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Abstract

This paper develops and empirically implements an arbitrage-free, dynamic term structure model with "priced" factor and regime-shift risks. The risk factors are assumed to follow a discrete-time Gaussian process, and regime shifts are governed by a discrete-time Markov process with state-dependent transition probabilities. This model gives closed-form solutions for zero-coupon bond prices and an analytic representation of the likelihood function for bond yields. Using monthly data on U.S. Treasury zero-coupon bond yields, we document notable differences in the behaviours of the market prices of factor risk across high and low volatility regimes. Additionally, the state-dependence of the regime-switching probabilities is shown to capture an interesting asymmetry in the cyclical behaviour of interest rates. The shapes of the term structures of bond yield volatilities are also very different across regimes, with the well-known hump in volatility being largely a low-volatility regime phenomenon.

1 Introduction

This paper develops and empirically implements an arbitrage-free, dynamic term structure model (*DTSM*) with "priced" factor and regime-shift risks. The risk factors are assumed to follow a discrete-time Gaussian process, and regime shifts are governed by a discrete-time Markov process with state-dependent transition probabilities. Agents are assumed to know both the current state of the economy and the regime they are currently in. This leads to regime-dependent pricing kernels and an equilibrium term structure that reflects the risks of both changes in the state and shifts in regimes.

There is an extensive empirical literature on bond yields (particularly short-term rates) that suggests that "switching regime" models describe the historical interest rate data better than single-regime models (see, for example, Cecchetti, Lam, and Mark [1993], Gray [1996], Garcia and Perron [1996], and Ang and Bekaert [2002a]).[1] In spite of this evidence, largely for reasons of tractability, most of the empirical literature on *DTSM*s has continued to focus on single-regime models (see Dai and Singleton [2003] for a survey). Recently Naik and Lee [1997], Landen [2000], and Dai and Singleton [2003] have proposed continuous-time regime-switching *DTSM*s that yield close-form solutions for zero-coupon bond prices, but multi-factor versions of their models have yet to be implemented empirically.

We develop a discrete-time multi-factor *DTSM* in which (i) there are two regimes characterized by low (L) and high (H) volatility, (ii) the regime-shift probabilities $\pi^{\mathbb{P}ij}$ $(i, j = H, L)$ under the historical measure $\mathbb{P}$ depend on the risk-factors underlying changes in the shapes of the yield curve, and (iii) regime-shift (and factor) risks are priced. This model yields exact closed-form solutions for bond prices, and an analytic representation of the likelihood function that we use in our empirical analysis of U.S. Treasury zero-coupon bond yields.

We find that the accommodation of economy-wide regime-shift risk is important for understanding the nature of the market prices of factor (*MPF*) risks that underlie variation in expected excess returns on bonds.[2] Duffee [2002] and Dai and Singleton [2002] have shown that sufficiently persistent and variable factor risk premiums in single-regime affine *DTSM*s resolve the expectations puzzles summarized in Campbell and Shiller [1991]. Nevertheless, consistent with the descriptive evidence on regime-switching models, Figure 1 suggests that these single-regime models fail to accurately represent expected excess returns (and implicitly, factor risk premiums) in U.S. Treasury markets. The swings in excess returns are notably larger in the two-regime model $A_0^{RS}(3)$ for those periods with largest absolute excess returns (e.g., during the period of the "monetary experiment" in the early 1980's). On the other hand, during more "normal" times, variation in the excess returns appears larger in the

[1]Ang and Bekaert [2002b] suggest that the mixing of regime-dependent state processes inherent in our *DTSM* can potentially replicate the nonlinear conditional means of short-term yields documented by Ait-Sahalia [1996] and Stanton [1997]. While the non-parametric evidence for non-linearity in the short-rate process is somewhat controversial (see, e.g., Chapman and Pearson [2000]), the findings of Ang and Bekaert for a Gaussian autoregressive model of a short rate suggest that our regime-dependent state process introduces the flexibility to match such nonlinearity if it is present.

[2]The analyses by Pan [2002] and Liu, Longstaff, and Pan [2002] are, in different contexts, premised on a similar point.

single-regime model $A_0(3)$. We document subsequently that the source of these differences is the very different behaviours of the *MPF* risks in regimes H and L, a difference that (by construction) is absent from single-regime models. This observation is robust to whether or not the $\pi^{\mathbb{P}ij}$ are state-dependent.

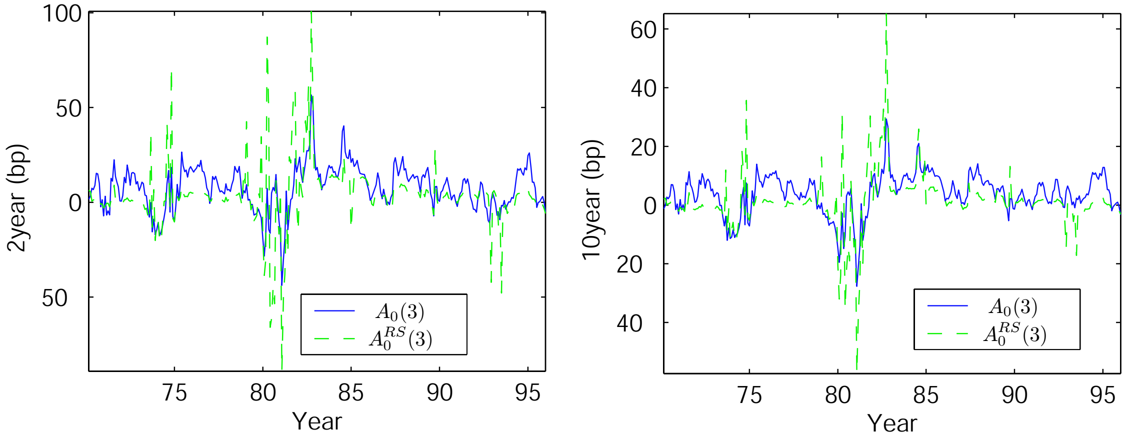

Figure 1: Expected excess returns on two- and ten-year zero-coupon Treasury bonds in Models $A_0(3)$ and $A_0^{RS}(3)$.

Where the state-dependence of $\pi^{\mathbb{P}ij}$ appears to matter most is in the persistence of regimes. A standard result in the empirical literature on regime switching models of interest rates with constant $\pi^{\mathbb{P}}$ (e.g., Ang and Bekaert [2002b] and Bansal and Zhou [2002]) is that $\pi_t^{\mathbb{P}HH} >> \pi_t^{\mathbb{P}HL}$ and $\pi^{\mathbb{P}LL} >> \pi^{\mathbb{P}LH}$; i.e., both regimes are highly persistent. On the other hand, with state-dependent $\pi^{\mathbb{P}}$, on average, we replicate the finding that $\pi^{\mathbb{P}LL} >> \pi^{\mathbb{P}LH}$, but now $\pi^{\mathbb{P}HL} > \pi^{\mathbb{P}HH}$– high volatility regimes are less persistent than low volatility regimes. Importantly, this asymmetry is equally present in a descriptive model of Treasury yields, suggesting that models (descriptive or pricing) that impose a constant $\pi^{\mathbb{P}}$ are missing an empirically important asymmetry in the cyclical behaviour of interest rates.

In developing our model we build upon a growing literature on *discrete-time DTSMs* by extending the Gaussian, discrete-time *DTSMs* in Bekaert and Grenadier [2001], Ang and Piazzesi [2002], and Gourieroux, Monfort, and Polimenis [2002] to allow for multiple regimes and priced regime-shift risk.[3] However, rather than adopting Hamilton [1989]'s convention of specifying the distribution of the state conditional on the *future* regime, we condition on the *current* regime. Under our convention, all of the conditioning variables at date t reside in agents' date t information set, which includes knowledge of the current regime. This leads to an intuitive interpretation of the components of agents' pricing kernel that parallels standard formulations in the continuous-time literature.

[3] To the extent that changes in regimes are related to business-cycle developments, multiple switches within a monthly, or even a quarterly, time frame seem unlikely. Therefore, we see little cost to a discrete-time framework, with the obvious benefit of being able to link our results directly with the descriptive literature on regime shifts in the distributions of interest rates.

Our analysis of a Gaussian *DTSM* is complementary to Bansal and Zhou [2002]'s study of an (approximate) discrete-time "*CIR*-style" *DTSM* with regime shifts. Model $A_0^{RS}(3)$ extends their framework by allowing for state-dependent $\pi_t^{\mathbb{P}}$ (Bansal and Zhou assumed that $\pi^{\mathbb{P}}$ = constant), and priced regime-shift risk (they assumed that the market price of regime-shift (*MPRS*) risk is zero).[4] Furthermore, though model $A_0^{RS}(3)$ precludes (by assumption) within-regime stochastic volatility, we find that it produces a level-dependence of the volatilities of yields conditional on the their past history. Finally, the added flexibility in the correlation structure of the risk factors in model $A_0^{RS}(3)$ (as contrasted with the independence in *CIR*-style models) allows us to replicate the well known hump in the term structure of volatility, and to explore the regime-dependence of the shape of this hump.

In a concurrent study, with a different objective, Ang and Bekaert [2003] also examine a regime-switching Gaussian *DTSM*.[5] They assume that regime-shift risk is not priced, $\pi^{\mathbb{P}}$ is constant, and the historical rates of mean reversion of the risk factors are the same across regimes. Model $A_0^{RS}(3)$ relaxes all of these assumptions thereby facilitating an exploration of the state-dependence of $\pi^{\mathbb{P}}$ and of the *MPRS* and *MPF* risks.

The remainder of this paper is organized as follows. Section 2 develops our model and derives the arbitrage-free bond pricing relations in the presence of regime shifts. We also compare the nature of the various market prices of risk in our setup to those in previous studies. The likelihood function that is used in estimation is derived in Section 3. Section 4 describes the data and presents the estimates of our models. The dependence of the model-implied market prices of risks on the shape of the yield curve and the regime of the economy are explored in more depth in Section 5. Finally, concluding remarks are presented in Section 6.

2 A Regime-Switching, Gaussian *DTSM*

There are $S+1$ "regimes" that govern the dynamic properties of the N-dimensional state (factor) vector Y_t. Formally, the joint process (Y, s) is modelled as a marked point process. Heuristically, the regime variable s_t may be thought of as a $(S+1)$-state Markov process, with the historical probability of switching from regime $s_t = j$ to regime $s_{t+1} = k$ given by $\pi_t^{\mathbb{P}jk}$, $0 \leq j, k \leq S$. In general $\pi_t^{\mathbb{P}jk}$ may be state-dependent (functions of Y_t), but the Markov process governing regime changes is assumed to be independent of the Y process. By definition, for all j, $\sum_{k=0}^{S} \pi_t^{\mathbb{P}jk} = 1$. Agents are presumed to know the current and past histories of *both* the N-dimensional state vector Y_t and regime the economy is in, s_t. Thus expectations $E_t[\cdot]$ are conditioned on the information set $\mathcal{I}_{t+1}$ generated by $\{Y_{t+1-\ell}, s_{t+1-\ell} : \ell \geq 0\}$. We use the notation $E_t[\cdot|s_t = j]$ in cases where we wish to highlight the current value of $s_t \in \mathcal{I}_t$.

Within regime $s_t = j$ the evolution of the economy under the physical (historical) measure

[4] As we explain more formally below, neither of our models is nested within the other with regard to the specifications of the *MPF* risks.

[5] In another related study, Wu and Zeng [2003] derive a general equilibrium, regime-switching model, building upon the one-factor *CIR*-style model of Naik and Lee [1997], with constant $\pi^{\mathbb{P}}$.

$\mathbb{P}$ is described by the discrete-time process

$$Y_{t+1} = \mu_t^{\mathbb{P}j} + \Sigma^j \epsilon_{t+1}, \tag{1}$$

where the conditional mean $\mu_t^{\mathbb{P}j}$ may depend on $\{Y_{t-\ell} : \ell \geq 0\}$, Σ^j is a constant conditional volatility matrix, and $\epsilon_{t+1} \sim N(0, I)$. We assume that the parameters determining $\mu_t^{\mathbb{P}j}$ and Σ^j depend on the regime $s_t = j$, but not on s_{t+1}, in which case

$$f(Y_{t+1}|Y_{t-\ell} : \ell \geq 0; s_t = j, s_{t+1} = k) = f(Y_{t+1}|Y_{t-\ell} : \ell \geq 0; s_t = j) \sim N(\mu_t^{\mathbb{P}j}, \Sigma^j \Sigma^{j\prime}); \tag{2}$$

equivalently, the conditional moment generating function (*MGF*) of Y_{t+1} is, given $s_t = j$,

$$E_t \left[e^{u' Y_{t+1}} \middle| s_t = j \right] = e^{u' \mu_t^{\mathbb{P}j} + \frac{u' \Sigma^j \Sigma^{j\prime} u}{2}}, \; u \in \mathbb{R}^N. \tag{3}$$

This differs from Hamilton [1989]'s formulation where $f(Y_{t+1}|Y_{t-k} : k \geq 0; s_{t+1} = k)$ is specified parametrically. As the sampling interval of the data shrinks toward zero (in the continuous time limit), these two formulations are equivalent. We adopt our discrete-time formulation for the tractability that it offers in constructing a *DTSM* with regime shifts.[6]

The pricing kernel underlying the time-t valuation of payoffs at date $t+1$ is denoted by $M_{t,t+1} = M(Y_t, s_t; Y_{t+1}, s_{t+1}) \in \mathcal{I}_{t+1}$. To accommodate regime-dependence of the pricing kernel, while staying within a discrete-time affine setting, we assume that

$$M_{t,t+1} = e^{-r_t - \Gamma_{t,t+1} - \frac{1}{2}\Lambda_t' \Lambda_t - \Lambda_t' \Sigma_t^{-1}(Y_{t+1} - \mu_t^{\mathbb{P}})}, \tag{4}$$

where $r_t = r(Y_t, s_t)$ is the one-period zero-coupon bond yield, $\Gamma_{t,t+1} = \Gamma(Y_t, s_t; s_{t+1})$ is the *MPRS* from s_t to s_{t+1}, $\Lambda_t = \Lambda(Y_t, s_t)$ is the *MPF* factor risk, $\mu_t^{\mathbb{P}} = \mu^{\mathbb{P}}(Y_t, s_t)$ is the conditional mean of Y_{t+1} and $\Sigma_t = \Sigma(s_t)$ is the conditional volatility of Y_t given current regime s_t. $M_{t,t+1} \in \mathcal{I}_{t+1}$ depends implicitly on the regimes (s_t, s_{t+1}), because agents know both the current regime s_{t+1} and the regime from which they transitioned, s_t.[7] For later development, we define $r_t^j \equiv r(Y_t, s_t = j)$, $\Gamma_t^{jk} \equiv \Gamma(Y_t, s_t = j; s_{t+1} = k)$, $\Lambda_t^j \equiv \Lambda(Y_t, s_t = j)$, $\mu_t^{\mathbb{P}j} \equiv \mu^{\mathbb{P}}(Y_t, s_t = j)$, and $\Sigma^j \equiv \Sigma(s_t = j)$.

Interpreting our formulation of the pricing kernel is facilitated by introducing the risk-neutral pricing measure $\mathbb{Q}$ for this setting. To this end, consider a security with payoff $P_{t+1} \equiv P(Y_{t+1}, s_{t+1})$. No arbitrage implies that its price at time t in regime $s_t = j$, P_t^j, is

$$P_t^j = E_t \left[M_{t,t+1} P_{t+1} | \, s_t = j \right] = e^{-r_t^j} E_t^{\mathbb{Q}} \left[P_{t+1} | \, s_t = j \right], \tag{5}$$

[6] A similar timing convention was adopted by Cecchetti, Lam, and Mark [1993] in their descriptive study of equity returns. In the context of descriptive regime-switching model (i.e., there is no pricing), the specification (2) and Hamilton's specification lead to identical likelihood functions, except for the interpretation of the initial values of certain conditional regime probabilities. Once pricing is introduced, the interpretations of the pricing kernels are not the same for reasons discussed subsequently.

[7] Expression (4) can be constructed by extending the specification of the exponential affine pricing kernel used by Gourieroux, Monfort, and Polimenis [2002] to $M_{t,t+1}^{j,s_{t+1}} = e^{\gamma_t^j + \lambda_t^{j\prime} Y_{t+1}}$. This differs from their single-regime formulation both in the regime dependence of $(\gamma_t^j, \lambda_t^j)$ and in the state-dependence of λ_t^j. The empirical relevance of these extensions is discussed subsequently.

where the risk-neutral measure $\mathbb{Q}$ is defined by[8]

$$\mathbb{Q}(dY_{t+1}, s_{t+1}|\mathcal{I}_t) = e^{-\Gamma_{t,t+1}} \times e^{-\frac{1}{2}\Lambda_t'\Lambda_t - \Lambda_t'\Sigma_t^{-1}(Y_{t+1}-\mu_t^{\mathbb{P}})} \times \mathbb{P}(dY_{t+1}, s_{t+1}|\mathcal{I}_t). \tag{6}$$

Under $\mathbb{Q}$, $\mu_t^{\mathbb{Q}} \equiv E_t^{\mathbb{Q}}[Y_{t+1}]$ is given by

$$\mu_t^{\mathbb{Q}} = \int Y_{t+1} \times e^{-\frac{1}{2}\Lambda_t'\Lambda_t - \Lambda_t'\Sigma_t^{-1}(Y_{t+1}-\mu_t)} \times \mathbb{P}(dY_{t+1}|I_t) = \frac{\partial}{\partial u} \log E_t[e^{u'Y_{t+1}}]\bigg|_{u=-\Sigma_t'^{-1}\Lambda_t}. \tag{7}$$

Substitution of (3) gives

$$\mu_t^{\mathbb{Q}} = \mu_t^{\mathbb{P}} - \Sigma_t\Lambda_t. \tag{8}$$

Similarly, under $\mathbb{Q}$, the regime switching probabilities are given by

$$\pi_t^{\mathbb{Q}jk} = E_t^{\mathbb{Q}}\left[1_{\{s_{t+1}=k\}}\middle|\, s_t = j\right] = \pi_t^{\mathbb{P}jk} e^{-\Gamma_t^{jk}}. \tag{9}$$

Thus given the physical measure $\mathbb{P}$, Λ_t and $\Gamma_{t,t+1}$ completely determine the risk-neutral measure $\mathbb{Q}$, and vice versa.

No arbitrage requires that $E_t[M_{t,t+1}|s_t = j] = e^{-r_t^j}$. Substituting (4), and using the *MGF* (3) of Y_{t+1}, it follows that

$$1 = E_t\left[e^{-\Gamma_{t,t+1}}\middle|\, s_t = j\right] = \sum_{k=0}^{S} \pi_t^{\mathbb{P}jk} e^{-\Gamma_t^{jk}},\ 0 \leq \forall j \leq S. \tag{10}$$

From (9) it follows that $\pi^{\mathbb{Q}jk} = \pi_t^{\mathbb{P}jk} e^{-\Gamma_t^{jk}}$. Therefore, the $(S+1)$ no-arbitrage restrictions (10) hold for any parameterisation that imposes (9) and $\sum_k \pi^{\mathbb{Q}jk} = 1$.

Equipped with $\mathbb{Q}$, we next link the market prices of risk to equilibrium expected excess returns. The security with payoff $e^{-b'Y_{t+1}}$, which has exposure only to factor risks at date $t+1$, has price

$$P_t^j = e^{-r_t^j} E_t^{\mathbb{Q}}[e^{-b'Y_{t+1}}|s_t = j] = e^{-r_t^j} e^{-b'\mu_t^{\mathbb{Q}j} + \frac{1}{2}b'\Sigma^j\Sigma^{j\prime}b} \tag{11}$$

and $\mathbb{P}$-expected payoff $E_t[e^{-b'Y_{t+1}}] = e^{-b'\mu_t^{\mathbb{P}j} + \frac{1}{2}b'\Sigma^j\Sigma^{j\prime}b}$ in regime $s_t = j$. Therefore, the expected excess return (continuously compounded) for this security is

$$\log \frac{E_t[e^{-b'Y_{t+1}}|s_t = j]}{P_t^j} - r_t^j = -b'\Sigma^j\Lambda_t^j. \tag{12}$$

Since $b'\Sigma^j$ is the "risk exposure" or volatility of the security associated with the factor risk, Λ_t^j – the *MPF* risk in regime $s_t = j$ – gives the expected excess return per unit of factor risk exposure.

Next, consider a security with payoff $1_{\{s_{t+1}=k\}}$, which has exposure only to the risk of shifting to regime k at date $t+1$. Conditional on the current regime $s_t = j$, its (risk-neutral)

[8] $\mathbb{Q}$ is strictly positive whenever $\mathbb{P}$ is strictly positive, and $\sum_{s_{t+1}} \int dY_{t+1}\mathbb{Q}(dY_{t+1}, s_{t+1}|I_t) = 1$. Thus, $\mathbb{Q}$ is an equivalent measure to $\mathbb{P}$.

expected payoff is $\pi_t^{\mathbb{Q}jk}$, and its current price is $P_t^j = e^{-r_t^j}\pi_t^{\mathbb{Q}jk}$. Thus, its expected excess return (continuously compounded) is given by

$$\log \frac{E_t[1_{\{s_{t+1}=k\}}|s_t = j]}{P_t^j} - r_t^j = \Gamma_t^{jk}. \tag{13}$$

That is, $\Gamma_{t,t+1}^{jk}$ is naturally defined as the *MPRS* risk from regime j to regime k.

To derive a regime-switching *DTSM* that gives closed-form solutions for zero-coupon bond prices,[9] we impose further structure on the dependence of r_t^j on Y_t and on the risk neutral distribution of (Y_{t+1}, s_{t+1}). Specifically, we assume that r_t is an affine function of Y_t:

Assumption Ar: $r_t^j = \delta_0^j + \delta_Y' Y_t$.

The regime-dependence of δ_0^j implies that the long-run mean of r_t^j is allowed to change under both the $\mathbb{P}$ and $\mathbb{Q}$ measures. However, to facilitate bond pricing, we constrain the "loadings" δ_Y on Y in this expression for r_t^j to be the same across regimes.

Additionally, the risk-neutral drifts of Y and risk-neutral regime-shift probabilities are assumed to satisfy:

Assumption A$\mu^{\mathbb{Q}}$: $\mu_t^{\mathbb{Q}j} = Y_t + \kappa^{\mathbb{Q}}(\theta^{\mathbb{Q}j} - Y_t)$, for constant $\theta^{\mathbb{Q}j} \in \mathbb{R}^N$, $j = 0, \ldots, S$ and $N \times N$ constant matrix $\kappa^{\mathbb{Q}}$ with $\kappa_{ij}^{\mathbb{Q}} <\in \mathbb{R}$;

Assumption A$\pi^{\mathbb{Q}}$: the $\pi_t^{\mathbb{Q}jk}$ are constants, for all j and k.

Assumption $(A\mu^{\mathbb{Q}})$ allows the long-run mean of Y under $\mathbb{Q}$ to be regime-dependent, while imposing the constraint that the state-dependent component of $\mu^{\mathbb{Q}j}$ is common across regimes.

These assumptions give us "risk-neutral" pricing. Specifically, let $D_{t,n} = D_n(Y_t, s_t)$ denote the time-t price on a zero-coupon bond with maturity of n periods, and $D_{t,n}^j$ denote the price when the current regime is $s_t = j$.

Proposition 1 (Zero-Coupon Bond Prices) *Assuming that Y_{t+1} follows the process (1) and Assumptions (Ar), $(A\mu^{\mathbb{Q}})$, and $(A\pi^{\mathbb{Q}})$ hold, zero-coupon bond prices are given by*

$$D_{t,n}^j = e^{-A_n^j - B_n' Y_t}, \tag{14}$$

$$\textit{where, } A_{n+1}^j = \delta_0^j + (\kappa^{\mathbb{Q}}\theta^{\mathbb{Q}j})' B_n - \frac{1}{2} B_n' \Sigma^j \Sigma^{j\prime} B_n - \log\left(\sum_{k=0}^{S} \pi^{\mathbb{Q}jk} e^{-A_n^k}\right), \tag{15}$$

$$B_{n+1} = \delta_Y + B_n - \kappa^{\mathbb{Q}\prime} B_n, \tag{16}$$

with initial conditions: $A_0^j = 0$ *and* $B_0 = \mathbf{0}$.

[9] As discussed in Dai and Singleton [2003] for the case of a continuous-time, regime-switching model, the affine structure does not in general admit closed-form solutions for bond prices in the presence of regime shifts. The additional structure imposed here parallels the restrictions imposed in Dai and Singleton for their continuous-time model.

Proof: Substituting (14) into the risk-neutral pricing equation,

$$D^j_{t,n+1} = E^{\mathbb{Q}}_t \left[e^{-r^j_t} D_{t+1,n} \middle| s_t = j \right],$$

we have

$$e^{-A^j_{n+1} - B'_{n+1} Y_t} = E^{\mathbb{Q}}_t \left[e^{-r^j_t} D_{t+1,n} | s_t = j \right] = e^{-r^j_t} \sum_{k=0}^{S} \pi^{\mathbb{Q}jk} E^{\mathbb{Q}}_t \left[D^k_{t+1,n} | s_t = j \right]$$

$$= \; e^{-r^j_t} \sum_{k=0}^{S} \pi^{\mathbb{Q}jk} e^{-A^k_n} E^{\mathbb{Q}}_t \left[e^{-B'_n Y_{t+1}} | s_t = j \right] = e^{-r^j_t} \left[\sum_{k=0}^{S} \pi^{\mathbb{Q}jk} e^{-A^k_n} \right] e^{-B'_n \mu^{\mathbb{Q}j} + \frac{1}{2} B'_n \Sigma^j \Sigma^{j\prime} B_n}.$$

Equations (15) and (16) are necessary and sufficient for the above equation to hold for any Y_t and $s_t = j$. It is easy to check that $A^j_0 = 0$, $B^j_0 = \mathbf{0}$, $A^j_1 = \delta^j_0$, and $B^j_1 = \delta_1$ satisfy the recursion. Thus, the recursion can start either at $n = 0$ or at $n = 1$.

To complete our econometric model of bond prices, it remains to specify the market prices of factor and regime-shift risks. Importantly, Assumption ($A\mu^{\mathbb{Q}}$) does *not* constrain the state- or regime-dependence of $\mu^{\mathbb{P}j}_t$. Given our parameterisation of $\mu^{\mathbb{Q}j}_t$ and the regime-dependence of Σ^j, we can match any desired state- and regime-dependence of $\mu^{\mathbb{P}j}_t$ under $\mathbb{P}$,

$$\mu^{\mathbb{P}j}_t = \mu^{\mathbb{Q}j}_t + \Sigma^j \Lambda^j_t, \; 0 \leq j \leq S, \tag{17}$$

by appropriate choice of the market prices of factor risks, Λ^j_t. Indeed, there is no requirement that $\mu^{\mathbb{P}j}_t$ be affine in Y_t.[10] Similarly Assumption ($A\pi^{\mathbb{Q}}$) does *not* restrict the state-dependence of $\pi^{\mathbb{P}jk}_t$. Given the $\pi^{\mathbb{Q}jk}$, by appropriate choice of the Γ^{jk}_t, we can match any desired state-dependence of the $\pi^{\mathbb{P}jk}_t$.

In our parametric *DTSM*, we extend Duffee [2002]'s essentially affine, Gaussian model to the case of multiple regimes by assuming that

$$\Lambda^j_t = \left(\Sigma^j \right)^{-1} \left(\lambda^j_0 + \lambda^j_Y Y_t \right). \tag{18}$$

Duffee [2002] and Dai and Singleton [2002] found that $A_0(3)$ models with *MPF* risks given by (18) (without the regime index) were able to match many features of historical expected excess returns on bonds. This formulation extends the specifications of the *MPF* risks in the $A_0(3)$ models of Duffee [2002], Dai and Singleton [2002], Ang and Piazzesi [2002], and Gourieroux, Monfort, and Polimenis [2002] by allowing both λ^j_0 and λ^j_Y to be regime-dependent, and it extends the regime-switching model of Naik and Lee [1997] by allowing for non-zero, regime-dependent λ^j_Y.[11]

[10] This same flexibility is, of course, present in single-regime, affine *DTSM*s. The assumption that the drift of the state is affine under both measures $\mathbb{P}$ and $\mathbb{Q}$ has been made for convenience in formulating estimation strategies.

[11] Relative to continuous-time Gaussian models, our discrete-time setting embodies the added flexibility of allowing $\mu^{\mathbb{P}j}_t$ to depend on a distributed lag function of $(Y_t, Y_{t-1}, \ldots, Y_{t-\ell})$, $\ell > 0$, and not only on Y_t. However, we do not explore this flexibility in the econometric analysis in this paper.

We are free to choose any parameterisations of state-dependent historical regime-shift probabilities $\pi^{\mathbb{P}}$ and *MPRS* risks Γ_t, subject to the constraint that (10) is satisfied. Following Gray [1996], Boudoukh, Richardson, Smith, and Whitelaw [1999], and many subsequent studies, we assume that (for two-regime case)

$$\pi_t^{\mathbb{P}jk} = \frac{1}{1 + e^{\eta_0^{jk} + \eta_Y^{jk} \cdot Y_t}}, \; j \neq k, \; \pi^{\mathbb{P}jj} = 1 - \sum_{k \neq j} \pi^{\mathbb{P}jk}. \tag{19}$$

Then, to assure that the no-arbitrage constraints are satisfied, we parameterise the *MPRS* risks as

$$\Gamma_t^{jk} = \log\left(\frac{\pi_t^{\mathbb{P}jk}}{\pi^{\mathbb{Q}jk}}\right), \; \forall j, k. \tag{20}$$

The unknown parameters to be estimated are the (constant) risk-neutral regime-shift probabilities $\pi^{\mathbb{Q}jk}$ and the $N \times 1$ vectors η_0^{jk} and $N \times N$ matrices η_Y^{jk}. Unlike in descriptive regime-switching models for interest rates, the elements of $\pi^{\mathbb{P}}$ in our *DTSM* depend directly on the latent risk factors Y.

The state-dependence of the Γ_t^{jk} implied by (19) and (20) is key to achieving our objective of an improved understanding of the nature of market prices of risk, and regime-shift risk in particular. As in Naik and Lee [1997] and Bansal and Zhou [2002], we assume that the $\pi^{\mathbb{Q}jk}$ are constant (Assumption ($A\pi^{\mathbb{Q}}$)). However, these studies also assume that regime-shift risk is not priced ($\Gamma_t^{jk} = 0$). The latter assumption means that the state- and regime-dependence of Λ_t^j and Σ^j (the volatility matrix of Y) must explain the time-series properties of expected excess returns. By allowing for priced regime-shift risk, we have added a third source of regime-dependence of expected returns.

A potential weakness of our Gaussian *DTSM*, relative to say multiple-regime versions of $A_M(N)$ *DTSM*s, with $0 < M$, is that the conditional variances of the Y's are constants. However, our experience with single-regime affine *DTSM*s is that the conditional volatility in bond yields induced by conditional volatility in the Y's is, in fact, very small relative to the volatility of excess returns. Furthermore, by overlaying regime shifts on top of a Gaussian state vector we introduce stochastic volatility into our *DTSM*, perhaps at least to the same degree as in square-root processes. The degree of time variation in the conditional variance of Y will depend on the nature of the state dependence of the $\pi_t^{\mathbb{P}}$ (equivalently, on the state dependence of the Γ_t^{jk}). Even in the case of constant $\pi_t^{\mathbb{P}}$, the conditional variances of bond yields will be time varying. This is the only source of time-varying volatility in models that assume that $\pi^{\mathbb{P}}$ is a constant matrix (e.g., Ang and Bekaert [2003]).

Additionally, Assumptions ($A\pi^{\mathbb{Q}}$) and (20) imply that our model cannot accommodate state-dependent regime-shift risk that is not priced. That is, we nest the special cases of $\pi^{\mathbb{Q}} = \pi^{\mathbb{P}} =$ constant with Γ^{jk} being either a non-zero constant (priced regime-shift risk) or zero (non-priced regime-shift risk). However, our formulation does not nest the case of state-dependent $\pi_t^{\mathbb{P}}$ with $\Gamma_t^{jk} = 0$. Nevertheless, we view the accommodation of state-dependent $\pi^{\mathbb{P}}$ and rich regime dependence of Λ_t^j as potentially important extensions of the literature on $A_0(3)$ models that are worthwhile exploring empirically.

Finally, a notable difference between our formulation and that in Bansal and Zhou [2002] and Ang and Bekaert [2003] is that we have assumed that $(\Lambda_t^j, \Gamma_t^{jk}) \in \mathcal{I}_t$, consistent with the continuous-time regime-switching model developed in Dai and Singleton [2003]. In contrast, using our notation for the one-factor case, these authors adopted the pricing kernel

$$M_{t,t+1} = \exp\left[-r_{f,t} - \lambda_{s_{t+1}}^2 \frac{Y_t}{2} - \lambda_{s_{t+1}}\epsilon_{t+1}\right], \tag{21}$$

in which $\lambda_{s_{t+1}}$ depends on s_{t+1}. In our formulation, the components Λ_t^j and Γ_t^{jk} can be directly interpreted as market prices of risk. On the other hand, under the formulation (21), $\lambda_{s_{t+1}}$ is not the *MPF* risk, since it is not in $\mathcal{I}_t$. The *MPF* risk depends on both $\lambda_{s_{t+1}}$ and the regime-switching probabilities $\pi_t^{\mathbb{P}jk}$.[12]

3 Maximum Likelihood Estimation

Given the Gaussian structure of the risk factors, we proceed with maximum likelihood (*ML*) estimation of the regime-switching *DTSM*s. Following common practice (e.g., Chen and Scott [1995], Duffie and Singleton [1997]), we assume that the yields on a collection of N zero-coupon bonds are priced without error, and the yields on a collection of M zero-coupon bonds are priced with error. Additionally, we let Δ denote the sampling interval (measured in years) of the data, and proceed to construct the likelihood function based on parameters in annual units.

Let $\hat{R}_t$ be the vector of yields for the bonds priced exactly by the model. In regime $s_t = j$, $\hat{R}_t = \hat{a}^j + \hat{b}Y_t^j$, where a^j is the $N \times 1$ regime-dependent vector with $a_n^j = A_n^j/n$, b is the $N \times N$ regime-independent matrix of factor loadings $[b_{ni} \equiv B_{ni}/n]$, and Y^j is the $N \times 1$ vector of state variables implied by the model. Inverting for fitted yields we obtain

$$Y_t^j = \hat{b}^{-1}(\hat{R}_t - \hat{a}^j). \tag{22}$$

Conditional on $s_t = j$ and $s_{t+1} = k$, we have

$$\begin{aligned} \hat{R}_{t+1} &= \hat{a}^k + \hat{b}Y_{t+1}^k = \hat{a}^k + \hat{b}\mu_t^{\mathbb{P}j} + \hat{b}\Sigma^j\sqrt{\Delta}\epsilon_{t+1} \\ &= \hat{R}_t + (\hat{a}^k - \hat{a}^j) + \hat{\kappa}^j(\hat{\theta}^j - \hat{R}_t)\Delta + \hat{\Sigma}^j\sqrt{\Delta}\epsilon_{t+1}, \end{aligned} \tag{23}$$

where, letting $\mu_t^{\mathbb{P}j} = Y_t + \kappa^{\mathbb{P}j}(\theta^{\mathbb{P}j} - Y_t)\Delta$, $\hat{\kappa}^j = \hat{b}\kappa^{\mathbb{P}j}\hat{b}^{-1}$, $\hat{\theta}^j = \hat{a}^j + \hat{b}\theta^{\mathbb{P}j}$, and $\hat{\Sigma}^j = \hat{b}\Sigma^j$. It

[12] More precisely, in their setting, the continuously compounded expected excess return for a security with regime-independent, time-$(t+1)$ payoff of $e^{-bY_{t+1}}$ is given by

$$\log \frac{\sum_{k=0}^{S} \pi^{\mathbb{P}jk} e^{-b\mu_t^k + \frac{Y_t}{2} b\Sigma^k\Sigma^k b}}{\sum_{k=0}^{S} \pi^{\mathbb{P}jk} e^{-b\mu_t^k + \frac{Y_t}{2} b\Sigma^k\Sigma^k b + b\lambda^k Y_t}}.$$

follows that

$$f(\hat{R}_{t+1}|\hat{R}_t, s_t = j, s_{t+1} = k)$$
$$= \frac{e^{-\frac{1}{2}(\hat{R}_{t+1}-\hat{R}_t-(\hat{a}^k-\hat{a}^j)-\hat{\kappa}^j(\hat{\theta}^j-\hat{R}_t)\Delta)'[\hat{\Sigma}^j\hat{\Sigma}^{j'}\Delta]^{-1}(\hat{R}_{t+1}-\hat{R}_t-(\hat{a}^k-\hat{a}^j)-\hat{\kappa}^j(\hat{\theta}^j-\hat{R}_t)\Delta)}}{\sqrt{(2\pi)^N\left|\hat{\Sigma}^j\hat{\Sigma}^{j\prime}\Delta\right|}}. \tag{24}$$

Notice that $f(\hat{R}_{t+1}|\hat{R}_t, s_t = j)$ is obtained by integrating out the dependence of (24) on s_{t+1}, so conditioning only on $s_t = j$ (and $\hat{R}_t$) gives a mixture of normals distribution.

The remaining M yields used in estimation are denoted by $\tilde{R}_t$, with corresponding loadings $\tilde{a}^j$ and $\tilde{b}$ when $s_t = j$: $\tilde{R}_t = \tilde{a}^j + \tilde{b}Y_t^j$. Conditional on $s_t = j$,

$$\tilde{R}_t = (\tilde{a}^j - \tilde{b}\hat{b}^{-1}\hat{a}^j) + \tilde{b}\hat{b}^{-1}\hat{R}_t + u_t^j, \tag{25}$$

where u_t is *i.i.d.* with zero mean and volatility Ω^j. Thus, the conditional density for $\tilde{R}_{t+1}$, conditional on $\hat{R}_{t+1}$, $s_t = j$ and $s_{t+1} = k$, is given by

$$f(\tilde{R}_{t+1}|\hat{R}_{t+1}, s_t = j, s_{t+1} = k) = \frac{e^{-\frac{1}{2}(\tilde{R}_{t+1}-(\tilde{a}^k-\tilde{b}\hat{b}^{-1}\hat{a}^k)-\tilde{b}\hat{b}^{-1}\hat{R}_{t+1})'[\Omega^k\Omega^{k'}]^{-1}(\tilde{R}_{t+1}-(\tilde{a}^k-\tilde{b}\hat{b}^{-1}\hat{a}^k)-\tilde{b}\hat{b}^{-1}\hat{R}_{t+1})}}{\sqrt{2\pi\left|\Omega^k\Omega^{k\prime}\right|}}. \tag{26}$$

To construct the likelihood function for the data, we introduce the econometrician's information set $J_t = \{\hat{R}_\tau, \tilde{R}_\tau, \tau \leq t\} \subset I_t$, and let $Q_t^j = f(s_t = j|J_t)$ be the probability of regime j given J_t. Define the following matrices:

$$\begin{aligned}
Q_t &= \begin{bmatrix} f(s_t = 0|J_t) & f(s_t = 1|J_t) \end{bmatrix}, \\
f_{t,t+1}^R &= \begin{bmatrix} f(\hat{R}_{t+1}|\hat{R}_t, s_t = 0, s_{t+1} = 0) & f(\hat{R}_{t+1}|\hat{R}_t, s_t = 0, s_{t+1} = 1) \\ f(\hat{R}_{t+1}|\hat{R}_t, s_t = 1, s_{t+1} = 0) & f(\hat{R}_{t+1}|\hat{R}_t, s_t = 1, s_{t+1} = 1) \end{bmatrix}, \\
f_{t,t+1}^u &= \begin{bmatrix} f(\tilde{R}_{t+1}|\hat{R}_{t+1}, s_{t+1} = 0) & f(\tilde{R}_{t+1}|\hat{R}_{t+1}, s_{t+1} = 1) \\ f(\tilde{R}_{t+1}|\hat{R}_{t+1}, s_{t+1} = 0) & f(\tilde{R}_{t+1}|\hat{R}_{t+1}, s_{t+1} = 1) \end{bmatrix}.
\end{aligned}$$

Using this notation, the conditional density of observed yields is

$$\begin{aligned}
f(\hat{R}_{t+1}, \tilde{R}_{t+1}|J_t) &= \sum_j f(\hat{R}_{t+1}, \tilde{R}_{t+1}|J_t, s_t = j)Q_t^j \\
&= \sum_{j,k} f(\hat{R}_{t+1}, \tilde{R}_{t+1}|J_t, s_t = j, s_{t+1} = k)Q_t^j\pi_t^{\mathbb{P}jk} \\
&= \sum_{j,k} f(\hat{R}_{t+1}, |\hat{R}_t, s_t = j, s_{t+1} = k)Q_t^j\pi_t^{\mathbb{P}jk} f(\tilde{R}_{t+1}|\hat{R}_{t+1}, s_{t+1} = k).
\end{aligned}$$

Q_t^j is updated using Bayes rule:

$$
\begin{aligned}
Q_{t+1}^k &= f(s_{t+1}=k|J_{t+1}) \\
&= \frac{\sum_j f(s_{t+1}=k,\hat{R}_{t+1},\tilde{R}_{t+1}|J_t,s_t=j)Q_t^j}{f(\hat{R}_{t+1},\tilde{R}_{t+1}|J_t)} \\
&= \frac{\sum_j Q_t^j f(\hat{R}_{t+1}|\hat{R}_t,\le t,s_t=j,s_{t+1}=k)\pi_t^{\mathbb{P}jk} f(\tilde{R}_{t+1}|\hat{R}_{t+1},s_{t+1}=k)}{f(\hat{R}_{t+1},\tilde{R}_{t+1}|J_t)}
\end{aligned}
$$

Thus, the log-likelihood function L is given by

$$
L = \frac{1}{T-1}\sum_{t=1}^{T-1}\log f(\hat{R}_{t+1},\tilde{R}_{t+1}|J_t), \tag{27}
$$

$$
f(\hat{R}_{t+1},\tilde{R}_{t+1}|J_t) = Q_t \times \left(f_{t,t+1}^R \odot f_{t,t+1}^u \odot \pi_t^{\mathbb{P}}\right)\times \mathbf{1}, \tag{28}
$$

$$
Q_{t+1} = \frac{Q_t \times \left(f_{t,t+1}^R \odot f_{t,t+1}^u \odot \pi_t^{\mathbb{P}}\right)}{f(\hat{R}_{t+1},\tilde{R}_{t+1}|J_t)}, \tag{29}
$$

where $A\odot B$ means element by element multiplication of matrix A and B with the same dimensions, and $\mathbf{1}$ is the two-dimensional unit vector.

In interpreting our empirical results, we follow the standard practice of using the "smoothed regime probabilities" $q_t^j \equiv f(s_t=j|J_T)$ to classify observations into regimes. (Recall that we do not observe which regime the economy is in at date t, s_t.) For our case of two regimes, we classify the yield observation at date t into regime j if $q_t^j > .5$, where

$$
q_t^j = \frac{g_t^j Q_t^j}{\sum_k g_t^k Q_t^k}, \tag{30}
$$

$g_T^j \equiv 1$ and, for $1 \le t \le T-1$,

$$
g_t^j \equiv f(\hat{R}_s,\tilde{R}_s : t+1 \le s \le T|s_t=j,J_t) = \sum_k \pi_t^{\mathbb{P}jk} f_{t,t+1}^{jk} g_{t+1}^k.
$$

In matrix notation, we have

$$
\begin{aligned}
q_t &\equiv \begin{bmatrix} q_t^0 \\ q_t^1 \end{bmatrix} = \frac{Q_t' \odot g_t}{Q_t g_t},\ 1\le t\le T, \\
g_t &\equiv \begin{bmatrix} g_t^0 \\ g_t^1 \end{bmatrix} = \left(\pi_t^{\mathbb{P}} \odot f_{t,t+1}^R \odot f_{t,t+1}^u\right)\times g_{t+1},\ 1\le t\le T-1;\ g_T = \begin{bmatrix} 1 \\ 1 \end{bmatrix}.
\end{aligned}
$$

4 Empirical Results

We estimate a two-regime, three-factor ($N=3$) model, $A_0^{RS}(3)$, using the Fama-Bliss monthly data on U.S. Treasury zero-coupon bond yields for the period 1970 through 1995.

The vector $\hat{R}$ includes the yields on bonds with maturities of 6, 24, and 120 months, and $M = 1$ with $\tilde{R}$ chosen to be the yield on the 60-month bond. The two regimes are denoted L and H, corresponding to "low" and "high" values of the diagonal entries of Σ^j.

In parameterising model $A_0^{RS}(3)$, we impose several normalisations. Analogous to the normalisations imposed in Dai and Singleton [2000] for single-regime affine *DTSM*s, in regime L, we set Σ^L to an identity matrix, $\kappa^{\mathbb{P}L}$ to a lower triangular matrix, and $\theta^{\mathbb{P}L}$ to zero. Second, in regime H, Σ^H was set to a lower diagonal matrix, because the Brownian motions in regime H can be rotated independent of any rotations on the Brownian motions in regime L. Third, across regimes, $\delta_0^L = \delta_0^H = \delta_0$, which allows the identification of the long-run mean of the state vector in regime H. Beyond these normalisations, the restrictions $\kappa^{\mathbb{Q}H} = \kappa^{\mathbb{Q}L} \equiv \kappa^{\mathbb{Q}}$ and $\delta_Y^H = \delta_Y^L = \delta_Y$ were imposed so that zero-coupon bonds are priced in closed form.

Even with these normalisations/constraints, the resulting maximally flexible $A_0^{RS}(3)$ model (with restrictions for analytical pricing) involves a high dimensional parameter space: there are 55 parameters in

$$\delta_0, \delta_Y, \kappa^{\mathbb{P}L}, \Lambda_0^L, \Lambda_Y^L, \theta^H, \Sigma^H, \Lambda_0^H, \Lambda_Y^H, \pi^Q, \eta_0^{LL}, \eta_0^{HH}, \eta_Y^{LL}, \eta_Y^{HH}, \Omega^L, \Omega^H.$$

To facilitate numerical identification of the free parameters, we imposed several additional over-identifying restrictions. First, we set $\delta_0^L = 6.86\%$, the historical mean of the one-month Treasury bill yield during our sample period. Additionally, the constraints $\lambda_0^L = (0, 0, \lambda_0^L(3))'$ and $\lambda_0^H = (0, 0, 0)'$ were imposed, because of the difficulty in distinguishing the levels of the *MPF* risks, Λ_0^j, from the level of short rate. The only free parameter in λ_0^j, $\lambda_0^L(3)$, allows the means of the slope of the yield curve to vary across regimes. Also, after a preliminary exploration of model $A_0^{RS}(3)$ we set the parameters $\kappa^{\mathbb{P}L}(2,1)$, $\Sigma^H(2,1)$, $\Sigma^H(3,1)$, $\Sigma^H(3,2)$, $\lambda_Y^L(1,1)$, $\lambda_Y^L(1,3)$, $\lambda_Y^L(2,1)$, $\lambda_Y^L(2,2)$, $\lambda_Y^H(1,3)$, $\lambda_Y^H(2,1)$, $\lambda_Y^H(3,2)$, $\lambda_Y^H(3,3)$, $\eta_Y^{LL}(2)$, and $\eta_Y^{HH}(1)$ to 0, because they were small relative to their estimated standard errors. Finally, preliminary estimates revealed that $\eta_Y^{LL}(3)$ was approximately equal to $-\eta_Y^{LL}(1)$, so we imposed equality of these parameters in estimation.

ML estimates for model $A_0^{RS}(3)$ (there are 34 free parameters after various normalisations and restrictions) and their associated asymptotic standard errors are reported in Tables 1 and 3, and equation (34) (note that δ_Y^H and $\kappa^{\mathbb{P}H}$ are not free parameters). The diagonal elements of Σ^H are all larger than their counterparts in Σ^L, which motivates our labelling of the two regimes. The estimates of the $\kappa^{\mathbb{P}}$ show that the rates of mean reversion of the risk factors Y change across regimes. Equivalently, there are statistically significant differences in the state-dependence of the *MPF* risks (in the estimated values of λ_Y) across regimes. Comparing diagonal elements of $\kappa^{\mathbb{P}L}$ and $\kappa^{\mathbb{P}H}$, it is seen that all three factors tend to exhibit less mean reversion in regime L. This finding, which we elaborate on subsequently, is consistent with past studies of descriptive regime-switching models (e.g., Gray [1996] and Ang and Bekaert [2002b]).

In arriving at model $A_0^{RS}(3)$, we initially estimated model $A_0^{RS}(3)'$ that allowed all six of the components of $(\eta_Y^{LL}, \eta_Y^{HH})$ to be free. However, we found that, in estimating our preferred model $A_0^{RS}(3)$ with the constraints $\eta_Y^{LL}(3) = -\eta_Y^{LL}(1)$ and $\eta_Y^{LL}(2) = \eta_Y^{HH}(1) = 0$ imposed, the value of the log-likelihood function changed very little (compare rows three and four of Table 2).

	$\log(L) = 19.70402$					
	Regime L			Regime H		
δ_0	6.86%	-	-	6.86%	-	-
δ_Y	**0.00515**	**-0.00297**	**0.0105**	0.00515	-0.00297	0.0105
	(0.00094)	(0.00118)	(0.00047)	(0.00094)	(0.00118)	(0.00047)
	0.486			**5.827**	**2.412**	
	(0.052)			(2.036)	(1.066)	
κ		**0.0615**			0.687	0.802
		(0.0266)			(0.346)	(0.415)
	-2.001	-0.352	**1.089**	-0.781	0.036	**1.498**
	(0.447)	(0.274)	(0.244)	(1.361)	(0.254)	(0.117)
θ				0.753	-1.909	1.246
				(0.610)	(1.398)	(1.005)
diag(Σ)	1	1	1	**2.082**	**1.785**	**4.425**
				(0.205)	(0.521)	(0.262)
λ_0			0.283			
			(0.658)			
		0.190		**-5.340**	**-2.222**	
		(0.0569)		(2.019)	(1.032)	
λ_Y			**0.0753**		-0.626	-0.727
			(0.0340)		(0.340)	(0.415)
	-0.177	**0.388**	0.409	-1.402		
	(0.450)	(0.173)	(0.258)	(1.345)		
Ω	**6.0** bp	-	-	7.9 bp	-	-
	(2.0 bp)			(4.5 bp)		

Table 1: *ML* estimates and asymptotic standard errors for model $A_0^{RS}(3)$. Parameters in bold face are significantly different from 0 at the 5% significance level.

We also investigated the nature of priced regime shift risk in two ways. First, the constraint that $\pi^{\mathbb{Q}} = \pi^{\mathbb{P}}$ (constant) – regime-shift risk is not priced and the regime switching probabilities are state-independent – was tested (Table 2, row 1). Second, the constraint that $\pi^{\mathbb{P}} =$ constant – regime-shift risk is priced, but the regime-shift probabilities, and hence the $MPRS$ risks, are constants – was tested (Table 2, row 2). Both null hypotheses were strongly rejected by the data at conventional significance levels. Accordingly, we focus primarily on model $A_0^{RS}(3)$, occasionally comparing the results for this model with those from model $A_0^{RS}(3)[\pi^{\mathbb{Q}} = \pi^{\mathbb{P}}]$.

The estimated values of the "intercepts" $\hat{a}_n^j$ and factor loadings $\hat{b}$ in (22) are displayed in Figure 2. The regime-dependence of the a^j translates into different slopes of the mean yield curves (see below) across regimes. The patterns across maturities exhibited by the estimated factor loadings suggest that state 1 is a "curvature" factor and that states 2 and 3 have the

Null	$\log \mathcal{L}$	$-2(T-1)\log\frac{\mathcal{L}}{\mathcal{L}'}$	d.f.	p-value (%)
$A_0^{RS}(3)[\pi^{\mathbb{P}}=\pi^{\mathbb{Q}}]$	19.63681	46.5318	8	1.9e-5
$A_0^{RS}(3)[\pi^{\mathbb{P}}\text{ constant}]$	19.63826	45.6299	6	3.5e-6
$A_0^{RS}(3)$	19.70423	4.5966	2	10.043
$A_0^{RS}(3)'$	19.71162			

Table 2: Likelihood ratio tests of constrained versions of model $A_0^{RS}(3)$.

characteristics of "slope" factors. We confirmed this impression by regressing the fitted state variables, $\hat{Y}_{jt}$ $(j=1,2,3)$, on the six-month, two-year, and ten-year bond yields,

$$
\begin{aligned}
Y_1 &= \text{const} - 75.15\times R^6 + 192.22\times R^{24} - 85.73\times R^{120} + u_1,\\
Y_2 &= \text{const} - 24.47\times R^6 + 109.83\times R^{24} - 183.41\times R^{120} + u_2,\\
Y_3 &= \text{const} + 200.11\times R^6 - 163.63\times R^{24} + 16.44\times R^{120} + u_3.
\end{aligned} \tag{31}
$$

The first factor, Y_1, is roughly the yield on the butterfly (long the 2-year and short both the 6-month and 10-year). The second factor is roughly minus the 2-year and 10-year slope (the "long slope"), and the third factor is approximately minus the 6-month and 2-year slope (the "short slope"). Figure 3 plots the fitted state variables against spreads on these slopes and the butterfly position.

This "rotation" of the factors obtained from the pricing model is different from what is obtained in standard principal component analyses, in that Y_{2t} and Y_{3t} are two different *slope* factors, rather than the typical finding of *level* and *slope* (e.g., $R_t^{120}-r_t$) from principal components. Of course, unlike in a principal components analysis, the components of Y exhibit substantial correlation induced by the non-zero off-diagonal elements of the $\kappa^{\mathbb{P}}$.

4.1 Regime Probabilities

The filtered regime probabilities ($Q_t^H = f(s_t = H|J_t)$ for models $A_0^{RS}(3)$ and $A_0^{RS}(3)[\pi^{\mathbb{Q}} = \pi^{\mathbb{P}}]$ are displayed in Figure 4. For comparison we also plot (dashed lines) the corresponding filtered probabilities from a descriptive regime-switching (*DRS*) model. To estimate the descriptive regime-switching model the vector PC_t of the first three principal components was computed using the covariance matrix of the 6-, 24-, and 120-month zero coupon bonds. Then a descriptive model for PC_t in which the state-dependent regime-switching probabilities π_t^{PC} were assumed to depend on PC_t as in (19) was estimated. The shaded periods in Figure 4 represent the periods of recessions according to NBER business-cycle dating.

Focusing first on model $A_0^{RS}(3)$, we confirm the widely documented observation that regime H tends to be associated with recessions: Q_t^H and its counterpart from the *DRS* model are larger during recessions. The model-based Q_t^H often gives the stronger signal in the sense that it is closer to one during recessions than the Q_t^H computed from model *DRS*. Additionally, both model $A_0^{RS}(3)$ and *DRS* suggest that the economy was in regime H in 1985, and they both signal a weak economy prior to the NBER's dating of the 1990 recession.

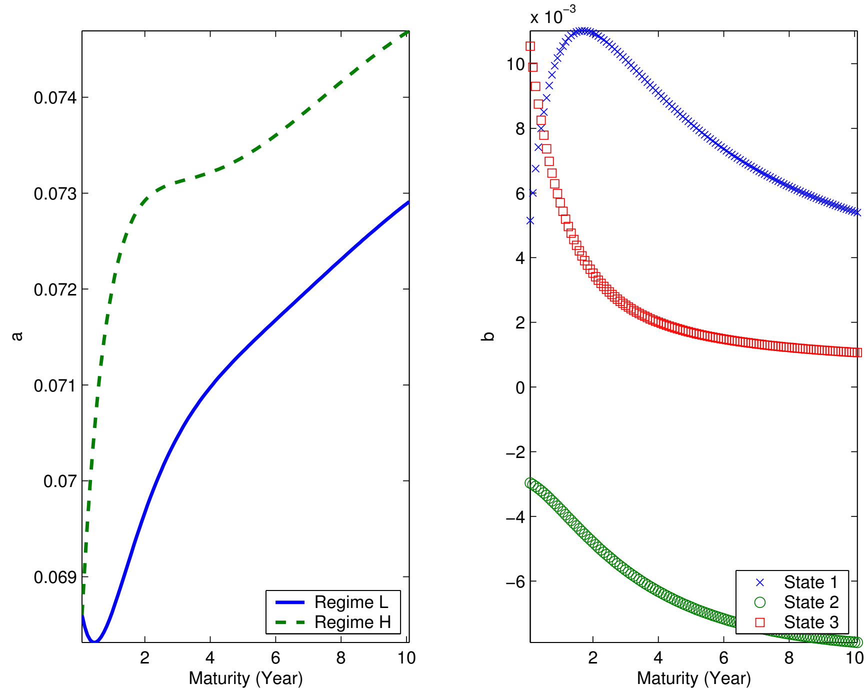

Figure 2: Estimates of the factor loadings, the a^j and b in the affine expression for bond yields.

At the same time, there are several notable differences between the Q^H implied by models $A_0^{RS}(3)$ and $A_0^{RS}(3)[\pi^{\mathbb{Q}} = \pi^{\mathbb{P}}]$. For the first three recessions, Q^H from model $A_0^{RS}(3)[\pi^{\mathbb{Q}} = \pi^{\mathbb{P}}]$ tends to predict longer periods in regime H; indeed, model $A_0^{RS}(3)$ tends to signal a shift out of regime H well before the NBER judges these recessions to be over. Further, throughout the late 1980's, model $A_0^{RS}(3)[\pi^{\mathbb{Q}} = \pi^{\mathbb{P}}]$ shows several periods of substantially increased likelihood of being in regime H compared to model $A_0^{RS}(3)$. The reverse is true during the 1990s. Some intuition for these differences will be developed subsequently.

The parameters governing $\pi_t^{\mathbb{P}ij}$ are displayed in Table 3.[13] Equation (19) and the OLS regression results in (31) imply that

$$\begin{aligned}
\pi^{\mathbb{P},LH} &= \left[1 + e^{3.98+296\times(R^{24}-R^{6})+110\times(R^{120}-0.78R^{24})}\right]^{-1} \\
\pi^{\mathbb{P},HL} &= \left[1 + e^{-13.57+167\times R^{6}+(151\times R^{6}-342\times R^{24}+191\times R^{120})}\right]^{-1}.
\end{aligned}$$

That is, the probability of switching from regime L to regime H increases as the short-term and long-term slopes flatten (particularly the slope of the short end of the curve), and the probability of switching from regime H to regime L increases as the short-term yields or

[13]As noted previously, for reasons of parsimony, we have constrained the $\pi_t^{\mathbb{P}ij}$ such that $\pi_t^{\mathbb{P}LL}$ or $\pi_t^{\mathbb{P}LH}$ does not depend on Y_2 and $\pi^{\mathbb{P}HH_t}$ or $\pi_t^{\mathbb{P}HL}$ does not depend on Y_1. If we free up these constraints, i.e., let $\pi^{\mathbb{P}}$ to be a function of all three state variables, the likelihood function increases from 19.70402 to 19.71162. Thus, these constraints are not rejected at 99% confidence interval by a likelihood ratio test.

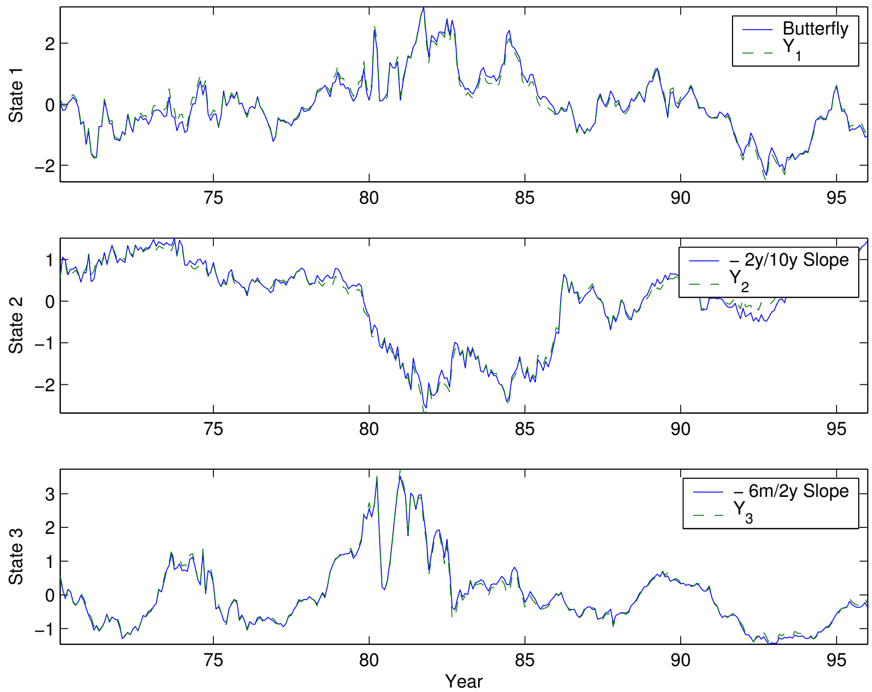

Figure 3: Implied state variables plotted against butterfly and slope spreads.

the butterfly spread decline. Given the relative magnitudes of the short rate and butterfly spread, it turns out that $\pi^{\mathbb{P},HL}$ is driven virtually entirely by R^6.

η_0^{LL}	$\eta_Y^{LL}(1)$	$\eta_Y^{LL}(2)$	$\eta_Y^{LL}(3)$	η_0^{HH}	$\eta_Y^{HH}(1)$	$\eta_Y^{HH}(2)$	$\eta_Y^{HH}(3)$
2.414	**1.074**		**-1.074**	-1.759		**-0.907**	1.479
(0.460)	(0.319)	-	(0.319)	(1.169)	-	(0.379)	(1.009))

Table 3: *ML* estimates of parameters governing the regime switching process, with asymptotic standard errors in parentheses.

Figure 5 displays the probabilities $\pi_t^{\mathbb{P}LH}$ and $\pi_t^{\mathbb{P}LH}$, evaluated at the ML estimates, from models $A_0^{RS}(3)$ and *DRS*. Both the $A_0^{RS}(3)$-based and *DRS*-based estimates of $\pi_t^{\mathbb{P}LH}$ are higher during the recessionary periods in our sample. Pursuing our interpretation of regimes H and L as different stages of the business cycle, towards the end of an expansionary phase of the economy, short-term rates are often rising faster than long-term rates as a central bank's concerns about inflation puts upward pressure on short-term yields. Consistent with these observations, our econometric model shows $\pi^{\mathbb{P}LH}$ increasing as the yield curve flattens, both at the short and long end of the curve. On the other hand, if we are already in regime H (a recession), then short-term rates typically have to come down far enough to induce an expansion. This is consistent with $\pi^{\mathbb{P}HL}$ rising as short-term rates fall.

During the recessionary periods in our sample, π^{LH} and π^{HL} tend to move in opposite

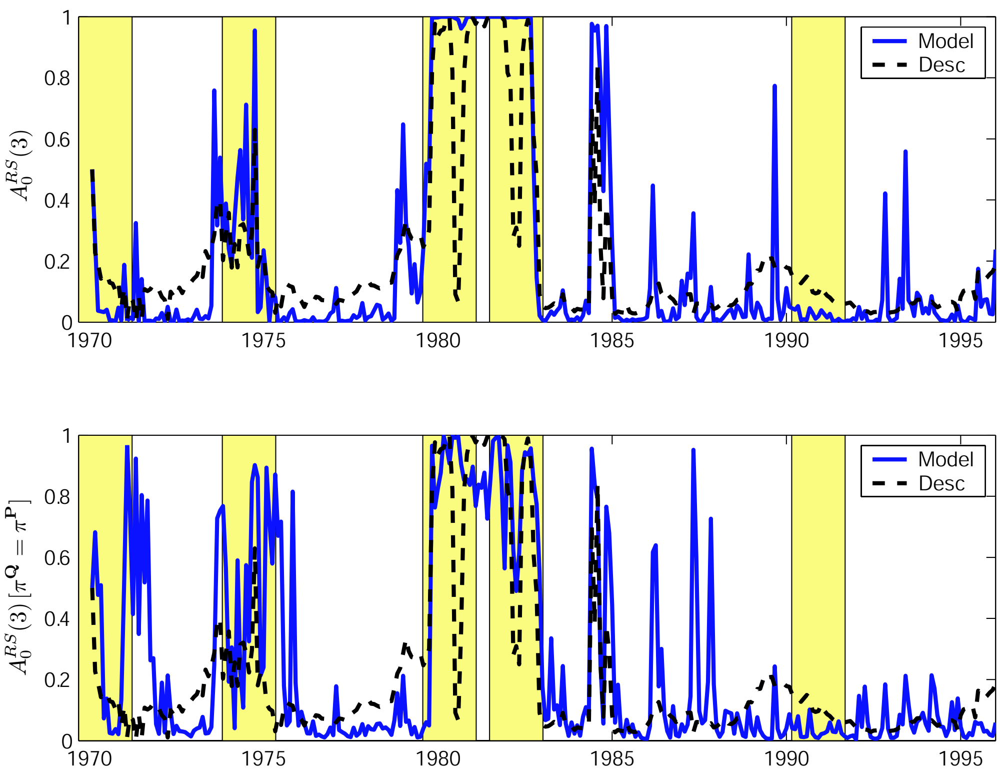

Figure 4: This graph displays the filtered probabilities Q_t^H from models $A_0^{RS}(3)$ (top panel) and $A_0^{RS}(3)[\pi^{\mathbb{Q}} = \pi^{\mathbb{P}}]$ (lower panel). In each panel we have overlayed the periods of recessions according to dating by the NBER (shaded portions) and the implied Q_t^H from our descriptive model (dashed lines).

directions. That is, when the U.S. economy was in a recession, the conditional probability of moving to regime L from regime H was lower. A notable departure from this inverse relationship occurred during 1985. As noted above, $\pi^{\mathbb{P}HL}$ was driven almost entirely by the short-term rate, R^6. During 1984 the Federal Reserve temporarily tightened monetary policy. Then in late 1984 and throughout 1985 there was a monetary easing and concurrent decline in short-term interest rates. Additionally, the striking decline in U.S. inflation rates, instigated by Volker's anti-inflation policy of the early 1980's, continued. These events show up in our model as an increase in $\pi^{\mathbb{P}HL}$ from near zero in 1984 to near unity by the end of 1985. The monetary easing in late 1984 also explains the sharp increases in the estimated filtered probabilities Q_t^H in all three models (model DRS and the two pricing models). Bond yield movements had many of the salient features of a shift to regime H.

One interesting difference between models $A_0^{RS}(3)$ and DRS is that $\pi_t^{\mathbb{P}HL}$ is larger in model DRS than in model $A_0^{RS}(3)$ during much of the period between 1983 and 1985. That is, the pricing model shows much more persistent risk of a staying in regime H during this period, suggesting that bond markets did not view the announced shift in monetary policy in 1982 as fully credible. We find this interesting in the light of the fact that the Federal

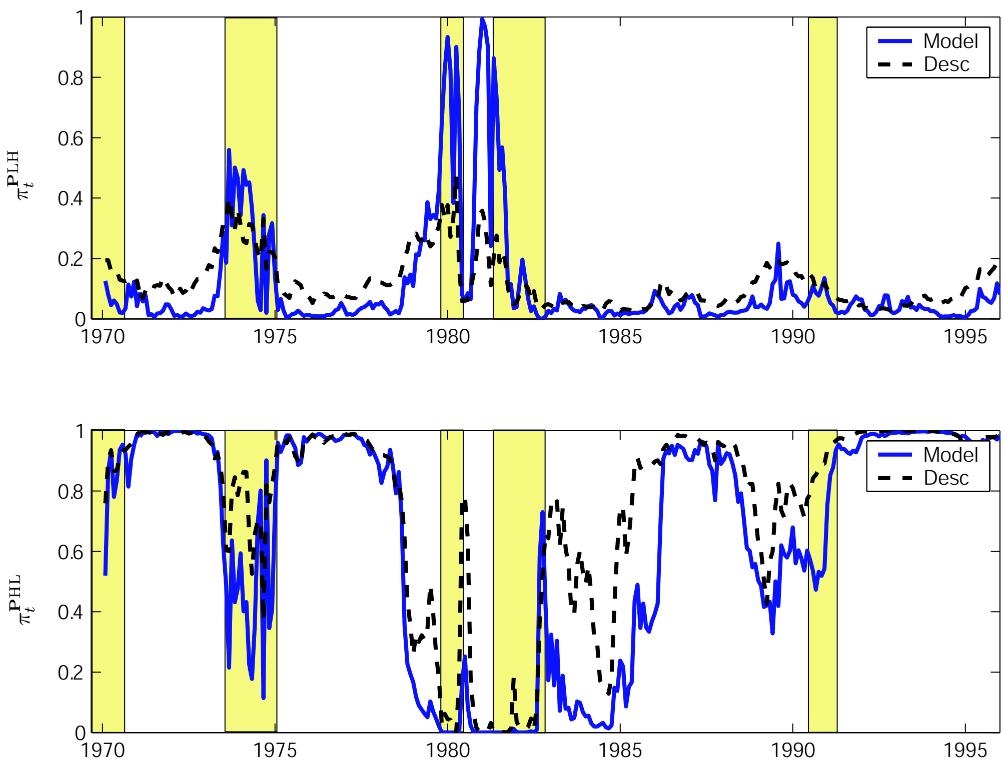

Figure 5: This graph displays the estimated probabilities $\pi_t^{\mathbb{P}LH}$ and $\pi_t^{\mathbb{P}HL}$, evaluated at the ML estimates, from model $A_0^{RS}(3)$. In each panel we have overlayed the periods of recessions according to dating by the NBER (shaded portions).

Reserve only weakened its dedication to monetary growth targets in October, 1982 (the ending date for the "monetary experiment") and, in fact, maintained a target for $M1$ until 1987 (Friedman [2000]).[14]

The relative sensitivities of the $\pi^{\mathbb{P}}$ to the level and slope of the yield curve may also be relevant for recent findings on the predictability of GDP growth using yield curve variables. Ang, Piazzesi, and Wei [2003] find that both *level* and *slope* have predictive content within a single-regime $DTSM$. Our two-regime model suggests that the relative predictive contents of these variables may vary with the stage of the business cycle.

The sample means of the fitted time-varying $\pi_t^{\mathbb{P}ij}$ (together with the sample means of the fitted transition matrix from the descriptive model, π^{DRS}) are

$$\bar{\pi}^{\mathbb{P}} = \begin{bmatrix} 89.34\% & 10.66\% \\ 64.36\% & 35.64\% \end{bmatrix}, \quad \bar{\pi}^{DRS} = \begin{bmatrix} 88.12\% & 11.88\% \\ 76.06\% & 23.94\% \end{bmatrix}. \tag{32}$$

[14]Based on their statistical analyses, Friedman and Kuttner [1996] argue that deviations from the Federal Reserve's target for $M1$ remained a significant determinant of their monetary policy rule until mid-1984, and deviations from $M2$ were significant until mid-1985.

The vector of stable probabilities implied by the mean transition matrix $\pi^{\mathbb{P}}$ is[15]

$$\tilde{\pi}^{\mathbb{P}} = \begin{bmatrix} 85.79\% \\ 14.21\% \end{bmatrix}, \quad \tilde{\pi}^{DRS} = \begin{bmatrix} 86.49\% \\ 13.51\% \end{bmatrix},$$

which match roughly the sample mean of the fitted probabilities (Q_t^L, Q_t^H), $(79.97\%, 20.03\%)$.

These findings are different than those for model $A_0^{RS}(3)[\pi^{\mathbb{Q}} = \pi^{\mathbb{P}}]$, which are:

$$\bar{\pi}^{\mathbb{P}} = \begin{bmatrix} 0.9525 & 0.0475 \\ 0.1355 & 0.8645 \end{bmatrix}, \quad \tilde{\pi}^{\mathbb{P}} = \begin{bmatrix} 0.7405 \\ 0.2595 \end{bmatrix}. \tag{33}$$

Notably, with $\pi^{\mathbb{Q}} = \pi^{\mathbb{P}} =$ constant, $\pi^{\mathbb{P}HH}$ is much larger than $\pi^{\mathbb{P}HL}$. This finding is similar to those in previous studies, both for descriptive and pricing models (e.g., Ang and Bekaert [2002b] and Bansal and Zhou [2002]) that assume constant ratings transitions probabilities. However, our descriptive model with time-varying regime-switching probabilities calls for $\pi_t^{\mathbb{P}HH}$ to be less than $\pi_t^{\mathbb{P}HL}$ on average. That is, the data on US treasury bonds suggests that regime H was less persistent on average than regime L. If we view regime H as capturing periods of downturns and regime L as periods of expansions, consistent with our previous discussion of NBER business cycles and the probabilities Q_t^H, then this finding can be viewed as a manifestation of the well documented asymmetry in U.S. business cycles: recoveries tend to take longer than contractions (see, e.g., Neftci [1984] and Hamilton [1989]). Model $A_0^{RS}(3)$ with priced, state-dependent regime shift risk captures this asymmetry, but model $A_0^{RS}(3)[\pi^{\mathbb{Q}} = \pi^{\mathbb{P}}]$ with constant regime-shift probabilities does not.

The estimated risk-neutral transition probabilities (from model $A_0^{RS}(3)$), and their associated asymptotic standard errors, are

$$\pi^{\mathbb{Q}} = \begin{bmatrix} 93.13\% & 6.87\% \\ (5.53\%) & - \\ 0.00\% & 100.00\% \\ - & (11.36\%) \end{bmatrix}, \tag{34}$$

with associated stable probabilities $\tilde{\pi}^{\mathbb{Q}} = [0.00\%, 100.00\%]'$. Comparing the stable probabilities $\tilde{\pi}^{\mathbb{Q}}$ and $\tilde{\pi}^{\mathbb{P}}$, it is seen that the economy spends more time in regime H under $\mathbb{Q}$ than under $\mathbb{P}$, but less time in regime L under $\mathbb{Q}$ than under $\mathbb{P}$. This is intuitive since, with risk-averse bond investors, risk-neutral pricing will recover market prices for bonds only if we treat the "bad" H regime as being more likely to occur than in actuality. The diagonal elements of $\pi^{\mathbb{Q}}$ are statistically not different from 1, but they are statistically different from the means of the corresponding elements in $\pi^{\mathbb{P}}$.

4.2 Model-Implied Means and Volatilities of Bond Yields

Figure 6 plots the model-implied population unconditional means and standard deviations (volatilities) of the Treasury yields implied by our model. These are obtained by computing

[15]For a constant transition matrix Π, the stable probabilities x are defined by the equation $\Pi' x = x$. Equivalently, x is the limit of $\Pi^{n'} x$, as $n \to \infty$.

the population moments implied by the distribution of the state in each regime, evaluated at the *ML* estimates of the model. To construct a sample counterpart, we computed the smoothed probabilities q_t^j given by (30), and then classified a date as being in regime L if $q_t^L > .5$ or in regime H if $q_t^H > .5$. After sorting the dates, we computed the sample means and volatilities of the yields in each regime. These are reported as *Sample* in Figure 6. Even if our model is correctly specified, the *Model* and *Sample* results need not coincide exactly, of course, because of sampling variation in data, the use of *ML* parameter estimates, and our allocation based on q_t^j is an approximation. Nevertheless, Figure 6 suggests that the model does a quite good job at matching the first and second unconditional moments in the data.[16] The mean yield curves are upward sloping in both regimes, with the yields being notably higher in regime H.

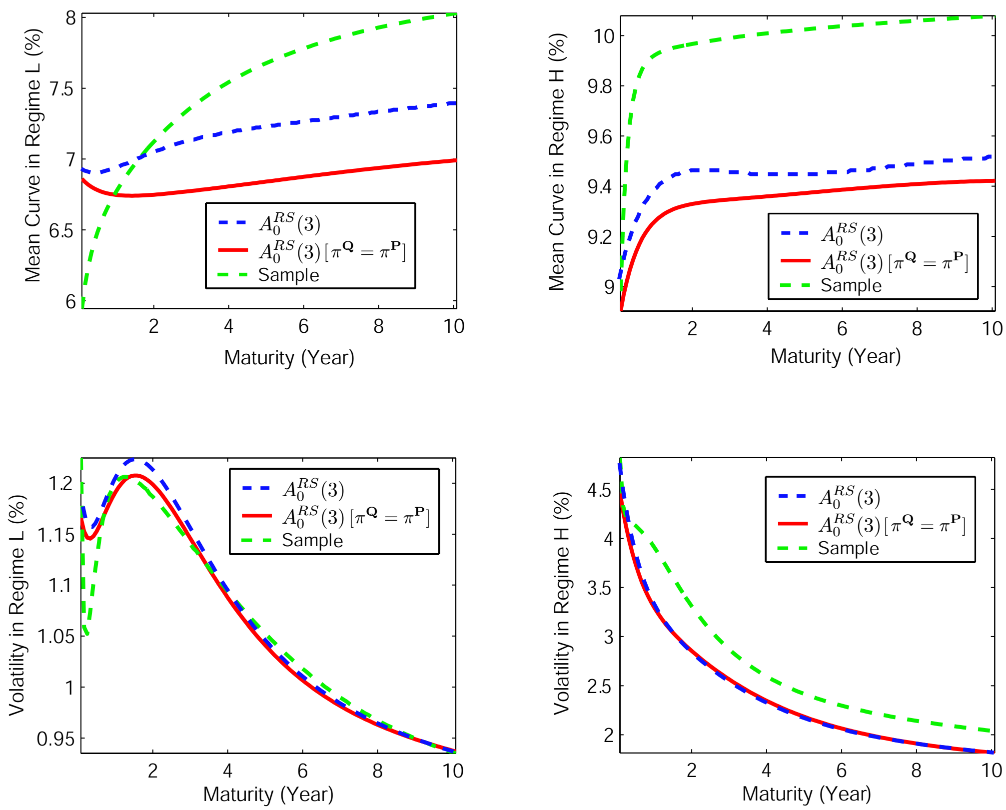

Figure 6: Term structures of unconditional means and volatilities of Treasury bond yields implied by models $A_0^{RS}(3)$ and $A_0^{RS}(3)[\pi^{\mathbb{Q}} = \pi^{\mathbb{P}}]$. Population moments for the models are evaluated at the *ML* estimates. "Sample" results are obtained by computing sample means and volatilities after allocating dates to regimes based on the smoothed probabilities q_t^j.

[16] We stress that we are comparing *population* moments implied by the model to *sample* moments in the data. If, instead, we computed say model-implied mean curves based on fitted yields from the model within each regime, then the *Model* and *Sample* results would lie virtually on top of each other. Thus, the comparisons in Figure 6 place greater demands on the model.

Of particular note are the shapes of the volatility curves in the two regimes. It is well known that in many U.S. fixed-income markets (e.g., Treasury bond, swaps, etc.), the term structures of unconditional yield volatilities are are humped-shaped (see, e.g., Litterman, Scheinkman, and Weiss [1988]), with the peak of the hump being approximately at two years to maturity.[17] Under our classification of dates into regimes, the hump in volatility is an L-regime phenomenon. Fleming and Remolona [1999] present evidence linking the hump to market reactions to macroeconomic announcements. Through the lens of our model, it appears that these, and possibly other, sources of yield volatility show up as a hump in volatility primarily during relatively tranquil, expansionary phases of the business cycle.

When the economy is in regime H volatility is high and the risk factors mean revert to their long-run means relatively quickly (Table 1). The fast mean reversion in regime H swamps a humped reaction (if any) to macroeconomic news, and induces the steeply downward sloping term structure of (unconditional) volatility.

Superimposed on the same graph are the corresponding results for Model $A_0^{RS}(3)[\pi^{\mathbb{Q}} = \pi^{\mathbb{P}}]$, in which regime-shift risk is not priced and regime-switching probabilities are state-independent. By and large, there is not a large difference between the model-implied first and second unconditional moments across these two models.

We examine the model-implied conditional volatilities in Section 6 as part of our assessment of the robustness of the properties of model $A_0^{RS}(3)$ to the presence of within-regime time-varying volatility.

5 Excess Returns and the Market Prices of Risk

In this section we return to one of the primary motivations for our analysis, namely, an investigation of the contributions of factor and regime-shift risk premiums to the temporal variation in expected excess returns. Figure 7 displays the *MPF* risks for model $A_0^{RS}(3)$ (first three quadrants), as well as the *MPF* risks from a single-regime $A_0(3)$ model (lower right quadrant). The three *MPF* risks from model $A_0^{RS}(3)$ are much smoother in regime L (solid lines) than in regime H (dashed lines). This is consistent with the common impression that expected returns should not fluctuate dramatically under "normal" circumstances. A very different impression comes from inspection of the *MPF* risks from the corresponding single-regime Gaussian $A_0(3)$ model. They look much more like those of regime H than those of regime L. This supports our remarks in the introduction to the effect that omission of the regime-switching process tends to distort the model-implied excess returns both in tranquil and turbulent times.

Turning to the *MPRS* risk (see Figure 8), on average the *MPRS* from H to L is higher than the *MPRS* from L to H. This implies that bond investors are more willing to buy insurance against an economic down turn (from L to H) than against an economic expansion (from

[17]Figure 6 also shows the "snake" shaped pattern in historical yield volatilities for very short-term bonds. This pattern is not captured by our three-factor model. However, the findings in both Longstaff, Santa-Clara, and Schwartz [2001] and Piazzesi [2001] suggest that the addition of a fourth factor would allow our model to replicate this pattern.

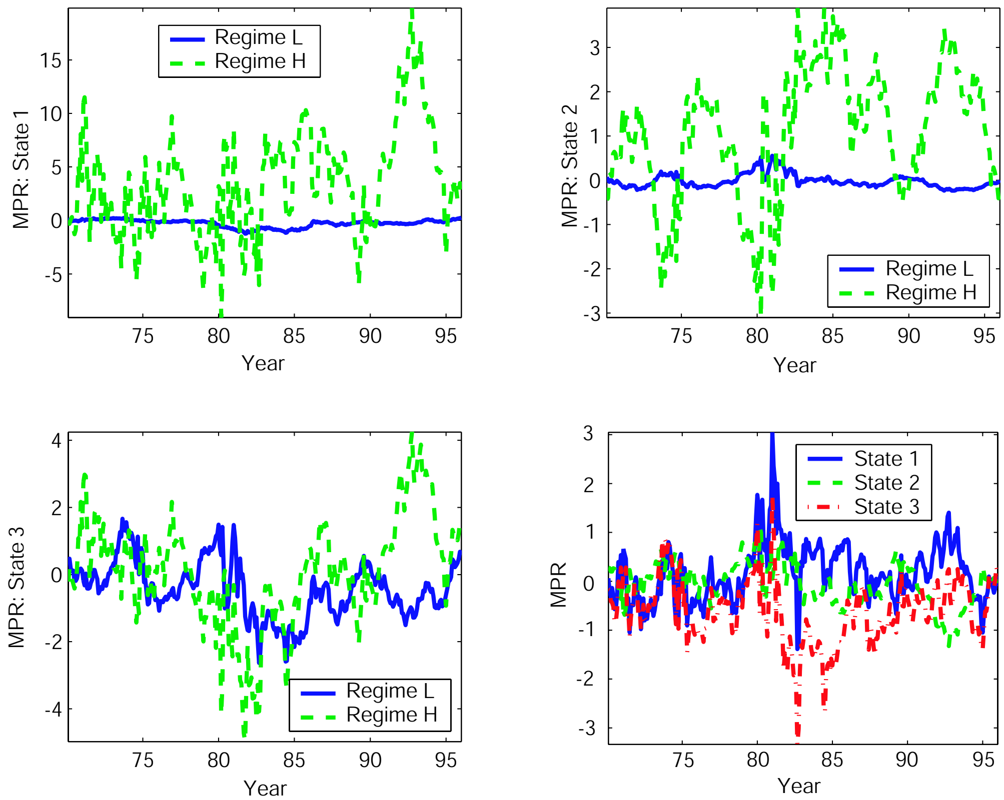

Figure 7: Market Prices of Factor Risks for Models $A_0^{RS}(3)$ and $A_0(3)$.

H to L). Intuitively, this may reflect the fact that agents' marginal rates of substitution of consumption tend to be low during economic expansions and high during recessions. Although insurance contracts with payoffs $1_{\{s_{t+1}=j\}}$ are not traded, the demand for (or the risk premium associated with) such contracts may be interpreted as inter-temporal hedging demands (in the sense of Merton [1973]) or the associated risk premium against business cycle fluctuations.

A comparison of the *MPF* risks in models $A_0^{RS}(3)$ and $A_0^{RS}(3)[\pi^{\mathbb{Q}} = \pi^{\mathbb{P}}]$ is revealing about the effects of state-dependent *MPRS* risks on the dynamic properties of the *MPF* risks. The estimated $\kappa^{\mathbb{Q}}$ in these models are similar. However, the estimated $\kappa^{\mathbb{P}}$ are different, particularly elements $\kappa_{31}^{\mathbb{P}}$ and $\kappa_{33}^{\mathbb{P}}$. From Figure 9 it is seen that these differences show up primarily in the dynamic properties of the market price of risk of Y_3 in regime L. The *MPF* risk for Y_3 is somewhat larger on average and notably less volatile in model $A_0^{RS}(3)$ with priced regime shift risk. A portion of the volatility in the *MPF* risk for Y_3 is shifted to variation in the *MPRS* risk in model $A_0^{RS}(3)$.

Since the elements of $\pi_t^{\mathbb{Q}}$ are constants, the dynamic properties of the *MPRS* risks are determined by those of the elements of $\pi_t^{\mathbb{P}}$. The correlation between Γ^{LH} and Y_3, which we recall is well described as $-(R^{24} - R^6)$, is 0.93. The third state variable, Y_3 has the fastest rate of mean reversion among the three state variables in regime L. This shows up in Γ^{LH} in the form of notable variation over short horizons (several months). At the same time, from

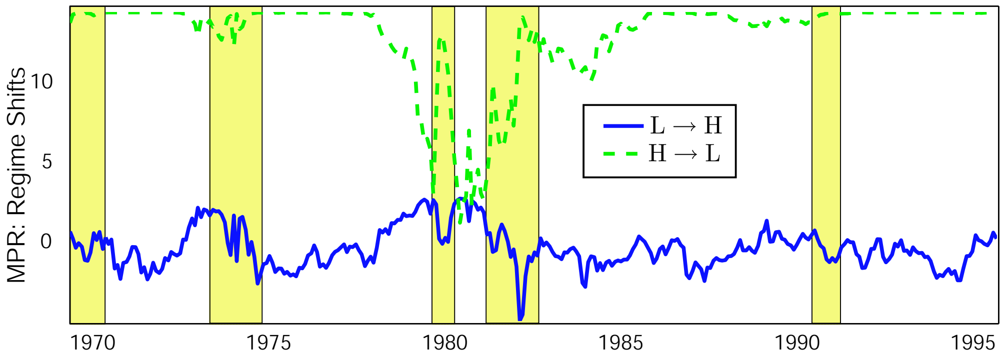

Figure 8: Market Prices of Regime-Shift Risks for Model $A_0^{RS}(3)$.

Figure 8 it is seen that there is clear cyclical component to Γ^{LH} induced by its dependence on the other state variables.

The *MPRS* risk of moving from H to L is most strongly correlated with $-R_t^6$, the level of the short rate ($corr(R^6, \Gamma^{HL}) = -0.88$). The dominant feature of the time path of Γ^{HL} is the big dip during the Fed experiment, though there are smaller dips around other recessions.

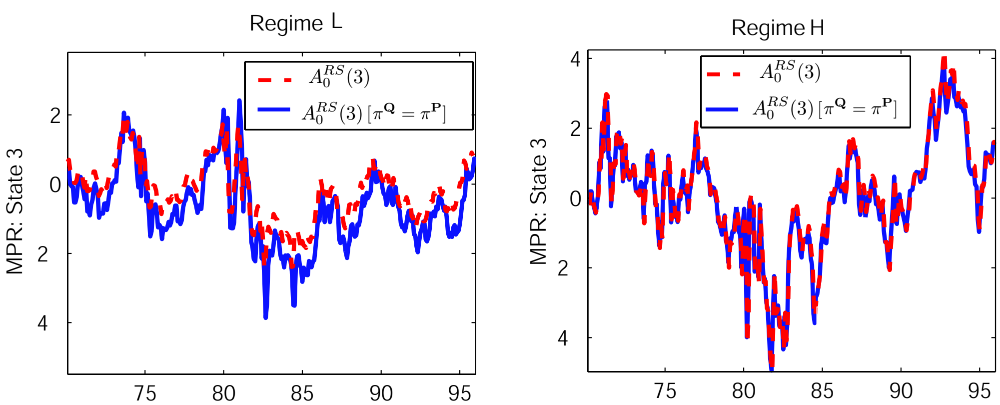

Figure 9: Market prices of risk for the third state variable in regime L for Models $A_0^{RS}(3)$ and $A_0^{RS}(3)[\pi^{\mathbb{Q}} = \pi^{\mathbb{P}}]$.

These findings reinforce our earlier discussion of the links between the temporal behaviour of the Γ^{ij} and business cycles. When the economy is expanding, the yield curve is typically upward sloping, and the excess return on a "down-turn" insurance contract is relatively low (investors are apprehensive about increasing rates and a potential downturn and are willing to buy insurance). When the economy is contracting, the yield curve is typically flat or even downward sloping, and so Γ^{LH} is relatively high. On the other hand, a large dip in Γ^{HL} reflects the expectation that the recession will be protracted and the expected return to a contract that pays off in the event of recovery is low.

Finally, regarding the predictability of excess returns on bonds, the empirical results in Duffee [2002] and Dai and Singleton [2002] suggest that, within the family of single-regime affine *DTSM*s, the rich state-dependence of the market prices of factor risks accommodated by Gaussian models is essential for predictability puzzles associated with violation of the "expectations theory" of the term structure (e.g., Campbell and Shiller [1991]). Since our $A_0^{RS}(3)$ model nests single-regime Gaussian models it is not surprising that it also does a reasonable job of matching the Campbell-Shiller evidence against the expectations theory.

6 Concluding Remarks

In this paper, we show that regime switching term structure models in which regime transition probabilities are constant and equal under both physical and risk-neutral measures may potentially give a mis-leading impression of the dynamics of expected bond returns and the relationship between the shape of the term structure and business cycle fluctuations. Likelihood ratio tests formally reject the case of constant regime transition probabilities in favor of a model with state-dependent regime transition probabilities and market prices of regime-shift risk. In concluding this paper, we point out some limitations/caveats of our analysis.

First, in order to price bonds analytically, we have imposed some parametric restrictions on the joint dynamics of the state vector and the Markov regime switching process under the risk-neutral measure. These restrictions preclude examination of a model in which regime-shift risk is priced and the regime transition probabilities are state-dependent under both physical and risk-neutral measures (as in Boudoukh, Richardson, Smith, and Whitelaw [1999]), or a model in which factor loadings on bond yields are allowed to be regime-dependent. We could relax these constraints, but at the cost of introducing approximations to both pricing and likelihood functions. Following the tradition of the large single-regime term structure literature, it seemed worthwhile to explore how far one could go in improving the fits over single-regime affine models, while preserving the analytical tractability of this family.

Perhaps of greater concern is the fact that our empirical study is based on the assumption that the state vector is an autoregressive Gaussian state process. The regime-dependence of both the level and the volatility of the short-term interest rates in model $A_0^{RS}(3)$ induce time varying, and in particular level-dependence, of the volatilities of bond yields of all maturities. However, we are unable to accommodate level-dependence of volatilities within each regime, as incorporated in the models of Naik and Lee [1997] and Bansal and Zhou [2002].

To gain some insight into how models $A_0^{RS}(3)$ and $A_0^{RS}(3)[\pi^{\mathbb{Q}} = \pi^{\mathbb{P}}]$ perform relative to a model with time-varying volatility within each regime, we extended our descriptive model for the first three principal components of bond yields to allow the volatility of each principal component in each regime to follow a $GARCH(1,1)$ process (model $DRSG$).[18] Figure 10 displays the one-month ahead conditional volatilities for the ten-year bond yield from our

[18] The parameters of the $GARCH$ processes were allowed to differ both across principal components and across regimes. The spirit of this analysis is a multi-variate version of the switching $GARCH$ model examined by Gray [1996]. However, we set up our switching $GARCH$ model using the same timing conventions as in our pricing model.

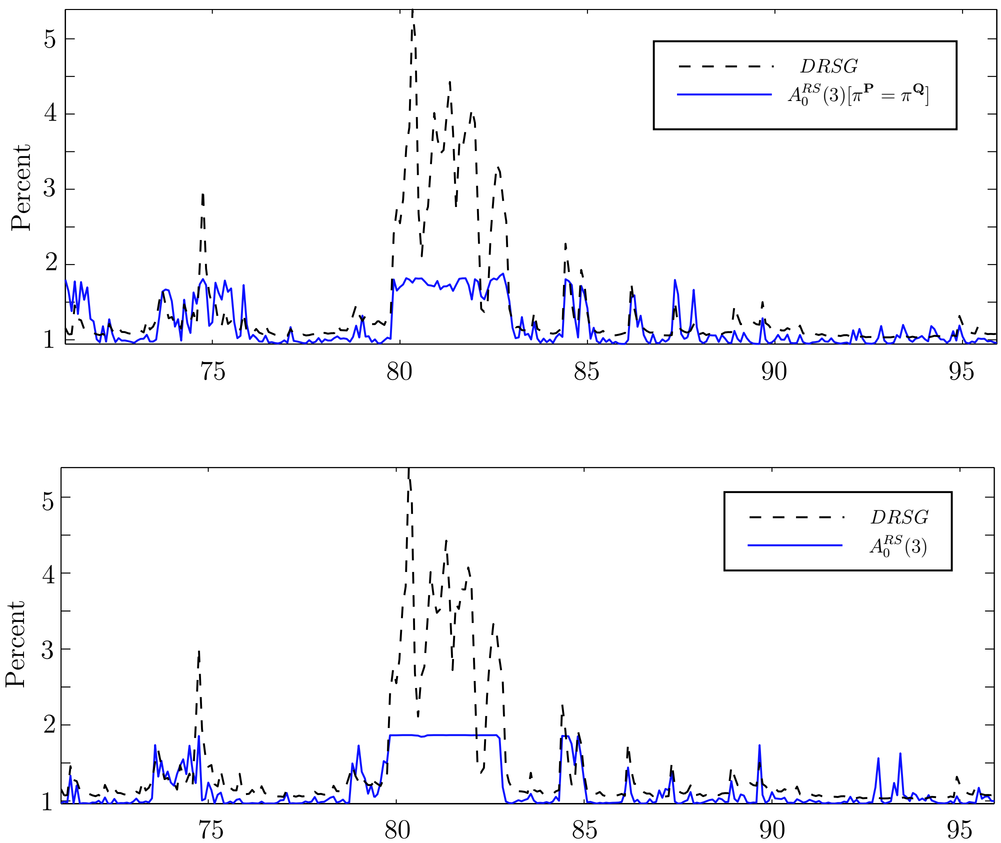

Figure 10: Conditional volatilities of ten-year bond yields from Models $A_0^{RS}(3)[\pi^{\mathbb{Q}} = \pi^{\mathbb{P}}]$ and $A_0^{RS}(3)$ plotted against the implied volatility from a descriptive regime-switching *GARCH* model (*DRSG*).

pricing models against those from model *DRSG*.[19] Perhaps the most striking feature of this figure is the fact that our pricing models understate conditional volatility relative to model *DRSG* during the monetary experiment of the early 1980's. (This is also true, but to a lessor degree, for the spike up in volatility during late 1974.)

However, we find it equally notable that model *ARS* captures (at least some of) the increased volatility associated with the Gulf War during 1990 and the turbulence in bond markets during 1994 (Borio and McCauley [1997]). In contrast, the volatility during this period is missed entirely by models $A_0^{RS}(3)[\pi^{\mathbb{Q}} = \pi^{\mathbb{P}}]$ and *DRSG*. Moreover, during the high volatility periods in the early 1970's and the mid-1980's, the implied volatilities from models $A_0^{RS}(3)$ and *DRSG* are very similar. Together, these observations suggest that the *GARCH*-based likelihood function of model *DRSG* may have over-weighted the "outliers" of 1980's at the expense of not capturing the volatility during the 1990's.[20]

[19]The analogous pictures for yields on bonds with shorter maturities show higher levels of volatility, but very similar temporal patterns.

[20]Comparing the time paths of volatility in model $A_0^{RS}(3)$ and $A_0^{RS}(3)[\pi^{\mathbb{Q}} = \pi^{\mathbb{P}}]$, it is evidently the absence of priced, state-dependent regime-shift risk from the latter model that underlies its failure to match

Of particular concern to us was the robustness (to the presence of time-varying volatility) of our finding that regime-switching *DTSM*s with state-independent regime switching probabilities (constant $\pi^{\mathbb{P}}$) are over-stating the persistence of the high volatility regime H. Equation (35) presents the average value of $\pi^{\mathbb{P}}$ from our pricing model $A_0^{RS}(3)$ along with the corresponding average from the descriptive model *DRSG*. The estimates are very similar. Indeed, the estimates of $\bar{\pi}^{\mathbb{P}}$ from model *DRSG* are even closer to those in model $A_0^{RS}(3)$ than are those from model *DRS*. Thus, the results from model *DRSG* reinforce the finding from model $A_0^{RS}(3)$ of an asymmetry in the persistence of regimes: $\pi^{\mathbb{P}HH} << \pi^{\mathbb{P}HL}$.

$$\bar{\pi}^{\mathbb{P}} = \begin{bmatrix} 89.34\% & 10.66\% \\ 64.36\% & 35.64\% \end{bmatrix}, \quad \bar{\pi}^{DRSG} = \begin{bmatrix} 90.35\% & 9.65\% \\ 68.39\% & 31.61\% \end{bmatrix}. \tag{35}$$

This extended descriptive analysis with model *DRSG* does not, of course, allow us to assess the implications of within-regime time-varying volatility for the structure of the market prices of factor or regime-shift risks. Such an assessment would require a regime-shifting *DTSM* that allows for both within regime stochastic volatility and state-dependent regime-shift probabilities. The development and implementation of such a model will be explored in future research.

the volatility during the 1990's. Further, when yield volatility was high, the conditional volatility of R^{10} tended to be much choppier in Model $A_0^{RS}(3)[\pi^{\mathbb{Q}} = \pi^{\mathbb{P}}]$ than in Model $A_0^{RS}(3)$. These findings are mirrored in the behaviours of the filtered probabilities Q_t^H. During the 1970's and 1980's, the Q_t^H implied by the pricing model without priced regime shift risk is much choppier. In contrast, during the 1990's, Q_t^H in Model $A_0^{RS}(3)$ exhibits notable upward spikes (see Figure 4) that are absent from the Q_t^H in Model $A_0^{RS}(3)[\pi^{\mathbb{Q}} = \pi^{\mathbb{P}}]$.

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