OPTIMAL PORTFOLIO SELECTION

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$S_t$: price of an asset at time $t$. $r_t$: interest rate at $t$.

Examples:

1. Constant coefficients (Merton, (1971)):

   \[ dS_i(t) = \mu_i S_i(t)dt + \sum_{j=1}^{n} S_i(t)\sigma_{i,j} \, dw_j(t) \quad i \leq n \]

2. Stochastic coefficients:

   \[ dS_i(t) = \mu_i(t) S_i(t)dt + \sum_{j=1}^{n} S_i(t)\sigma_{i,j}(t) \, dw_j(t), \quad i \leq n \]

Example ($n=1$ for simplicity):

   \[ dS_t = \mu_t S_t dt + \sigma S_t dw_t \]

   \[ d\mu_t = \alpha(\delta - \mu_t)dt + \beta dw_1(t) \]

$\mu$ is a mean-reverting Ornstein-Uhlenbeck process.

3. Diffusion (a special case of 2):

   \[ dS_t = \mu(S_t) dt + \sigma(S_t) dw_t \]
4. A mixed process:

\[ dS_t = \mu_t dt + \sigma_t dw_t + g_t dN_t \]

\( N \) is a counting process.

A special case:

\[ dS_t = g_t dN_t \]

Discount process:

\[ \beta_t = \exp\left\{ -\int_0^t r_u du \right\} \]

Discounted price process:

\[ S^*_t = \beta_t S_t \]
**Condition NA:** There exists a $Q \sim P$ such that $S_t^*$ is a martingale under $Q$. This will be assumed henceforth.

**Condition C:** Every $Q$-martingale $\{M_t, \ t \leq T\}$ can be represented as

$$M_t = M_0 + \sum_{i=0}^{n} \int_{0}^{t} \phi_i(u)dS_i^*(u), \quad t \leq T \quad (1)$$

(may or may not hold in a particular model).
\( \pi_i(t) \): amount of money invested in the \( i \)th asset.

\( X_t \): wealth at time \( t \).

\( X_t^* = \beta_t X_t \), discounted wealth.

\[
X_t^* = x + \sum_{i=1}^{n} \int_{0}^{t} \frac{\pi_i(u)}{S_i(u)} dS_i^*(u), \quad t \leq T \quad (2)
\]

Admissibility requirement: \( X_t \geq 0 \), a.s., \( t \in [0, T] \).

**Objective:** Maximize: \( E[U(X_T)] \) over all admissible trading strategies.

\( U(\cdot) \) is in \( C^1((0, \infty)) \), strictly concave, increasing.

Under \( Q \) the discounted wealth process \( X_t^* \) is a supermartingale, so

\[
E_Q X_T^* \leq x \quad \text{(budget constraint)}
\]
Under condition C for every RV $V \geq 0$ satisfying $E_Q \beta_T V = x$ there exists an admissible trading strategy such that $X_T = V$ and $X_0 = x$. Indeed, let

$$X_t^* = E_Q[\beta_T V \mid \mathcal{F}_t]$$

then look at (2) again:

$$X_t^* = x + \sum_{i=1}^{n} \int_{0}^{t} \frac{\pi_i(u)}{S_i(u)} dS_i^*(u), \quad t \leq T \quad (2)$$

$I(\cdot)$ is the inverse function of $U'(\cdot)$.

$$Z_T = \frac{dQ}{dP}$$

**THEOREM:** Under condition C the optimal terminal wealth is given by

$$X_T = I(y\beta_T Z_T)$$

where $y > 0$ is a constant determined by the constraint

$$E_Q \beta_T X_T = x$$
Computation of $Z_T$ in the case of example 2.

$$dS_i(t) = \mu_i(t)S_i(t)dt + \sum_{j=1}^{n} S_i(t)\sigma_{i,j}(t)dw_j(t), \ i \leq n$$

This model (stochastic coefficients) satisfies both conditions if the coefficients are adapted to the Brownian motion, and the volatility matrix $\sigma(t)$ is invertible. We assume that this is the case.

$$dS_i^*(t) = S_i^*(t)\left[ (\mu_i(t) - r_t)dt + \sum_{j=1}^{n} \sigma_{i,j}(t)dw_j(t) \right]$$

$$dR_i^*(t) = \frac{1}{S_i^*(t)}dS_i^*(t), \ R_i^*(0) = 0$$

$S_i^*$ is a martingale under $Q$ if $R^*$ is a martingale.

$$dR_t^* = \sigma_t\left[ \sigma_t^{-1}(\mu_t - r_t1)dt + dw_t \right] = \sigma_t d\tilde{w}_t$$

$\tilde{w}$ becomes a BM on $[0,T]$ under $Q$ if

$$\frac{dQ}{dP} = Z_T = \exp\{-\int_0^T \theta_u d\tilde{w}_u + \frac{1}{2} \int_0^T \|\theta_u\|^2 du\}$$

$$\theta_t = \sigma_t^{-1}(\mu_t - r_t1)$$
Computation of the optimal trading strategy

Assume $\sigma_t = \sigma$, $\mu_t = \mu$, $r_t = r$ constants, $\sigma$ is invertible.

$$X_T = I \left( ye^{-rT} \exp\{-\theta \tilde{w}_T + \frac{1}{2} \| \theta \|^2 T \} \right)$$

$$e^{-rT} X_T = x + \int_0^T e^{-rt} \pi_t^T r \sigma d\tilde{w}_t$$

Let $e^{-rT} X_T = G(\tilde{w}_T)$. We have the representation

$$G(\tilde{w}_T) = x + \int_0^T \phi_t^T r d\tilde{w}_t$$

hence

$$\pi_t = e^{rt} (\sigma^{Tr})^{-1} \phi_t$$

By the Markov property of the BM

$$E_Q[G(\tilde{w}_T) \mid \mathcal{F}_t] = g(t, \tilde{w}_t)$$

If $g(\cdot, \cdot)$ is sufficiently smooth then Ito’s rule yields

$$\phi_t = \nabla g(t, \tilde{w}_t)$$
Here $\nabla$ is the gradient with respect to the $w$ variable. Exchange the gradient and the conditional expectation:

$$\phi_t = E_Q[\nabla G(\tilde{w}_T) \mid \mathcal{F}_t]$$

(3)

If $U(x) = \frac{1}{\lambda} x^\lambda$ for some $\lambda \in (0, 1)$:

$$e^{-r^T X_T} = C_1(T) \exp\left\{ \frac{\theta}{1 - \lambda} \tilde{w}_T - \frac{1}{2} \frac{||\theta||^2}{1 - \lambda} T \right\}$$

(3) yields

$$\phi(t) = E_Q\left[ \frac{1}{1 - \lambda} \theta e^{-r^T X_T} \mid \mathcal{F}_t \right] = \frac{1}{1 - \lambda} \theta e^{-r t} X_t$$

$$\pi_t = \frac{1}{1 - \lambda} (\sigma \sigma^{Tr})^{-1} (\mu - r1) X_t$$

A fixed proportion of the wealth is invested in each asset. Substitution of $\lambda = 0$ gives the optimal trading strategy for $U(x) = \log x$. 
An example in which condition C does not hold.


\[
dS_t = S_t \left( \mu_t dt + \sigma dw_1(t) \right)
\]

\[
d\mu_t = \alpha (\delta - \mu_t) dt + \beta dw_2(t)
\]

\(n = 1, \ r, \alpha, \beta, \sigma > 0\). We require that \(\pi_t\) is \(\mathcal{F}_t^S\)-measurable.

\[
d(e^{-rt}S_t) = \sigma e^{-rt} S_t d\tilde{w}_t
\]

where

\[
d\tilde{w}_t = \frac{1}{\sigma} (\mu_t - r) dt + dw_1(t)
\]

\[
Z_T = \frac{dQ}{dP} = \exp \left\{ - \int_0^T \frac{1}{\sigma} (\mu_t - r) d\tilde{w}_t 
\right. \\
\left. + \frac{1}{2\sigma^2} \int_0^T (\mu_t - r)^2 dt \right\}
\]
However, the there is no such trading strategy in this case that $X_T = I(C(T)Z_T)$. Define

$$\zeta_T = E[Z_T \mid \mathcal{F}_T^S]$$

The optimal terminal wealth is

$$X_T = I(C(T)\zeta_T)$$

where $C(T)$ is a constant determined by the budget constraint.

Computation of $\zeta_T$:

$$\zeta_t = E_Q[\zeta_T \mid \mathcal{F}_t^S]$$

Define $Z_t$ for $t \leq T$ by writing $t$ instead of $T$ in the definition of $Z_T$. Then

$$d(Z_t^{-1}) = Z_t^{-1}\frac{1}{\sigma}(\mu_t - r)d\tilde{w}_t$$

$$d(\zeta_t^{-1}) = \zeta_t^{-1}\frac{1}{\sigma}(m_t - r)d\tilde{w}_t$$

where

$$m_t = E[\mu_t \mid \mathcal{F}_t^S]$$
It follows that

\[
\zeta_T = \exp\left\{ -\frac{1}{\sigma} \int_0^T (m_u - r) \, d\tilde{w}_u \right. \\
\left. + \frac{1}{2\sigma^2} \int_0^T (m_u - r)^2 \, du \right\}
\]

Compute \( m_t \) using the Kalman-Bucy filter:

\[
dm_t = \\
\left( \left[ -\alpha - \frac{1}{\sigma^2} \gamma(t) \right] m_t + \left[ \alpha \delta + \frac{r}{\sigma^2} \gamma(t) \right] \right) dt + \frac{1}{\sigma} \gamma(t) d\tilde{w}_t
\]

where \( \gamma(t) = E(\mu_t - m_t)^2 \)

\[
\frac{d}{dt} \gamma(t) = -\frac{1}{\sigma^2} \gamma^2(t) - 2\alpha \gamma(t) + \beta^2 \quad \text{(Riccati eq.)}
\]
Computation of the optimal trading strategy using the gradient operator.


Lakner & Ma, Portfolio Optimization with Downsize Constraints, *Mathematical Finance*.

Lakner & Ma, Partial Hedging Using Malliavin Calculus.

The previous method does not work because $I\left(C(T)\zeta_T\right)$ is not a function of $\tilde{w}_T$. 
Gradient operator.

\( \tilde{w} \): 1-dimensional BM.

Random variables of the form

\[ F = f(\tilde{w}(t_1), \ldots, \tilde{w}(t_m)) \]

where \( f \in C^\infty_b(\mathbb{R}^m) \) are called simple functionals. The class of simple functionals are denoted by \( S \).

\[ D_t F = \sum_{j=1}^{m} \frac{\partial f}{\partial x^j}(\tilde{w}(t_1), \ldots, \tilde{w}(t_m))1\{t \leq t_j\} \]

For example, if \( F = f(\tilde{w}_s) \) then

\[ D_t F = \begin{cases} f'(\tilde{w}_s), & \text{if } t \leq s; \\ 0, & \text{otherwise}. \end{cases} \]

We regard \( DF \) as a \( L^2[0, T] \)-valued random variable.

\[ D : S \subset L^1(\omega) \mapsto L^1(\Omega, L^2[0, T]) \]

is closable. The domain of the closure of \( D \) is denoted by \( D_{1,1} \).
Clark-Ocone formula: For every $F \in D_{1,1}$

$$F = EF + \int_0^T E[D_tF \mid \mathcal{F}_t^\tilde{w}]d\tilde{w}_t$$

Apply this to $F = e^{-rt}X_T$. In the $U(x) = \log x$ case we get

$$\pi_t = \frac{1}{\sigma^2}(m_t - r)X_t$$

Compare this to the constant coefficient case:

$$\pi_t = \frac{1}{\sigma^2}(\mu - r)X_t$$

Formally substitute $\mu$ by the estimate $m_t$. However, this formal derivation of the optimal trading strategy does not work for $U(x) = \frac{1}{\lambda}x^\lambda$. 