M_2 M_4}\right) F^E A_r^{E,A} d\tau}{1 + \gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T \alpha^E \left[\int_0^\tau A_r^{E,A} ds\right] A_r^{E,A} d\tau} \left[\frac{1}{\tau}\int_0^\tau A_r^{E,A}(s) ds\right] \Delta S_{t,\tau^*}^C$$
$$+ \frac{\gamma^E \sigma_\beta^2 \int_0^T \left(\frac{M_3 M_4}{M_1 M_5 - M_2 M_4}\right) F^E A_\beta^{E,A} d\tau}{1 + \gamma^E \sigma_\beta^2 \int_0^T \alpha^E \left[\int_0^\tau A_\beta^{E,A} ds\right] A_\beta^{E,A} d\tau} \left[\frac{1}{\tau}\int_0^\tau A_\beta^{E,A}(s) ds\right] \Delta S_{t,\tau^*}^C$$

Proof of Proposition 12. (Supply shock transmission, market E → C)

Consider the impact of an exogenous supply shock $\Delta S_{t,\tau^*}^E$ of bonds with maturity τ^* in market E. Suppose $\Delta S_{t,\tau^*}^E$ induces a change of $\Delta X_{t,\tau}^C = \Delta\beta_t^C F^C$ and $\Delta X_{t,\tau}^E = \Delta\beta_t^E F^E$ in the cross-market arbitrageur's demand.

$$C^{E,A\prime} - z_r^{E,A} A_r^{E,A} - z_\beta^{E,A} A_\beta^{E,A} + \frac{1}{2}[\sigma_r^2 + \mathrm{Var}(\Delta r)] {A_r^{E,A}}^2 + \frac{1}{2}\sigma_\beta^2 {A_\beta^{E,A}}^2 = 0$$

From the above equation we know that

$$\begin{aligned}\Delta\left(C^{E,A\prime}\right) &= \Delta\left(z_r^{E,A}\right) A_r^{E,A} + \Delta\left(z_\beta^{E,A}\right) A_\beta^{E,A} \\ &= \frac{\gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)] A_r^{E,A} \int_0^T \left(\Delta S_{t,\tau^*}^E - F^E \Delta\beta_t^E\right) A_r^{E,A} d\tau}{1 + \gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T \alpha^E \left[\int_0^\tau A_r^{E,A} ds\right] A_r^{E,A} d\tau} + \frac{\gamma^E \sigma_\beta^2 A_\beta^{E,A} \int_0^T \left(\Delta S_{t,\tau^*}^E - F^E \Delta\beta_t^E\right) A_\beta^{E,A} d\tau}{1 + \gamma^E \sigma_\beta^2 \int_0^T \alpha^E \left[\int_0^\tau A_\beta^{E,A} ds\right] A_\beta^{E,A} d\tau}\end{aligned}$$

Substitute into (3.3.2) and equate the induced changes on the L.H.S. and the R.H.S. to get

$$\begin{aligned}&\frac{\gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)] A_r^{E,A} \int_0^T \left(\Delta S_{t,\tau^*}^E - F^E \Delta\beta_t^E\right) A_r^{E,A} d\tau}{1 + \gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T \alpha^E \left[\int_0^\tau A_r^{E,A} ds\right] A_r^{E,A} d\tau} + \frac{\gamma^E \sigma_\beta^2 A_\beta^{E,A} \int_0^T \left(\Delta S_{t,\tau^*}^E - F^E \Delta\beta_t^E\right) A_\beta^{E,A} d\tau}{1 + \gamma^E \sigma_\beta^2 \int_0^T \alpha^E \left[\int_0^\tau A_\beta^{E,A} ds\right] A_\beta^{E,A} d\tau} \\ &= \gamma^A[\sigma_r^2 + \mathrm{Var}(\Delta r)] A_r^{E,A} \left(\Delta\beta_t^E \int_0^T F^E A_r^{E,A} d\tau + \Delta\beta_t^C \int_0^T F^C A_r^{C,A} d\tau\right) + \gamma^A \sigma_\beta^2 A_\beta^{E,A} \left(\Delta\beta_t^E \int_0^T F^E A_\beta^{E,A} d\tau\right)\end{aligned}$$

Equate the induced changes on the LHS and the RHS in (3.3.1) to get

$$-\frac{\gamma^C \Delta\beta_t^C \int_0^T A_r^{C,A} F^C d\tau}{1 + \gamma^C[\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T \alpha^C \left[\int_0^\tau A_r^{C,A}(s) ds\right] A_r^{C,A} d\tau} = \gamma^A \left(\Delta\beta_t^C \int_0^T A_r^{C,A} F^C d\tau + \Delta\beta_t^E \int_0^T A_r^{E,A} F^E d\tau\right)$$

The above two equations can be written as

$$\begin{aligned}N_1 \Delta\beta_t^C + N_2 \Delta\beta_t^E &= N_3 \Delta S_{t,\tau^*}^E \\ N_4 \Delta\beta_t^C + N_5 \Delta\beta_t^E &= 0\end{aligned}$$

where

$$N_1 \equiv \gamma^A[\sigma_r^2 + \mathrm{Var}(\Delta r)] \left(\int_0^T F^C A_r^{C,A} d\tau\right) A_r^{E,A} = M_4$$

$$\begin{aligned}N_2 \equiv &\left\{\gamma^A + \frac{\gamma^E}{1 + \gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T \alpha^E \left[\int_0^\tau A_r^{E,A} ds\right] A_r^{E,A} d\tau}\right\} [\sigma_r^2 + \mathrm{Var}(\Delta r)] \left(\int_0^T F^E A_r^{E,A} d\tau\right) A_r^{E,A} \\ &+ \left\{\gamma^A + \frac{\gamma^E}{1 + \gamma^E \sigma_\beta^2 \int_0^T \alpha^E \left[\int_0^\tau A_\beta^{E,A} ds\right] A_\beta^{E,A} d\tau}\right\} \sigma_\beta^2 \left(\int_0^T F^E A_\beta^{E,A} d\tau\right) A_\beta^{E,A} = M_5\end{aligned}$$

$$N_3 \equiv \left\{\frac{\gamma^E}{1+\gamma^E[\sigma_r^2+\mathrm{Var}(\Delta r)]\int_0^T \alpha^E\left[\int_0^\tau A_r^{E,A} ds\right] A_r^{E,A} d\tau}\right\}[\sigma_r^2+\mathrm{Var}(\Delta r)]\left(\int_0^T A_r^{E,A} d\tau\right) A_r^{E,A}$$
$$+\left\{\frac{\gamma^E}{1+\gamma^E\sigma_\beta^2\int_0^T \alpha^E\left[\int_0^\tau A_\beta^{E,A} ds\right] A_\beta^{E,A} d\tau}\right\}\sigma_\beta^2\left(\int_0^T A_\beta^{E,A} d\tau\right) A_\beta^{E,A}$$
$$N_4 \equiv \left\{\gamma^A+\frac{\gamma^C}{1+\gamma^C[\sigma_r^2+\mathrm{Var}(\Delta r)]\int_0^T \alpha^C\left[\int_0^\tau A_r^{C,A}(s) ds\right] A_r^{C,A} d\tau}\right\}[\sigma_r^2+\mathrm{Var}(\Delta r)]\left(\int_0^T F^C A_r^{C,A} d\tau\right) A_r^{E,A} = M_1$$
$$N_5 \equiv \gamma^A[\sigma_r^2+\mathrm{Var}(\Delta r)]\left(\int_0^T F^E A_r^{E,A} d\tau\right) A_r^{E,A} = M_2$$

Following from the proof in Corollary 6,

$$\Delta\beta_t^C = \frac{N_3 N_5}{N_1 N_5 - N_2 N_4}\Delta S_{t,\tau^*}^E = -\frac{N_3 M_2}{M_5 M_1 - M_4 M_2}\Delta S_{t,\tau^*}^E < 0$$
$$\Delta\beta_t^E = \frac{-N_3 N_4}{N_1 N_5 - N_2 N_4}\Delta S_{t,\tau^*}^E = \frac{N_3 M_1}{M_5 M_1 - M_4 M_2}\Delta S_{t,\tau^*}^E > 0$$

Secondly, I investigate how the supply shock $\Delta S_{t,\tau^*}^E$ in market *E* affects the term structures of market *C* and of market *E*. Because the mean-reversion parameters $\kappa_r^{C,A}$, $\kappa_r^{E,A}$ and $\kappa_\beta^{E,A}$ are independently pinned down and are not susceptible to supply and demand shocks, but only to shocks to the short-rate, supply and demand shocks affect the two term structures through the intercepts $C^{C,A}$ and $C^{E,A}$.

For market *C*, the intercept term *increases* by

$$\Delta C^{C,A}\left(\Delta S_{t,\tau^*}^E\right) = \Delta z_r^{C,A}\int_0^\tau A_r^{C,A}(s) ds$$
$$= -\frac{\gamma^C[\sigma_r^2+\mathrm{Var}(\Delta r)]\int_0^T \Delta\beta_t^C F^C A_r^{C,A} d\tau}{1+\gamma^C[\sigma_r^2+\mathrm{Var}(\Delta r)]\int_0^T \alpha^C\left[\int_0^\tau A_r^{C,A}(s) ds\right] A_r^{C,A} d\tau}\int_0^\tau A_r^{C,A}(s) ds$$
$$= \frac{\gamma^C[\sigma_r^2+\mathrm{Var}(\Delta r)]\int_0^T \left(\frac{N_3 M_2}{M_5 M_1 - M_4 M_2}\right) F^C A_r^{C,A} d\tau}{1+\gamma^C[\sigma_r^2+\mathrm{Var}(\Delta r)]\int_0^T \alpha^C\left[\int_0^\tau A_r^{C,A}(s) ds\right] A_r^{C,A} d\tau}\left[\int_0^\tau A_r^{C,A}(s) ds\right]\Delta S_{t,\tau^*}^E$$

so that yields *increase* by

$$\Delta y_{t,\tau}^{C,A} = \frac{1}{\tau}\Delta C^{C,A}\left(\Delta S_{t,\tau^*}^E\right) = \frac{\gamma^C[\sigma_r^2+\mathrm{Var}(\Delta r)]\int_0^T \left(\frac{N_3 M_2}{M_5 M_1 - M_4 M_2}\right) F^C A_r^{C,A} d\tau}{1+\gamma^C[\sigma_r^2+\mathrm{Var}(\Delta r)]\int_0^T \alpha^C\left[\int_0^\tau A_r^{C,A}(s) ds\right] A_r^{C,A} d\tau}\left[\frac{1}{\tau}\int_0^\tau A_r^{C,A}(s) ds\right]\Delta S_{t,\tau^*}^E$$

For market *E*, the intercept term *increases* by

$$\Delta C^{E,A}\left(\Delta S_{t,\tau^*}^E\right) = \Delta z_r^{E,A}\int_0^\tau A_r^{E,A}(s) ds + \Delta z_\beta^{E,A}\int_0^\tau A_\beta^{E,A}(s) ds$$
$$= \frac{\gamma^E[\sigma_r^2+\mathrm{Var}(\Delta r)]\int_0^T \left(\Delta S_{t,\tau^*}^E - F^E\Delta\beta_t^E\right) A_r^{E,A} d\tau}{1+\gamma^E[\sigma_r^2+\mathrm{Var}(\Delta r)]\int_0^T \alpha^E\left[\int_0^\tau A_r^{E,A} ds\right] A_r^{E,A} d\tau}\int_0^\tau A_r^{E,A}(s) ds$$
$$+\frac{\gamma^E\sigma_\beta^2\int_0^T \left(\Delta S_{t,\tau^*}^E - F^E\Delta\beta_t^E\right) A_\beta^{E,A} d\tau}{1+\gamma^E\sigma_\beta^2\int_0^T \alpha^E\left[\int_0^\tau A_\beta^{E,A} ds\right] A_\beta^{E,A} d\tau}\int_0^\tau A_\beta^{E,A}(s) ds$$

$$= \frac{\gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T \left[1 - \left(\frac{N_3 M_1}{M_5 M_1 - M_4 M_2}\right) F^E\right] A_r^{E,A} d\tau}{1 + \gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T \alpha^E\left[\int_0^\tau A_r^{E,A} ds\right] A_r^{E,A} d\tau} \left[\int_0^\tau A_r^{E,A}(s) ds\right] \Delta S_{t,\tau^*}^E$$

$$+ \frac{\gamma^E \sigma_\beta^2 \int_0^T \left[1 - \left(\frac{N_3 M_1}{M_5 M_1 - M_4 M_2}\right) F^E\right] A_\beta^{E,A} d\tau}{1 + \gamma^E \sigma_\beta^2 \int_0^T \alpha^E \left[\int_0^\tau A_\beta^{E,A} ds\right] A_\beta^{E,A} d\tau} \left[\int_0^\tau A_\beta^{E,A}(s) ds\right] \Delta S_{t,\tau^*}^E$$

so that yields *increase* by

$$\Delta y_{t,\tau}^{E,A} = \frac{1}{\tau} \Delta C^{E,A}\left(\Delta S_{t,\tau^*}^E\right)$$

$$= \frac{\gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T \left[1 - \left(\frac{N_3 M_1}{M_5 M_1 - M_4 M_2}\right) F^E\right] A_r^{E,A} d\tau}{1 + \gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T \alpha^E\left[\int_0^\tau A_r^{E,A} ds\right] A_r^{E,A} d\tau} \left[\frac{1}{\tau}\int_0^\tau A_r^{E,A}(s) ds\right] \Delta S_{t,\tau^*}^E$$

$$+ \frac{\gamma^E \sigma_\beta^2 \int_0^T \left[1 - \left(\frac{N_3 M_1}{M_5 M_1 - M_4 M_2}\right) F^E\right] A_\beta^{E,A} d\tau}{1 + \gamma^E \sigma_\beta^2 \int_0^T \alpha^E \left[\int_0^\tau A_\beta^{E,A} ds\right] A_\beta^{E,A} d\tau} \left[\frac{1}{\tau}\int_0^\tau A_\beta^{E,A}(s) ds\right] \Delta S_{t,\tau^*}^E$$

Proof of Proposition 13. (Demand shock transmission, market C → E)

Consider the impact of a demand shock $\Delta d^C(\tau)$ from market C. Suppose $\Delta d^C(\tau)$ induces a change of $\Delta X_{t,\tau}^C = \Delta\beta_t^C F^C$ and $\Delta X_{t,\tau}^E = \Delta\beta_t^E F^E$ in the cross-market arbitrageur's demand.

$$C^{C,A'} - z_r^{C,A} A_r^{C,A} + \frac{1}{2}[\sigma_r^2 + \mathrm{Var}(\Delta r)] {A_r^{C,A}}^2 = 0$$

From the above equation,

$$\Delta\left(C^{C,A'}\right) = \Delta\left(z_r^{C,A}\right) A_r^{C,A} = \frac{\gamma^C[\sigma_r^2 + \mathrm{Var}(\Delta r)] A_r^{C,A} \int_0^T A_r^{C,A}(\Delta d^C - F^C \Delta\beta_t^C) d\tau}{1 + \gamma^C[\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T \alpha^C\left[\int_0^\tau A_r^{C,A}(s) ds\right] A_r^{C,A} d\tau}$$

Substitute into (3.3.1) and equate the induced changes on the L.H.S. and the R.H.S. to get

$$\frac{\gamma^C \int_0^T (\Delta d^C - F^C \Delta\beta_t^C) A_r^{C,A} d\tau}{1 + \gamma^C[\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T \alpha^C\left[\int_0^\tau A_r^{C,A}(s) ds\right] A_r^{C,A} d\tau} = \gamma^A \left(\Delta\beta_t^C \int_0^T A_r^{C,A} F^C d\tau + \Delta\beta_t^E \int_0^T A_r^{E,A} F^E d\tau\right)$$

Equate the induced changes on the LHS and the RHS in (3.3.2) to get

$$- \frac{\gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)] A_r^{E,A} \Delta\beta_t^E \int_0^T A_r^{E,A} F^E d\tau}{1 + \gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T \alpha^E\left[\int_0^\tau A_r^{E,A} ds\right] A_r^{E,A} d\tau} - \frac{\gamma^E \sigma_\beta^2 A_\beta^{E,A} \Delta\beta_t^E \int_0^T A_\beta^{E,A} F^E d\tau}{1 + \gamma^E \sigma_\beta^2 \int_0^T \alpha^E \left[\int_0^\tau A_\beta^{E,A} ds\right] A_\beta^{E,A} d\tau}$$

$$= \gamma^A[\sigma_r^2 + \mathrm{Var}(\Delta r)] A_r^{E,A} \left(\Delta\beta_t^E \int_0^T F^E A_r^{E,A} d\tau + \Delta\beta_t^C \int_0^T F^C A_r^{C,A} d\tau\right) + \gamma^A \sigma_\beta^2 A_\beta^{E,A} \left(\Delta\beta_t^E \int_0^T F^E A_\beta^{E,A} d\tau\right)$$

The above two equations can be written as

$$\begin{aligned} U_1 \Delta\beta_t^C + U_2 \Delta\beta_t^E &= U_3(\Delta d^C) \\ U_4 \Delta\beta_t^C + U_5 \Delta\beta_t^E &= 0 \end{aligned}$$

where

$$U_1 \equiv \left\{\gamma^A + \frac{\gamma^C}{1+\gamma^C[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^T \alpha^C\left[\int_0^\tau A_r^{C,A}(s)ds\right]A_r^{C,A}d\tau}\right\}[\sigma_r^2 + \mathrm{Var}(\Delta r)]\left(\int_0^T F^C A_r^{C,A} d\tau\right)A_r^{E,A} = M_1$$

$$U_2 \equiv \gamma^A[\sigma_r^2 + \mathrm{Var}(\Delta r)]\left(\int_0^T F^E A_r^{E,A} d\tau\right)A_r^{E,A} = M_2$$

$$U_3(\Delta d^C) \equiv \frac{\gamma^C}{1+\gamma^C[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^T \alpha^C\left[\int_0^\tau A_r^{C,A}(s)ds\right]A_r^{C,A}d\tau}[\sigma_r^2 + \mathrm{Var}(\Delta r)]\left(\int_0^T \Delta d^C A_r^{C,A} d\tau\right)A_r^{E,A}$$

$$U_4 \equiv \gamma^A[\sigma_r^2 + \mathrm{Var}(\Delta r)]\left(\int_0^T F^C A_r^{C,A} d\tau\right)A_r^{E,A} = M_4$$

$$U_5 \equiv \left\{\gamma^A + \frac{\gamma^E}{1+\gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^T \alpha^E\left[\int_0^\tau A_r^{E,A}ds\right]A_r^{E,A}d\tau}\right\}[\sigma_r^2 + \mathrm{Var}(\Delta r)]\left(\int_0^T F^E A_r^{E,A} d\tau\right)A_r^{E,A}$$
$$+\left\{\gamma^A + \frac{\gamma^E}{1+\gamma^E\sigma_\beta^2\int_0^T \alpha^E\left[\int_0^\tau A_\beta^{E,A}ds\right]A_\beta^{E,A}d\tau}\right\}\sigma_\beta^2\left(\int_0^T F^E A_\beta^{E,A} d\tau\right)A_\beta^{E,A} = M_5$$

The induced demand shocks are

$$\Delta\beta_t^C = \frac{U_3(\Delta d^C)M_5}{M_1M_5 - M_2M_4} \quad \text{and} \quad \Delta\beta_t^E = -\frac{U_3(\Delta d^C)M_4}{M_1M_5 - M_2M_4}$$

Because $M_1M_5 - M_2M_4 > 0$, the signs of the two induced demand shocks depend only on $U_3(\Delta d^C)$, which depends on $\int_0^T \Delta d^C A_r^{C,A} d\tau$. Following the same argument in Corollary 5,

$$\begin{cases} \Delta\beta_t^C > 0, \Delta\beta_t^E < 0 & \Delta d^C(\tau) > 0\ \forall\tau \\ \Delta\beta_t^C < 0, \Delta\beta_t^E > 0 & \Delta d^C(\tau) < 0\ \forall\tau \\ \text{undetermined} & otherwise \end{cases}$$

Secondly, I investigate how the supply shock $\Delta S_{t,\tau^*}^C$ in market C affects the term structures of market C and of market E. Because the mean-reversion parameters $\kappa_r^{C,A}$, $\kappa_r^{E,A}$ and $\kappa_\beta^{E,A}$ are independently pinned down and are not susceptible to supply and demand shocks, but only to shocks to the short-rate, supply and demand shocks affect the two term structures through the intercepts $C^{C,A}$ and $C^{E,A}$.

For market C, Δd^C induces a *change* of

$$\begin{aligned}\Delta C^{C,A}(\Delta d^C) &= \Delta z_r^{C,A}\int_0^\tau A_r^{C,A}(s)ds \\ &= \frac{\gamma^C[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^T(\Delta d^C - F^C\Delta\beta_t^C)A_r^{C,A}d\tau}{1+\gamma^C[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^T \alpha^C\left[\int_0^\tau A_r^{C,A}(s)ds\right]A_r^{C,A}d\tau}\int_0^\tau A_r^{C,A}(s)ds \\ &= \frac{\gamma^C[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^T\left[\Delta d^C - \left(\frac{U_3(\Delta d^C)M_5}{M_1M_5 - M_2M_4}\right)F^C\right]A_r^{C,A}d\tau}{1+\gamma^C[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^T \alpha^C\left[\int_0^\tau A_r^{C,A}(s)ds\right]A_r^{C,A}d\tau}\left[\int_0^\tau A_r^{C,A}(s)ds\right]\end{aligned}$$

As pointed out in Corollary 2, the sign of $\int_0^T \Delta d^C A_r^{C,A} d\tau$ cannot be pinned down when Δd^C takes both positive and negative values on the interval $[0, T]$. Thus, yields *change* by

$$\Delta y_{t,\tau}^{C,A}(\Delta d^C) = \frac{1}{\tau}\Delta C^{C,A}(\Delta d^C) = \frac{\gamma^C[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^T\left[\Delta d^C - \left(\frac{U_3(\Delta d^C)M_5}{M_1M_5 - M_2M_4}\right)F^C\right]A_r^{C,A}d\tau}{1+\gamma^C[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^T \alpha^C\left[\int_0^\tau A_r^{C,A}(s)ds\right]A_r^{C,A}d\tau}\left[\frac{1}{\tau}\int_0^\tau A_r^{C,A}(s)ds\right]$$

For market E, Δd^C induces a *change* of

$$
\begin{aligned}
\Delta C^{E,A}(\Delta d^C) &= \Delta z_r^{E,A} \int_0^\tau A_r^{E,A}(s)ds + \Delta z_\beta^{E,A} \int_0^\tau A_\beta^{E,A}(s)ds \\
&= -\frac{\gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T F^E \Delta\beta_t^E A_r^{E,A} d\tau}{1 + \gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T \alpha^E \left[\int_0^\tau A_r^{E,A} ds\right] A_r^{E,A} d\tau} \int_0^\tau A_r^{C,A}(s)ds \\
&\quad - \frac{\gamma^E \sigma_\beta^2 \int_0^T F^E \Delta\beta_t^E A_\beta^{E,A} d\tau}{1 + \gamma^E \sigma_\beta^2 \int_0^T \alpha^E \left[\int_0^\tau A_\beta^{E,A} ds\right] A_\beta^{E,A} d\tau} \int_0^\tau A_\beta^{E,A}(s)ds \\
&= \frac{\gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T \left(\frac{U_3(\Delta d^C) M_4}{M_1 M_5 - M_2 M_4}\right) F^E A_r^{E,A} d\tau}{1 + \gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T \alpha^E \left[\int_0^\tau A_r^{E,A} ds\right] A_r^{E,A} d\tau} \left[\int_0^\tau A_r^{C,A}(s)ds\right] \\
&\quad + \frac{\gamma^E \sigma_\beta^2 \int_0^T \left(\frac{U_3(\Delta d^C) M_4}{M_1 M_5 - M_2 M_4}\right) F^E A_\beta^{E,A} d\tau}{1 + \gamma^E \sigma_\beta^2 \int_0^T \alpha^E \left[\int_0^\tau A_\beta^{E,A} ds\right] A_\beta^{E,A} d\tau} \left[\int_0^\tau A_\beta^{E,A}(s)ds\right]
\end{aligned}
$$

so that yields *change* by

$$
\begin{aligned}
\Delta y_{t,\tau}^{E,A}(\Delta d^C) &= \frac{1}{\tau} \Delta C^{E,A}(\Delta d^C) \\
&= \frac{\gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T \left(\frac{U_3(\Delta d^C) M_4}{M_1 M_5 - M_2 M_4}\right) F^E A_r^{E,A} d\tau}{1 + \gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T \alpha^E \left[\int_0^\tau A_r^{E,A} ds\right] A_r^{E,A} d\tau} \left[\frac{1}{\tau}\int_0^\tau A_r^{C,A}(s)ds\right] \\
&\quad + \frac{\gamma^E \sigma_\beta^2 \int_0^T \left(\frac{U_3(\Delta d^C) M_4}{M_1 M_5 - M_2 M_4}\right) F^E A_\beta^{E,A} d\tau}{1 + \gamma^E \sigma_\beta^2 \int_0^T \alpha^E \left[\int_0^\tau A_\beta^{E,A} ds\right] A_\beta^{E,A} d\tau} \left[\frac{1}{\tau}\int_0^\tau A_\beta^{E,A}(s)ds\right]
\end{aligned}
$$

Proof of Proposition 14. (Demand shock transmission, market E → C)

Consider the impact of a demand shock $\Delta d^E(\tau)$ from market *E*. Suppose $\Delta d^E(\tau)$ induces a change of $\Delta X_{t,\tau}^C = \Delta\beta_t^C F^C$ and $\Delta X_{t,\tau}^E = \Delta\beta_t^E F^E$ in the cross-market arbitrageur's demand.

$$
C^{E,A\prime} - z_r^{E,A} A_r^{E,A} - z_\beta^{E,A} A_\beta^{E,A} + \frac{1}{2}[\sigma_r^2 + \mathrm{Var}(\Delta r)] {A_r^{E,A}}^2 + \frac{1}{2}\sigma_\beta^2 {A_\beta^{E,A}}^2 = 0
$$

From the above equation we know that

$$
\begin{aligned}
\Delta\left(C^{E,A\prime}\right) &= \Delta\left(z_r^{E,A}\right) A_r^{E,A} + \Delta\left(z_\beta^{E,A}\right) A_\beta^{E,A} \\
&= \frac{\gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)] A_r^{E,A} \int_0^T (\Delta d^E - F^E \Delta\beta_t^E) A_r^{E,A} d\tau}{1 + \gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T \alpha^E \left[\int_0^\tau A_r^{E,A} ds\right] A_r^{E,A} d\tau} + \frac{\gamma^E \sigma_\beta^2 A_\beta^{E,A} \int_0^T (\Delta d^E - F^E \Delta\beta_t^E) A_\beta^{E,A} d\tau}{1 + \gamma^E \sigma_\beta^2 \int_0^T \alpha^E \left[\int_0^\tau A_\beta^{E,A} ds\right] A_\beta^{E,A} d\tau}
\end{aligned}
$$

Substitute into (3.3.2) and equate the induced changes on the L.H.S. and the R.H.S. to get

$$
\begin{aligned}
&\frac{\gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)] A_r^{E,A} \int_0^T (\Delta d^E - F^E \Delta\beta_t^E) A_r^{E,A} d\tau}{1 + \gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)] \int_0^T \alpha^E \left[\int_0^\tau A_r^{E,A} ds\right] A_r^{E,A} d\tau} + \frac{\gamma^E \sigma_\beta^2 A_\beta^{E,A} \int_0^T (\Delta d^E - F^E \Delta\beta_t^E) A_\beta^{E,A} d\tau}{1 + \gamma^E \sigma_\beta^2 \int_0^T \alpha^E \left[\int_0^\tau A_\beta^{E,A} ds\right] A_\beta^{E,A} d\tau} \\
&= \gamma^A[\sigma_r^2 + \mathrm{Var}(\Delta r)] A_r^{E,A} \left(\Delta\beta_t^E \int_0^T F^E A_r^{E,A} d\tau + \Delta\beta_t^C \int_0^T F^C A_r^{C,A} d\tau\right) + \gamma^A \sigma_\beta^2 A_\beta^{E,A} \left(\Delta\beta_t^E \int_0^T F^E A_\beta^{E,A} d\tau\right)
\end{aligned}
$$

Equate the induced changes on the LHS and the RHS in (3.3.1) to get

$$-\frac{\gamma^C \Delta\beta_t^C \int_0^T A_r^{C,A} F^C d\tau}{1+\gamma^C[\sigma_r^2+\mathrm{Var}(\Delta r)]\int_0^T \alpha^C\left[\int_0^\tau A_r^{C,A}(s)ds\right]A_r^{C,A}d\tau} = \gamma^A\left(\Delta\beta_t^C\int_0^T A_r^{C,A}F^C d\tau + \Delta\beta_t^E\int_0^T A_r^{E,A}F^E d\tau\right)$$

The above two equations can be written as

$$\begin{aligned} V_1\Delta\beta_t^C + V_2\Delta\beta_t^E &= V_3(\Delta d^E) \\ V_4\Delta\beta_t^C + V_5\Delta\beta_t^E &= 0 \end{aligned}$$

where

$$\begin{aligned}
V_1 &\equiv \gamma^A[\sigma_r^2+\mathrm{Var}(\Delta r)]\left(\int_0^T F^C A_r^{C,A}d\tau\right)A_r^{E,A} = M_4 = U_4 \\
V_2 &\equiv \left\{\gamma^A + \frac{\gamma^E}{1+\gamma^E[\sigma_r^2+\mathrm{Var}(\Delta r)]\int_0^T \alpha^E\left[\int_0^\tau A_r^{E,A}ds\right]A_r^{E,A}d\tau}\right\}[\sigma_r^2+\mathrm{Var}(\Delta r)]\left(\int_0^T F^E A_r^{E,A}d\tau\right)A_r^{E,A} \\
&\quad + \left\{\gamma^A + \frac{\gamma^E}{1+\gamma^E\sigma_\beta^2\int_0^T \alpha^E\left[\int_0^\tau A_\beta^{E,A}ds\right]A_\beta^{E,A}d\tau}\right\}\sigma_\beta^2\left(\int_0^T F^E A_\beta^{E,A}d\tau\right)A_\beta^{E,A} = M_5 = U_5 \\
V_3(\Delta d^E) &\equiv \frac{\gamma^E}{1+\gamma^E[\sigma_r^2+\mathrm{Var}(\Delta r)]\int_0^T \alpha^E\left[\int_0^\tau A_r^{E,A}ds\right]A_r^{E,A}d\tau}[\sigma_r^2+\mathrm{Var}(\Delta r)]\left(\int_0^T \Delta d^E A_r^{E,A}d\tau\right)A_r^{E,A} \\
&\quad + \frac{\gamma^E}{1+\gamma^E\sigma_\beta^2\int_0^T \alpha^E\left[\int_0^\tau A_\beta^{E,A}ds\right]A_\beta^{E,A}d\tau}\sigma_\beta^2\left(\int_0^T \Delta d^E A_\beta^{E,A}d\tau\right)A_\beta^{E,A} \\
V_4 &\equiv \left\{\gamma^A + \frac{\gamma^C}{1+\gamma^C[\sigma_r^2+\mathrm{Var}(\Delta r)]\int_0^T \alpha^C\left[\int_0^\tau A_r^{C,A}(s)ds\right]A_r^{C,A}d\tau}\right\}[\sigma_r^2+\mathrm{Var}(\Delta r)]\left(\int_0^T A_r^{C,A}F^C d\tau\right)A_r^{E,A} = M_1 = U_1 \\
V_5 &\equiv \gamma^A[\sigma_r^2+\mathrm{Var}(\Delta r)]\left(\int_0^T A_r^{E,A}F^E d\tau\right)A_r^{E,A} = M_2 = U_2
\end{aligned}$$

Following from the proof in Corollary 6,

$$\begin{aligned}
\Delta\beta_t^C &= \frac{V_3(\Delta d^E)V_5}{V_1V_5 - V_2V_4} = -\frac{V_3(\Delta d^E)M_2}{M_5M_1 - M_4M_2} < 0 \\
\Delta\beta_t^E &= -\frac{V_3(\Delta d^E)V_4}{V_1V_5 - V_2V_4} = \frac{V_3(\Delta d^E)M_1}{M_5M_1 - M_4M_2} > 0
\end{aligned}$$

Secondly, I investigate how the demand shock Δd^E in market E affects the term structures of market C and of market E. Because the mean-reversion parameters $\kappa_r^{C,A}$, $\kappa_r^{E,A}$ and $\kappa_\beta^{E,A}$ are independently pinned down and are not susceptible to supply and demand shocks, but only to shocks to the short-rate, supply and demand shocks affect the two term structures through the intercepts $C^{C,A}$ and $C^{E,A}$.

For market C, Δd^E induces a *change* of

$$\begin{aligned}
\Delta C^{C,A}(\Delta d^E) &= \Delta z_r^{C,A}\int_0^\tau A_r^{C,A}(s)ds \\
&= -\frac{\gamma^C[\sigma_r^2+\mathrm{Var}(\Delta r)]\int_0^T F^C\Delta\beta_t^C A_r^{C,A}d\tau}{1+\gamma^C[\sigma_r^2+\mathrm{Var}(\Delta r)]\int_0^T \alpha^C\left[\int_0^\tau A_r^{C,A}(s)ds\right]A_r^{C,A}d\tau}\int_0^\tau A_r^{C,A}(s)ds \\
&= \frac{\gamma^C[\sigma_r^2+\mathrm{Var}(\Delta r)]\int_0^T\left(\frac{V_3(\Delta d^E)M_2}{M_5M_1-M_4M_2}\right)F^C A_r^{C,A}d\tau}{1+\gamma^C[\sigma_r^2+\mathrm{Var}(\Delta r)]\int_0^T \alpha^C\left[\int_0^\tau A_r^{C,A}(s)ds\right]A_r^{C,A}d\tau}\left[\int_0^\tau A_r^{C,A}(s)ds\right]
\end{aligned}$$

so that yields *change* by

$$\Delta y_{t,\tau}^{C,A}(\Delta d^E) = \frac{1}{\tau}\Delta C^{C,A}(\Delta d^E) = \frac{\gamma^C[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^T \left(\frac{V_3(\Delta d^E)M_2}{M_5M_1 - M_4M_2}\right) F^C A_r^{C,A} d\tau}{1 + \gamma^C[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^T \alpha^C\left[\int_0^\tau A_r^{C,A}(s)ds\right]A_r^{C,A} d\tau}\left[\frac{1}{\tau}\int_0^\tau A_r^{C,A}(s)ds\right]$$

For market E, Δd^E induces a *change* of

$$\begin{aligned}
\Delta C^{E,A}(\Delta d^E) &= \Delta z_r^{E,A}\int_0^\tau A_r^{E,A}(s)ds + \Delta z_\beta^{E,A}\int_0^\tau A_\beta^{E,A}(s)ds \\
&= \frac{\gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^T (\Delta d^E - F^E\Delta\beta_t^E)A_r^{E,A}d\tau}{1 + \gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^T \alpha^E\left[\int_0^\tau A_r^{E,A}ds\right]A_r^{E,A}d\tau}\int_0^\tau A_r^{E,A}(s)ds \\
&\quad + \frac{\gamma^E\sigma_\beta^2\int_0^T (\Delta d^E - F^E\Delta\beta_t^E)A_\beta^{E,A}d\tau}{1 + \gamma^E\sigma_\beta^2\int_0^T \alpha^E\left[\int_0^\tau A_\beta^{E,A}ds\right]A_\beta^{E,A}d\tau}\int_0^\tau A_\beta^{E,A}(s)ds \\
&= \frac{\gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^T \left[\Delta d^E - \left(\frac{V_3(\Delta d^E)M_1}{M_5M_1 - M_4M_2}\right)F^E\right]A_r^{E,A}d\tau}{1 + \gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^T \alpha^E\left[\int_0^\tau A_r^{E,A}ds\right]A_r^{E,A}d\tau}\left[\int_0^\tau A_r^{E,A}(s)ds\right] \\
&\quad + \frac{\gamma^E\sigma_\beta^2\int_0^T \left[\Delta d^E - \left(\frac{V_3(\Delta d^E)M_1}{M_5M_1 - M_4M_2}\right)F^E\right]A_\beta^{E,A}d\tau}{1 + \gamma^E\sigma_\beta^2\int_0^T \alpha^E\left[\int_0^\tau A_\beta^{E,A}ds\right]A_\beta^{E,A}d\tau}\left[\int_0^\tau A_\beta^{E,A}(s)ds\right]
\end{aligned}$$

so that yields *change* by

$$\begin{aligned}
\Delta y_{t,\tau}^{E,A}(\Delta d^E) &= \frac{1}{\tau}\Delta C^{E,A}(\Delta d^E) \\
&= \frac{\gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^T \left[\Delta d^E - \left(\frac{V_3(\Delta d^E)M_1}{M_5M_1 - M_4M_2}\right)F^E\right]A_r^{E,A}d\tau}{1 + \gamma^C[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^T \alpha^E\left[\int_0^\tau A_r^{E,A}ds\right]A_r^{E,A}d\tau}\left[\frac{1}{\tau}\int_0^\tau A_r^{E,A}(s)ds\right] \\
&\quad + \frac{\gamma^E\sigma_\beta^2\int_0^T \left[\Delta d^E - \left(\frac{V_3(\Delta d^E)M_1}{M_5M_1 - M_4M_2}\right)F^E\right]A_\beta^{E,A}d\tau}{1 + \gamma^E\sigma_\beta^2\int_0^T \alpha^E\left[\int_0^\tau A_\beta^{E,A}ds\right]A_\beta^{E,A}d\tau}\left[\frac{1}{\tau}\int_0^\tau A_\beta^{E,A}(s)ds\right]
\end{aligned}$$

Proof of Theorem 1. (Shock internalization and transmission)

Part 1)

A cross-market arbitrageur's induced demand shocks due to a market-C supply shock are

$$\Delta\beta_{t,\tau}^C = \frac{M_3(\tau)M_5(\tau)}{M_1(\tau)M_5(\tau) - M_2(\tau)M_4(\tau)}\Delta S_{t,\tau^*}^C$$
$$\Delta\beta_{t,\tau}^E = -\frac{M_3(\tau)M_4(\tau)}{M_1(\tau)M_5(\tau) - M_2(\tau)M_4(\tau)}\Delta S_{t,\tau^*}^C$$

For infinitely risk-averse cross-market arbitrageur, i.e., $\gamma^A \to \infty$,

$$\lim_{\gamma^A\to\infty}\Delta\beta_{t,t}^C = \frac{\lim_{\gamma^A\to\infty} M_3(\tau)M_5(\tau)}{\lim_{\gamma^A\to\infty}[M_1(\tau)M_5(\tau) - M_2(\tau)M_4(\tau)]}\Delta S_{t,\tau^*}^C \sim \lim_{\gamma^A\to\infty}\frac{O(\gamma^A)}{O({\gamma^A}^2)} = 0$$

$$\lim_{\gamma^A\to\infty}\Delta\beta_{t,t}^E=\frac{\lim_{\gamma^A\to\infty}M_3(\tau)M_4(\tau)}{\lim_{\gamma^A\to\infty}[M_1(\tau)M_5(\tau)-M_2(\tau)M_4(\tau)]}\Delta S_{t,\tau^*}^C\sim\lim_{\gamma^A\to\infty}\frac{O(\gamma^A)}{O\left({\gamma^A}^2\right)}=0$$

Therefore, the *CMA*'s induced allocations are $X_{t,\tau}^C\equiv F^C(\tau,\gamma^A)\beta_t^C=0$ and $X_{t,\tau}^E\equiv F^E(\tau,\gamma^A)\beta_t^E=0$. The *CMA*'s inaction implies that the exogenous supply shock is fully internalized by the preferred-habitat investors in market *C*, leading to a maximal change in $\Delta y_{t,\tau}^{C,A}$. The market-*E* term structure is unaffected. From the proof of Proposition 11,

$$\lim_{\gamma^A\to\infty}\Delta y_{t,\tau}^{C,A}\left(\Delta S_{t,\tau^*}^C\right)=\frac{\gamma^C[\sigma_r^2+\mathrm{Var}(\Delta r)]\int_0^T A_r^{C,A}d\tau}{1+\gamma^C[\sigma_r^2+\mathrm{Var}(\Delta r)]\int_0^T\alpha^C\left[\int_0^\tau A_r^{C,A}(s)ds\right]A_r^{C,A}d\tau}\left[\frac{1}{\tau}\int_0^\tau A_r^{C,A}(s)ds\right]\Delta S_{t,\tau^*}^C$$

$$\lim_{\gamma^A\to\infty}\Delta y_{t,\tau}^{C,E}\left(\Delta S_{t,\tau^*}^C\right)=0$$

Part 2)

If the cross-market arbitrageur is risk-neutral, i.e., $\gamma^A=0$, $M_2=0=M_4$ and $M_1=M_3$,

$$\Delta\beta_{t,t}^C\big|_{\gamma^A=0}=\frac{M_3(\tau)M_5(\tau)}{M_1(\tau)M_5(\tau)-M_2(\tau)M_4(\tau)}\Delta S_{t,\tau^*}^C\bigg|_{\gamma^A=0}=\Delta S_{t,\tau^*}^C$$

$$\Delta\beta_{t,t}^E\big|_{\gamma^A=0}=\frac{M_3(\tau)M_4(\tau)}{M_1(\tau)M_5(\tau)-M_2(\tau)M_4(\tau)}\Delta S_{t,\tau^*}^C\bigg|_{\gamma^A=0}=0$$

Therefore, the *CMA* absorbs fully the exogenous market-*C* supply shock, leaving the term structure of market *C* unaffected. As the shock is not transmitted to market *E*, the term structure of market *E* is also unaffected.

Part 3)

If the cross-market arbitrageur is finitely risk-averse, *i.e.* neither risk-neutral nor infinitely risk-averse, she internalizes a certain percentage of the market-*C* supply shock

$$\Delta\beta_{t,t}^C\big|_{\gamma^A\in(0,\infty)}=\frac{M_3(\tau)M_5(\tau)}{M_1(\tau)M_5(\tau)-M_2(\tau)M_4(\tau)}\Delta S_{t,\tau^*}^C\bigg|_{\gamma^A\in(0,\infty)}\in\left(0,\Delta S_{t,\tau^*}^C\right)$$

The induced demand shock in market *E* generated from the transmitted supply shock through the *CMA* to market *E* is

$$\Delta\beta_{t,t}^C\big|_{\gamma^A\in(0,\infty)}=\frac{M_3(\tau)M_4(\tau)}{M_1(\tau)M_5(\tau)-M_2(\tau)M_4(\tau)}\Delta S_{t,\tau^*}^C\bigg|_{\gamma^A\in(0,\infty)}\in(0,\infty)$$

Thus, the market-*C* supply shock is transmitted partially to market *E* when the *CMA* is finitely risk-averse.

Proof of Theorem 2. (CMA's intervention)

Part 1)

For infinitely risk-averse cross-market arbitrageur, i.e., $\gamma^A\to\infty$, consider the situation with γ^A approaching infinity. To maintain the right hand sides of equations (3.3.1) and (3.3.2) finite, her optimal allocations in the two markets are both zero: ${X^{C}_{t,\tau}}^* = 0, {X^{E}_{t,\tau}}^* = 0$. The two markets remain *completely* segmented, and the gap between the two term structures remains unchanged.

Part 2)

If the cross-market arbitrageur is risk-neutral, i.e., $\gamma^A = 0$, her utility function becomes $U(W) = W$. Since infinite short-selling is allowed, she will buy as much as she can the cheaper sovereign bond in the market where the yield curve is higher and short as much as she can the more expensive one in the other market. The cross-market arbitrageur's optimal allocations are to make the two yield curves collapse, until there is no more room to arbitrage from. Thus, ${X^{C}_{t,\tau}}^*$ and ${X^{E}_{t,\tau}}^*$ solve

$$C^{E,A}(\tau) + A_r^{E,A}(\tau)r_t + A_\beta^{E,A}(\tau)\beta_t = C^{C,A}(\tau) + A_r^{C,A}(\tau)r_t$$

Express the constant terms $C^{C,A}(\tau)$ and $C^{E,A}(\tau)$ in terms of $X^C_{t,\tau}$ and $X^E_{t,\tau}$ to get

$$\left[z_r^{E,A}\left(X^E_{t,\tau}\right)\int_0^\tau A_r^{E,A}ds + z_\beta^{E,A}\left(X^E_{t,\tau}\right)\int_0^\tau A_\beta^{E,A}ds - \frac{1}{2}[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^\tau {A_r^{E,A}}^2 ds - \frac{1}{2}\sigma_\beta^2\int_0^\tau {A_\beta^{E,A}}^2 ds\right] + A_r^{E,A}r_t + A_\beta^{E,A}\beta_t$$
$$= \left[z_r^{C,A}\left(X^C_{t,\tau}\right)\int_0^\tau A_r^{C,A}ds - \frac{1}{2}[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^\tau {A_r^{C,A}}^2 ds\right] + A_r^{C,A}r_t$$

Suppose $C^{E,A}(\tau) + A_r^{E,A}(\tau)r_t + A_\beta^{E,A}(\tau)\beta_t > C^{C,A}(\tau) + A_r^{C,A}(\tau)r_t$, the cross-market arbitrageur longs the bonds in market *E* and shorts the bonds in market *C*, so that $X^C_{t,\tau} < 0$ and $X^E_{t,\tau} > 0$. Substitute $X^C_{t,\tau} = -X^E_{t,\tau}$ into the above equation to get

$$\left[z_r^{E,A}\left(X^E_{t,\tau}\right)\int_0^\tau A_r^{E,A}ds + z_\beta^{E,A}\left(X^E_{t,\tau}\right)\int_0^\tau A_\beta^{E,A}ds - \frac{1}{2}[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^\tau {A_r^{E,A}}^2 ds - \frac{1}{2}\sigma_\beta^2\int_0^\tau {A_\beta^{E,A}}^2 ds\right] + A_r^{E,A}r_t + A_\beta^{E,A}\beta_t$$
$$= \left[z_r^{C,A}\left(-X^E_{t,\tau}\right)\int_0^\tau A_r^{C,A}ds - \frac{1}{2}[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^\tau {A_r^{C,A}}^2 ds\right] + A_r^{C,A}r_t$$

Note that $z_r^{E,A}$, $z_\beta^{E,A}$ and $z_r^{C,A}$ are linear in $X^E_{t,\tau}$, and are thus linear in β_t^E and can be written as

$$z_r^{E,A}(\beta_t^E) = b_r^{E,A}\beta_t^E + d_r^{E,A}$$
$$z_\beta^{E,A}(\beta_t^E) = b_\beta^{E,A}\beta_t^E + d_\beta^{E,A}$$
$$z_r^{C,A}(\beta_t^E) = b_r^{C,A}\beta_t^E + d_r^{C,A}$$

where $b_r^{E,A}$, $b_\beta^{E,A}$ and $b_r^{C,A}$ are all positive. Thus, collecting terms to get a closed-form expression of $X^E_{t,\tau}$:

$${X^E_{t,\tau}}^* = F^E{\beta_t^E}^* = \frac{F^E}{-b_r^{C,A}\int_0^\tau A_r^{C,A}ds + b_r^{E,A}\int_0^\tau A_r^{E,A}ds + b_\beta^{E,A}\int_0^\tau A_\beta^{E,A}ds}\Bigg\{A_r^{C,A}r_t - A_r^{E,A}r_t - A_\beta^{E,A}\beta_t$$
$$+\frac{1}{2}[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^\tau {A_r^{E,A}}^2 ds + \frac{1}{2}\sigma_\beta^2\int_0^\tau {A_\beta^{E,A}}^2 ds - \frac{1}{2}[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^\tau {A_r^{C,A}}^2 ds$$
$$+ d_r^{C,A}\int_0^\tau A_r^{C,A}ds - d_r^{E,A}\int_0^\tau A_r^{E,A}ds - d_\beta^{E,A}\int_0^\tau A_\beta^{E,A}ds\Bigg\}$$

Part 3)

If the cross-market arbitrageur is risk-averse, i.e., $\gamma^A > 0$, and infinity short-selling is allowed, she is able to arbitrage from the yield differential across the two markets, while the gap between the two term structures can never be fully closed. This corresponds to the standard case and is solved following the steps in the case of weak-form segmentation. In the end, the two induced demand shocks are solved from the following two equations:

$$\gamma^C \int_0^T A_r^{C,A}\left[S_{t,\tau}^C - F^C\beta_t^C - \alpha^C y_{t,\tau}^{C,A}(\beta_t^C) + d^C\right]d\tau = \gamma^A\left(\beta_t^C \int_0^T F^C A_r^{C,A} d\tau + \beta_t^E \int_0^T F^E A_r^{E,A} d\tau\right)$$

$$\begin{aligned}\gamma^E[\sigma_r^2 + \mathrm{Var}(\Delta r)]A_r^{E,A} &\int_0^T A_r^{E,A}\left[S_{t,\tau}^E - F^E\beta_t^E - \alpha^E y_{t,\tau}^{E,A}(\beta_t^E) + \theta\beta_t + d^E\right]d\tau \\ &+ \gamma^E\sigma_\beta^2 A_\beta^{E,A}\int_0^T A_\beta^{E,A}\left[S_{t,\tau}^E - F^E\beta_t^E - \alpha^E y_{t,\tau}^{E,A}(\beta_t^E) + \theta\beta_t + d^E\right]d\tau \\ = \gamma^A[\sigma_r^2 + \mathrm{Var}(\Delta r)]A_r^{E,A}&\left(\beta_t^E\int_0^T F^E A_r^{E,A}d\tau + \beta_t^C\int_0^T F^C A_r^{C,A}d\tau\right) + \gamma^A\sigma_\beta^2 A_\beta^{E,A}\left(\beta_t^E\int_0^T F^E A_\beta^{E,A}d\tau\right)\end{aligned}$$

Note that $y_{t,\tau}^{C,A}$ is linear in $C^{C,A}$, which is linear in β_t^C. This implies that $y_{t,\tau}^{C,A}$ is linear in β_t^C, so is it true that $y_{t,\tau}^{E,A}$ is linear in β_t^E. Thus, closed-form solutions can be obtained from the above two equations.

Part 4)

If the cross-market arbitrageur is risk-neutral, i.e., $\gamma^A = 0$, and infinity short-selling is not allowed, she is still able to arbitrage from the yield differential across the two markets, while the gap between the two term structures can be either fully or partially closed. Suppose the cross-market arbitrageur cannot borrow more than a fraction λ of her endowment, such that $B_t \geq -\lambda W_t$. If the optimal demand ${X_{t,\tau}^E}^*$ in part 2) is less than λW_t, then she is able to short ${X_{t,\tau}^C}^* \geq -\lambda W_t$ in market C, in which case the two yield curves will collapse. If, however, ${X_{t,\tau}^E}^* > \lambda W_t$, she will not be able to short as much as she would like to and will choose to short λW_t.

In the latter case, the gap between the two term structures will only be partially closed. However, I am able to calculate the remaining yield differential after the cross-market arbitrageur intervenes. Suppose sovereign bonds in market E yields higher than those in market C, so that she will short the sovereign bonds in market C and long those in market E. Substitute ${X_{t,\tau}^E}^* = \lambda W_t$ and ${X_{t,\tau}^C}^* = -\lambda W_t$ into the expression to obtain the remaining un-closed gap between the two yield curves

$$\begin{aligned}\text{gap} &= {y_{t,\tau}^E}^*\left({X_{t,\tau}^C}^*, {X_{t,\tau}^E}^*\right) - {y_{t,\tau}^C}^*\left({X_{t,\tau}^C}^*, {X_{t,\tau}^E}^*\right) \\ &= \left[z_r^{E,A}\left({X_{t,\tau}^E}^*\right)\int_0^\tau A_r^{E,A}ds + z_\beta^{E,A}\left({X_{t,\tau}^E}^*\right)\int_0^\tau A_\beta^{E,A}ds - \frac{1}{2}[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^\tau {A_r^{E,A}}^2 ds - \frac{1}{2}\sigma_\beta^2\int_0^\tau {A_\beta^{E,A}}^2 ds \right. \\ &\qquad \left. + A_r^{E,A}r_t + A_\beta^{E,A}\beta_t\right] - \left[z_r^{C,A}\left(-{X_{t,\tau}^E}^*\right)\int_0^\tau A_r^{C,A}ds - \frac{1}{2}[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^\tau {A_r^{C,A}}^2 ds + A_r^{C,A}r_t\right] \\ &= \left[z_r^{E,A}(\lambda W_t)\int_0^\tau A_r^{E,A}ds + z_\beta^{E,A}(\lambda W_t)\int_0^\tau A_\beta^{E,A}ds - \frac{1}{2}[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^\tau {A_r^{E,A}}^2 ds - \frac{1}{2}\sigma_\beta^2\int_0^\tau {A_\beta^{E,A}}^2 ds \right. \\ &\qquad \left. + A_r^{E,A}r_t + A_\beta^{E,A}\beta_t\right] - \left[z_r^{C,A}(-\lambda W_t)\int_0^\tau A_r^{C,A}ds - \frac{1}{2}[\sigma_r^2 + \mathrm{Var}(\Delta r)]\int_0^\tau {A_r^{C,A}}^2 ds + A_r^{C,A}r_t\right]\end{aligned}$$

	$\gamma^A = 0$	$\gamma^A \in (0, \infty)$	$\gamma^A \to \infty$
infinite short-selling	*fully*	*weak-form, partially*	*semi-form* (0,0)
$B_t \geq -\lambda W_t$	*fully or partially*	*weak-form, partially*	*semi-form* (0,0)